УДК 517.53

M. M. Sheremeta, M. T. Bordulyak

A PROBLEM IN THE THEORY OF ENTIRE FUNCTIONS OF BOUNDED INDEX

M. M. Sheremeta, M. T. Bordulyak. *A problem in the theory of entire functions of bounded index*, Matematychni Studii, **15** (2001) 105–108.

A problem on the relation between the l-index boundedness of an entire function and the growth of its index in the sense of G.Frank is stated.

М. Н. Шеремета, М. Т. Бордуляк. *Одна проблема в теории целых функций ограниченного индекса* // Математичні Студії. – 2001. – Т.15, №1. – С.105–108.

Поставлена задача о связи между ограниченностью l-индекса целой функции и ростом ее индекса в смысле Г.Франка.

Definitions. Let l be a positive continuous function on $[0, +\infty)$. An entire function f is said to be of bounded l-index [1] if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \leq \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \leq k \leq N\right\}.$$

The least of such N is called the l-index of f and denoted by $N(l; f)$. For $l(r) \equiv 1$ whence we obtain the definition [2] of entire function of bounded index $N(f) = N(l; f)$.

An entire function f is said to be of bounded l-M-index [3] if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $r \in [0, +\infty)$

$$\frac{M(r, f^{(n)})}{n!l^n(r)} \leq \max\left\{\frac{M(r, f^{(k)})}{k!l^k(r)} : 0 \leq k \leq N\right\},$$

where $M(r, f) = \max\{|f(z)| : |z| = r\}$. For $l(r) \equiv 1$ whence we obtain the definition [4] of entire function of bounded M-index.

Clearly, if f is an entire function of bounded l-index then f is an entire function of bounded l-M-index.

As in [5], we denote $c_f(a) = \max\left\{\frac{|f^{(n)}(a)|}{n!} : n \in \mathbb{Z}_+\right\}$, $k_f(a) = \max\left\{k : \frac{|f^{(k)}(a)|}{k!} = = c_f(a)\right\}$ and $I(r, f) = \sup\{k_f(a) : |a| \leq r\}$. Then f is an entire function of bounded index if and only if $I(r, f) = O(1)$, $r \to +\infty$.

2000 *Mathematics Subject Classification:* 30D15.

Finally, let $J(r,f) = \max\left\{j : \frac{M(r,f^{(j)})}{j!} \geq \frac{M(r,f^{(n)})}{n!} \quad \text{for all } n \in \mathbb{Z}_+\right\}$. Then [6] $J(r,f) \leq I(r,f)$.

Some results on the growth. In [7–8] it is shown that if an entire function f is of bounded index then

$$\varlimsup_{r\to+\infty} \frac{\ln M_f(r)}{r} \leq N(f)+1.$$

Let K be the class of positive analytic on $[0,+\infty)$ functions l such that $l'(x) = o(l^2(x))$, $x \to +\infty$, and Q be the class of positive continuous on $[0,+\infty)$ functions l such that $l(x+O(1/l(x))) = O(l(x))$, $x \to +\infty$. It is known [1; 9, p. 72] that if $l \in K$ and f is of bounded l-index then

$$\varlimsup_{r\to+\infty} \frac{\ln M(r,f)}{L(r)} \leq N(l;f)+1, \quad L(r) = \int\limits_0^r l(x)dx, \tag{1}$$

If $l \in Q$ and f is of bounded l-index then [9, p. 71] $\ln M(r,f) = O(L(r))$, $r \to +\infty$.

Finally, in [5-6] it is shown that if an entire function f has the order $\rho < \infty$ then

$$\max\{0,\ \rho-1\} \leq \varlimsup_{r\to+\infty} \frac{\ln^+ J(r,f)}{\ln r} \leq \varlimsup_{r\to+\infty} \frac{\ln^+ I(r,f)}{\ln r} \leq \rho.$$

Problems. The main problem consists in establishment of a relation between the behaviour of $I(r,f)$ and the boundedness of the l-index of an entire function f.

If $l(r) \equiv 1$ and f is of bounded l-index then $I(r,f) = O(l(r))$, $r \to +\infty$. Therefore, the following problem is actual: *for which functions l the boundedness of the l-index of an entire function f implies the relation $I(r,f) = O(l(r))$, $r \to +\infty$?*

We remark, that if $l \in K$ and $l(r) \to 0$, $r \to +\infty$ then there is not exist an entire function f of bounded l-index such that $I(r,f) = O(l(r))$, $r \to +\infty$. Indeed, if such function f exists then in view of (1) $\ln M(r,f) = o(r)$, $r \to +\infty$, and, thus, f has zeroes. Therefore, $I(r,f) \geq 1$ for greater r. On other hand, since $I(r,f)$ is nondecreasing and $I(r,f) = O(l(r)) = o(1)$, $r \to +\infty$, then $I(r,f) \equiv 0$.

We can show only certain conditions on l in order that the boundedness of the l-index of an entire function f implies the relation $J(r,f) = O(l(r))$, $r \to +\infty$.

By Ω we denote a class of positive on $(-\infty,+\infty)$ functions Φ such that the derivative Φ' is continuous, positive and increasing to $+\infty$ on $(-\infty,+\infty)$. For $\Phi \in \Omega$ let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be a function associated with Φ in sense of Newton. Then Ψ is continuous and increasing to $+\infty$ on $(-\infty,+\infty)$.

Theorem. *Let $\Phi \in \Omega$, $\Phi'(x) = O(\Phi'(\Psi(x))$ and $\Phi'(x+O(e^{-x})) = O(\Phi'(x))$ as $x \to +\infty$. Let $l \in K$ and $l(r) = \Phi'(\ln r)/r$ for $r \geq r_0$. If an entire function f is of bounded l-index then $J(r,f) = O(l(r))$, $r \to +\infty$.*

Proof. Since $l \in K$ and $l(r) = \dfrac{\Phi'(\ln r)}{r}$ for $r \geq r_0$, we have $L(r) = \Phi(\ln r) + O(1)$, $r \to +\infty$, and from (1) we obtain

$$\varlimsup_{r\to+\infty} \frac{\ln M(r,f)}{\Phi(\ln r)} \leq N(l;f)+1. \tag{2}$$

By Hadamard's theorem the function $\ln M(e^x, f)$ is convex on $(-\infty, +\infty)$, i.e.

$$\ln M(e^x, f) = A + \int_{x_0}^{x} \omega(t)dt, \tag{3}$$

where ω is nondecreasing function on $(-\infty, +\infty)$. □

In [10] the following lemma is proved.

Lemma 1. *Let C be the class of convex functions P on $[a, b)$ and let Φ be a continuously differentiable function on $[a, b)$ such that $\Phi'(x) \to +\infty$, $x \to b$. In order that for each $P \in C$*

$$\left(\overline{\lim_{x\to b}} \frac{P(x)}{\Phi(x)} < +\infty\right) \implies \left(\overline{\lim_{x\to b}} \frac{P'(x)}{\Phi'(x)} < +\infty\right),$$

where P' is right-side derivatives of P, it is necessary and sufficient that

$$\overline{\lim_{x\to b}} \frac{1}{\Phi'(x)} \inf_{t>x} \frac{\Phi'(t)}{t-x} < +\infty.$$

If $\Phi \in \Omega$ then

$$\overline{\lim_{x\to+\infty}} \frac{1}{\Phi'(x)} \inf_{t>x} \frac{\Phi'(t)}{t-x} = \overline{\lim_{x\to+\infty}} \frac{1}{\Phi'(x)} \frac{\Phi(\Psi^{-1}(x))}{\Psi^{-1}(x)-x} = \overline{\lim_{x\to+\infty}} \frac{\Phi(x)}{\Phi'(\Psi(x))(x-\Psi(x))} =$$
$$= \overline{\lim_{x\to+\infty}} \frac{\Phi'(x)}{\Phi'(\Psi(x))}.$$

Therefore, since $\Phi'(x) = O(\Phi'(\Psi(x))$, $x \to +\infty$, by Lemma from (2) and (3) we obtain $\omega(t) = O(\Phi'(t))$, $t \to +\infty$.

In [6] it is proved that $J(r, f) \ln 2 \le \ln \dfrac{M(r+2, f)}{M(r, f)}$ for all $r \ge 0$. Hence, since $\Phi'(x + O(e^{-x})) = O(\Phi'(x))$ as $x \to +\infty$,

$$J(r, f) \le \frac{1}{\ln 2}(\ln M(e^{\ln(r+2)}, f) - \ln M(e^{\ln r}, f)) = \frac{1}{\ln 2} \int_{\ln r}^{\ln(r+2)} \omega(t)dt \le$$
$$\le C_1 \int_{\ln r}^{\ln(r+2)} \Phi'(t)dt \le C_1 \Phi'(\ln(r+2)) \ln\left(1 + \frac{2}{r}\right) \le$$
$$\le 2C_1 \frac{\Phi'(\ln r)}{r} \frac{\Phi'(\ln r + 2/r)}{\Phi'(\ln r)} \le C_2 \frac{\Phi'(\ln r)}{r} = C_2 l(r), \quad C_j = \text{const}.$$

Remark. The condition $l \in K$ can be replaced by the condition $l \in Q$.

REFERENCES

1. Kuzyk A. D., Sheremeta M. M. *Entire functions of bounded value l-distribution,* Matem. Zametki. **39** (1986), no. 1, 3–13 (in Russian).

2. Lepson B. *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index,* Proc. Sympos. Pure Math. V.2. Amer. Math. Soc., Providence, Phode Island, 1968, 298–307.

3. Abuarabi Sh., Sheremeta M. M. *Entire functions of bounded l-M-index,* Dopovidi AN URSR, ser.A.(1989), 3–5 (in Ukrainian).

4. Fricke G. H., Shah S. M., Sisarcik W. S.*A characterization of entire functions of exponential type and M-bounded index,* Indiana Univ. Math. J. **23** (1973), 405–412.

5. Frank G. *Uber den Index einer ganzen Funktionen,* Ark. der Math. **22** (1971), 175–180.

6. Frank G., Mues E. *Uber das Wachstum des Index ganzer Funktionen,* Math. Ann. **195** (1972), 114–120.

7. Hayman W. K. *Differential inequalities and local valency,* Pacific J. Math. (1973), 117–137.

8. Shah S. M. *Entire functions of bounded index,* Lect. Notes in Math. **589** (1977), 117–145.

9. Sheremeta M. M. *Analytic functions of bounded index,* Lviv: VNTL Publishers., 1999, 141 pp.

10. Bratishchev A. V. *On the converse of L'Hospital's rule,* Meh. sploshnoj sredy, Rostov-na-Donu: Izd. RSU, 1985, 28–42 (in Russian).

Faculty of Mechanics and Mathematics, Lviv National University

Received 15.05.2000