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A FOURIER PROBLEM FOR QUASI-LINEAR PARABOLIC EQUATIONS OF ARBITRARY ORDER IN NONCYLINDRIC DOMAINS

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The well-posedness of the Fourier Problem defined in noncylindric domains has been proved for some class of quasi-linear parabolic equations of higher order. No conditions on the behaviour of a solution and increasing of the data-in at $t \to -\infty$ are required.

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Установлена корректность задачи Фурье для некоторого класса квазилинейных параболических уравнений высших порядков в нецилиндрических областях. Не накладывается никаких условий на поведение решения и рост начальных условий при $t \to -\infty$.

1. Introduction. The boundary value problems in unbounded domains for linear and variety of nonlinear parabolic equations are well-posed if there are some restrictions on behaviour of the a solution and increasing the data-in at infinity (see [7,9–10] and other).

But there are a number of papers (see [1–6,11] and other) where the classes of nonlinear equations which have unique solutions of the corresponding boundary value problems without any restrictions on its behaviour at infinity whereas data-in has free arriving at infinity are founded.

Individually, for the Fourier Problem such results have been obtained for quasi-linear parabolic equations of arbitrary degree with strong monotone space part [4-5]. In these papers the problem is posed in cylindric domain. In our paper analogous results are transmitted into the same problem posed in noncylindric domains.

2. Statement of problem and formulation of main results. Let $Q \subset \mathbb{R}^{n+1}_{x,t}$ be a domain in $\{(x,t) : t < T\}$, where $0 < T < +\infty$. Suppose that for every $\tau \in (-\infty; T)$ set $\Omega_\tau = Q \cap \{(x,t) : t = \tau\}$ is bounded and nonempty; $\Sigma = \partial Q \cap \{(x,t) : t < T\}$ is an n-dimensional hyperplane of class C^m and $\nu_{n+1} \neq \pm 1$ on Σ, where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the external unit vector normal to the surface Σ in the space $\mathbb{R}^{n+1}_{x,t}$.

We consider the problem

$$u_t + \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha a_\alpha(x,t,\delta u) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha(x,t) \qquad \text{in} \quad Q, \tag{1}$$

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$$\frac{\partial^j u}{\partial \nu^j} = 0 \qquad \text{on} \quad \Sigma, \quad j = 0, 1, \dots, m-1. \tag{2}$$

Here m is an arbitrary natural number, δu is a vector consisting of all possible derivatives $D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}$ of the degree less than or equal to m, and N is the dimension of this vector. From now on, the letters α, β, γ denote multiindexes of length n.

We impose the following conditions on the data-in:

(A1) the functions $a_\alpha(x,t,\xi)$ $(|\alpha| \leqslant m)$ are defined for almost all points $(x,t) \in Q$ and all vectors $\xi \in \mathbb{R}^N$ with coordinates ξ_β $(|\beta| \leqslant m)$ and of Caratheodory type, that is they are measurable on (x,t) for all ξ and continuous on ξ for almost all (x,t); $a_\alpha(x,t,0) = 0$ $(|\alpha| \leqslant m)$;

(A2) there is some number $p \geqslant 2$ such that for all α $(|\alpha| \leqslant m)$

$$|a_\alpha(x,t,\xi)| \leqslant h_\alpha(x,t) \sum_{|\beta| \leqslant m} |\xi_\beta|^{p-1} + k_\alpha(x,t),$$

where $h_\alpha \in L_{\infty,\mathrm{loc}}(\overline{Q})$, $k_\alpha \in L_{p',\mathrm{loc}}(\overline{Q})$, $\frac{1}{p} + \frac{1}{p'} = 1$;

(A3) for arbitrary vectors $\xi, \eta \in \mathbb{R}^N$ and for almost all $(x,t) \in Q$ the following inequality holds:

$$\sum_{|\alpha| \leqslant m} (a_\alpha(x,t,\xi) - a_\alpha(x,t,\eta))(\xi_\alpha - \eta_\alpha) \geqslant K_0 \sum_{|\alpha| = m} |\xi_\alpha - \eta_\alpha|^p,$$

where K_0 is a positive constant;

(A4) $f_\alpha \in L_{p',\mathrm{loc}}(\overline{Q})$ $(|\alpha| \leqslant m)$.

Remark 1. Let us note that as an example of an equation satisfying the conditions (A1)-(A4), we may consider the equation

$$u_t + \sum_{|\alpha| = m} (-1)^{|\alpha|} D^\alpha (|D^\alpha u|^{p-2} D^\alpha u) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha(x,t),$$

where $p \geqslant 2$, $f_\alpha \in L_{p',\mathrm{loc}}(\overline{Q})$ $(|\alpha| \leqslant m)$. It follows from Lemma 1.2 [4] that condition (A3) is valid with $K_0 = 2^{2-p}$.

For a domain Ω in $\mathbb{R}^n_x$ and a natural number k, we denote by $\overset{\circ}{W}{}^k_p(\Omega)$ the completion of the space $C_0^\infty(\Omega)$ by the norm $\|v\| = \sum_{|\alpha|=k} \|D^\alpha v\|_{L^p(\Omega)}$, and by $W^{-m}_{p'}(\Omega)$ the dual space to $\overset{\circ}{W}{}^m_p(\Omega)$.

Let $Q_{t_1,t_2} = Q \cap \{(x,t) : t_1 < t < t_2\}$, where t_1, t_2 are arbitrary numbers, $-\infty < t_1 < t_2 \leqslant T$. $\overset{\circ}{W}{}^{k,0}_p(Q_{t_1,t_2})$ stands for the completion of the space $C_0^\infty(Q_{t_1,t_2})$ by the norm $\|v\| = \sum\limits_{|\alpha|=k} \|D^\alpha v\|_{L^p(Q_{t_1,t_2})}$. Denote by $\overset{\circ}{W}{}^{m,0}_{p,\mathrm{loc}}(\overline{Q})$ the space of measurable on Q functions whose restrictions on Q_{t_1,t_2} for arbitrary numbers t_1, t_2, $-\infty < t_1 < t_2 \leqslant T$, belong to the space $\overset{\circ}{W}{}^{m,0}_p(Q_{t_1,t_2})$.

Definition 1. *A generalized solution of Problem* (1), (2) is a function $u \in \overset{\circ}{W}{}^{m,0}_{p,\mathrm{loc}}(\overline{Q})$, satisfying for any functions $\psi \in C_0^\infty(Q)$ the following integral equality

$$\iint_Q \Big\{ -u\psi_t + \sum_{|\alpha| \leqslant m} a_\alpha(x,t,\delta u) D^\alpha \psi \Big\}\, dxdt = \iint_Q \sum_{|\alpha| \leqslant m} f_\alpha D^\alpha \psi \, dxdt. \tag{3}$$

In the sequel we assume that conditions (A1)–(A4) hold. We shall consider the question of well-posedness of Problem (1), (2) i.e. we shall study conditions under which a generalized solution 1) exists for each set of functions $\{f_\alpha \in L_{p',\mathrm{loc}}(\overline{Q}),\ |\alpha| \leqslant m\}$; 2) is unique; 3) continuously depends on data-in.

Continuous dependence of the generalized solution of Problem (1), (2) on data-in is understood as follows: for arbitrary sequences $\{f_{\alpha,k}\}_{k=1}^{\infty} \subset L_{p,\mathrm{loc}}(\overline{Q})$ $(|\alpha| \leqslant m)$ such that $f_{\alpha,k} \underset{k\to\infty}{\to} f_\alpha$ in $L_{p,\mathrm{loc}}(\overline{Q})$ $(|\alpha| \leqslant m)$ the corresponding sequence of generalized solutions $\{u_k\}$ of Problem (1), (2) with f_α replaced by $f_{\alpha,k}$ $(|\alpha| \leqslant m)$ converges to a generalized solution u of Problem (1), (2) in $\overset{\circ}{W}{}^{m,0}_{p,\mathrm{loc}}(\overline{Q})$.

Note that $g_k \to g$ in $L_{p,\mathrm{loc}}(\overline{Q})$ $(\overset{\circ}{W}{}^{m,0}_{p,\mathrm{loc}}(\overline{Q}))$, if for any numbers t_1, t_2 $(-\infty < t_1 < t_2 \leqslant T)$ $g_k \to g$ in $L_p(Q_{t_1,t_2})$ $(\overset{\circ}{W}{}^{m,0}_{p}(Q_{t_1,t_2}))$.

Let for any multiindex α $(|\alpha| = m)$ $\lambda_\alpha \in L_{\infty,\mathrm{loc}}((-\infty, T])$ be a positive function, such that

$$\|v\|_{L_2(\Omega_t)} \leqslant \lambda_\alpha(t)\|D^\alpha v\|_{L_p(\Omega_t)} \tag{4}$$

for arbitrary $t \in (-\infty, T)$ and $v \in \overset{\circ}{W}{}^{m}_{p}(\Omega_t)$ (see [4, Lemma 1]).

Remark 2. Let $\Omega_t \subset K_t = \{x : -\infty < a_j(t) < x_j < b_j(t) < \infty, j = 1, ..., n\}$ for any $t \in (-\infty, T)$. In this case we can choose $\lambda_\alpha(t) = \prod_{l=1}^{n}(b_l(t) - a_l(t))^{p\alpha_l},\ t \in (-\infty, T)$.

Let for any natural number k $\varkappa_k \in L_\infty((-\infty, T])$ be a positive function such that

$$\|v\|_{L_p(\Omega_t)} \leqslant \varkappa_k(t)\Big(\sum_{|\alpha|=k} \|D^\alpha v\|^p_{L_p(\Omega_t)}\Big)^{\frac{1}{p}} \tag{4$'$}$$

for arbitrary $t \in (-\infty, T]$ and $v \in \overset{\circ}{W}{}^{k}_{p}(\Omega_t)$. Put $\varkappa_0(t) = 1$ for all $t \in (-\infty, T]$.

Theorem. *Suppose that $p > 2$ and there exists a multiindex γ $(|\gamma| = m$) such that $\lambda_\gamma(t) \sim K_\gamma|t|^r$ at $t \to -\infty$, where K_γ, r are constants satisfying $K_\gamma > 0$, $r < \frac{1}{p}$.*

Then Problem (1), (2) is well-posed. Its generalized solution u belongs to the space $L_{\infty,loc}((-\infty; T]; L_2(\Omega_t))$, besides, $\|u(\cdot, t)\|_{L_2(\Omega_t)} \in C((-\infty, T])$. Moreover, for arbitrary numbers t_1, t_2, δ such that $-\infty < t_1 < t_2 \leqslant T$, $\delta > 0$, u satisfies the estimate:

$$\sup_{[t_1,t_2]} \int_{\Omega_t} |u(x,t)|^2\, dx + \iint_{Q_{t_1,t_2}} \sum_{|\alpha|=m} |D^\alpha u|^p\, dxdt \leqslant$$
$$\leqslant C_1\delta^{-\frac{p}{p-2}} \int_{t_1-\delta}^{t_1} [\lambda_\gamma(t)]^{\frac{2p}{p-2}} dt + C_2 \iint_{Q_{t_1-\delta,t_2}} \varkappa^{p'}_{m-|\alpha|}(t) \sum_{|\alpha|\leqslant m} |f_\alpha(x,t)|^{p'}\, dxdt, \tag{5}$$

where C_1, C_2 are positive constants depending only on n, m, K_0, γ, p .

Corollary. *Suppose that $\varkappa_{m-|\alpha|} \cdot f_\alpha \in L_{p'}(Q)$ for every $\alpha(|\alpha| \leqslant m)$ and the conditions of Theorem hold. Then the generalized solution of Problem (1), (2) satisfies the following inequality*

$$\sup_{(-\infty,T]} \int_{\Omega_t} |u(x,t)|^2\, dx + \iint_Q \sum_{|\alpha|=m} |D^\alpha u|^p\, dxdt \leqslant C_2 \iint_Q \varkappa^{p'}_{m-|\alpha|}(t) \sum_{|\alpha|\leqslant m} |f_\alpha(x,t)|^{p'}\, dxdt, \tag{6}$$

where C_2 is the constant from (5).

3. Auxiliary statements.

Lemma 1. *Let* $v \in \overset{\circ}{W}{}_p^{m,0}(Q_{\tau_1,\tau_2})$ *and* $g_\alpha \in L_{p'}(Q_{\tau_1,\tau_2})$ $(|\alpha| \leqslant m)$ *satisfy the integral equality*

$$\iint\limits_{Q_{\tau_1,\tau_2}} \Big\{ -v\psi_t + \sum_{|\alpha|\leqslant m} g_\alpha D^\alpha \psi \Big\}\, dxdt = 0 \tag{7}$$

for an arbitrary function $\psi \in C_0^\infty(Q_{\tau_1,\tau_2})$, *where* τ_1, τ_2 *are arbitrary numbers such that* $-\infty < \tau_1 < \tau_2 \leqslant T$.

Then function $y(t) = \int_{\Omega_t} v^2(x,t)\, dx$ *is absolutely-continuous on* $[\tau_1, \tau_2]$ *and for arbitrary piece-smooth on* $[\tau_1, \tau_2]$ *function* θ *and for any* $t_1, t_2,\ \tau_1 \leqslant t_1 < t_2 \leqslant \tau_2$, *it holds the following equality*

$$\theta(t_2) \int\limits_{\Omega_{t_2}} v^2(x,t_2)\, dx - \theta(t_1) \int\limits_{\Omega_{t_1}} v^2(x,t_1)\, dx - \iint\limits_{Q_{t_1,t_2}} v^2 \theta'\, dxdt + 2 \iint\limits_{Q_{t_1,t_2}} \theta \sum_{|\alpha|\leqslant m} g_\alpha D^\alpha v\, dxdt = 0. \tag{8}$$

Proof. Let t_1, t_2 be any numbers, $\tau_1 \leqslant t_1 < t_2 \leqslant \tau_2$. Denote by $\hat v, \hat g_\alpha$ $(|\alpha| \leqslant m)$ the restrictions of v and g_α $(|\alpha| \leqslant m)$ on Q_{t_1,t_2}, respectively. The function $\hat v$ belongs to the space $\overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2})$, and $\hat g_\alpha$ belongs to the space $L_{p'}(Q_{t_1,t_2})$ $(|\alpha| \leqslant m)$.

It follows from the identity (7) that

$$\iint\limits_{Q_{t_1,t_2}} \hat v \psi_t\, dxdt = \iint\limits_{Q_{t_1,t_2}} \sum_{|\alpha|\leqslant m} \hat g_\alpha D^\alpha \psi\, dxdt \tag{9}$$

for an arbitrary $\psi \in C_0^\infty(Q_{t_1,t_2})$. We can consider the right side of (9) as a linear functional on $C_0^\infty(Q_{t_1,t_2}) : \psi \to \iint_{Q_{t_1,t_2}} \sum_{|\alpha|\leqslant m} \hat g_\alpha D^\alpha \psi\, dxdt$, which is continuous in the norm of the space $\overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2})$. Let us extend this functional on the whole space $\overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2})$ by the continuity and denote this extension by F. It is evident that F is an element of the space $(\overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2}))^*$, which is dual to the space $\overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2})$. So we can rewrite the identity (9) as

$$-\iint\limits_{Q_{t_1,t_2}} \hat v \psi_t\, dxdt = \langle F, \psi \rangle \tag{10}$$

for any $\psi \in C_0^\infty(Q_{t_1,t_2})$, where $\langle \cdot, \cdot \rangle$ is a canonical bilinear form on $(\overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2}))^* \times$ $\times\ \overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2})$, $\langle F, \hat\psi \rangle = \iint_{Q_{t_1,t_2}} \hat g_\alpha D^\alpha \hat\psi\, dxdt$ for an arbitrary $\hat\psi \in \overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2})$. ¿From correlation (10) it follows that $\hat v_t \in (\overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2}))^*$ and

$$\langle \hat v_t, \hat\psi \rangle = \langle F, \hat\psi \rangle \quad \forall \hat\psi \in \overset{\circ}{W}{}_p^{m,0}(Q_{t_1,t_2}). \tag{11}$$

Substituting $\hat\psi = \hat v$ in (11), we get

$$\langle \hat v_t, \hat v \rangle = \langle F, \hat v \rangle. \tag{12}$$

Taking into account the formula of integration by the parts for the functions of type $\hat v$ (see, for example, [8;p.359]) we obtain

$$\langle \hat v_t, \hat v \rangle = \frac{1}{2} \Big(\int\limits_{\Omega_{t_2}} \hat v^2(x,t_2)\, dx - \int\limits_{\Omega_{t_1}} \hat v^2(x,t_1)\, dx \Big). \tag{13}$$

It follows from (12) and (13) that

$$\frac{1}{2}\Big(\int\limits_{\Omega_{t_2}} v^2(x,t_2)\,dx - \int\limits_{\Omega_{t_1}} v^2(x,t_1)\,dx\Big) = \iint\limits_{Q_{t_1,t_2}} \sum_{|\alpha|\leqslant m} g_\alpha D^\alpha v\,dxdt. \tag{14}$$

Since the numbers t_1, t_2 are arbitrary, from (14) we obtain that the function $y(t) = \int\limits_{\Omega_t} v^2(x,t)\,dx$ is absolutely-continuous on $[\tau_1, \tau_2]$ and

$$\frac{1}{2}\frac{dy(t)}{dt} + \int\limits_{\Omega_t} \sum_{|\alpha|\leqslant m} g_\alpha D^\alpha v\,dx = 0 \tag{15}$$

for almost all $t \in [-\tau_1, \tau_2]$. Multiplying the identity (15) by θ and integrating by parts from t_1 to t_2 we obtain (8). Lemma 1 is proved. □

Let $Q_0 = Q \cap \{0 < t < T\}$, $\Sigma_0 = \partial Q \cap \{0 < t < T\}$. We consider the problem

$$u_t + \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha a_\alpha(x,t,\delta u) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha(x,t) \quad \text{in} \quad Q_0, \tag{16}$$

$$\frac{\partial^j u}{\partial \nu^j} = 0 \quad \text{on} \quad \Sigma_0, \quad j = 0,1,...,m-1, \tag{17}$$

$$u|_{t=0} = 0. \tag{18}$$

Definition 2. *A generalized solution of problem (16)–(18)* is a function $u \in \overset{\circ}{W}{}^{m,0}_{p}(Q_0)$, satisfying the following integral equality

$$\iint\limits_{Q_0} \Big\{-u\psi_t + \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u) D^\alpha \psi\Big\}\,dxdt = \iint\limits_{Q_0} \sum_{|\alpha|\leqslant m} f_\alpha D^\alpha \psi\,dxdt \tag{19}$$

for arbitrary functions $\psi \in C_0^\infty(Q)$.

Remark 3. Let us note that in (19) we take ψ from the space $C_0^\infty(Q)$. It provides the correctness of condition (18) in some sense.

Lemma 2. *Problem (16)–(18) has a unique generalized solution.*

Proof. Let us prove with the help of the method of fine. Let $G_0 = \Omega \times (0,T)$, where Ω is a bounded domain in $\mathbb{R}^n_x$ with partly smooth boundary $\partial\Omega$, such that $Q_0 \subset G_0$. We consider an auxiliary problem

$$u_{\varepsilon t} + \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha a^0_\alpha(x,t,\delta u_\varepsilon) + \frac{1}{\varepsilon} M u_\varepsilon = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f^0_\alpha(x,t) \quad \text{in} \quad G_0, \tag{20}$$

$$\frac{\partial^j u_\varepsilon}{\partial \nu^j} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \quad j = 0,1,...,m-1, \tag{21}$$

$$u_\varepsilon(x,0) = 0, \tag{22}$$

where $\varepsilon > 0$ is any real number; the function $M \in L_\infty(G_0)$ is equal to zero on Q_0 and to one on $G_0 \setminus Q_0$; $f^0_\alpha(x,t) = f_\alpha(x,t)$, if $(x,t) \in Q_0$, and $f^0_\alpha(x,t) = 0$, if $(x,t) \in G_0 \setminus Q_0$; $a^0_\alpha(x,t,\xi) = a_\alpha(x,t,\xi)$, if $(x,t) \in Q_0$, $\xi \in \mathbb{R}^N$, and $a^0_\alpha(x,t,\xi) = K_0 \cdot 2^{2-p}|\xi_\alpha|^{p-2}\xi_\alpha$, if $(x,t) \in G_0 \setminus Q_0$, $\xi \in \mathbb{R}^N$ $(|\alpha| \leqslant m)$. It is evident that the functions $a^0_\alpha(x,t,\xi)$ $(|\alpha| \leqslant m)$ have Caratheodory type and for arbitrary vectors $\xi, \eta \in \mathbb{R}^N$ and for almost all $(x,t) \in Q_0$ the following inequality holds:

(A′3) $\sum\limits_{|\alpha|\leqslant m}(a^0_\alpha(x,t,\xi)-a^0_\alpha(x,t,\eta))(\xi_\alpha-\eta_\alpha)\geqslant K_0\sum\limits_{|\alpha|=m}|\xi_\alpha-\eta_\alpha|^p$,

where $K_0=\text{const}>0$ is from condition (A3).

Definition 3. *A generalized solution of problem (20)–(22) is a function* $u_\varepsilon\in\overset{\circ}{W}{}^{m,0}_p(G_0)\cap C([0,T];L_2(\Omega))$*, satisfying condition (22) and the following integral equality*

$$\iint\limits_{G_0}\Big\{-u_\varepsilon\psi_t+\sum_{|\alpha|\leqslant m}a^0_\alpha(x,t,\delta u_\varepsilon)D^\alpha\psi+\frac{1}{\varepsilon}Mu_\varepsilon\psi\Big\}\,dxdt=\iint\limits_{G_0}\sum_{|\alpha|\leqslant m}f^0_\alpha(x,t)D^\alpha\psi\,dxdt \tag{23}$$

for arbitrary functions $\psi\in C^\infty_0(G_0)$.

Existence and uniqueness of the solution of problem (16)–(18) is easily proved using Faedo-Galerkin method (see [5,8]).

From [5, Lemma 2] for arbitrary $\tau\in(0,T]$ we have

$$\frac{1}{2}\int\limits_\Omega[u_\varepsilon(x,\tau)]^2\,dx+\int\limits_0^\tau\int\limits_\Omega\Big\{\sum_{|\alpha|\leqslant m}a^0_\alpha(x,t,\delta u_\varepsilon)D^\alpha u_\varepsilon+\frac{1}{\varepsilon}Mu_\varepsilon^2-\sum_{|\alpha|\leqslant m}f^0_\alpha D^\alpha u_\varepsilon\Big\}\,dxdt=0. \tag{24}$$

From (A′3) we get for all $\tau\in(0,T]$

$$\frac{1}{2}\int\limits_\Omega[u_\varepsilon(x,\tau)]^2\,dx+\int\limits_0^\tau\int\limits_\Omega\{K_0\sum_{|\alpha|=m}|D^\alpha u_\varepsilon|^p+\frac{1}{\varepsilon}Mu_\varepsilon^2\}\,dxdt\leqslant\int\limits_0^\tau\int\limits_\Omega\sum_{|\alpha|\leqslant m}f^0_\alpha D^\alpha u_\varepsilon\,dxdt. \tag{25}$$

Using Young inequality $ab\leqslant\varepsilon a^p+M(\varepsilon,p)b^{p'}$, where $a\geqslant0$, $b\geqslant0$, $\varepsilon>0$, $p>1$, $\frac{1}{p}+\frac{1}{p'}=1$, $M(\varepsilon,p)=\varepsilon^{-1/(p-1)}p^{-p'}(p-1)$, we get

$$\int\limits_0^\tau\int\limits_\Omega\sum_{|\alpha|\leqslant m}f^0_\alpha D^\alpha u_\varepsilon\,dxdt\leqslant\sum_{|\alpha|\leqslant m}\varepsilon_\alpha\int\limits_0^\tau\int\limits_\Omega|D^\alpha u_\varepsilon|^p\,dxdt+\sum_{|\alpha|\leqslant m}M(\varepsilon_\alpha,p)\int\limits_0^\tau\int\limits_\Omega|f^0_\alpha|^{p'}\,dxdt\leqslant$$

$$\leqslant\sum_{|\alpha|\leqslant m}\varepsilon_\alpha\cdot H_\alpha\int\limits_0^\tau\int\limits_\Omega\sum_{|\beta|=m}|D^\beta u_\varepsilon|^p\,dxdt+\sum_{|\alpha|\leqslant m}M(\varepsilon_\alpha,p)\int\limits_0^\tau\int\limits_\Omega|f^0_\alpha|^{p'}\,dxdt,$$

where $\varepsilon_\alpha>0$ is any real number, $H_\alpha>0$ is a constant such that

$$\int\limits_\Omega|D^\alpha v|^p\,dx\leqslant H_\alpha\int\limits_\Omega\sum_{|\beta|=m}|D^\beta v|^p\,dx$$

for all $v\in\overset{\circ}{W}{}^m_p(\Omega)\,(|\alpha|\leqslant m)$, besides $H_\alpha=1$ for all such α that $|\alpha|=m$. Hence, taking ε_α ($|\alpha|\leqslant m$) such that $\sum\limits_{|\alpha|\leqslant m}\varepsilon_\alpha H_\alpha=\frac{1}{2}K_0$, and from (25) we get

$$\int\limits_\Omega[u_\varepsilon(x,\tau)]^2\,dx+K_0\sum_{|\alpha|=m}\int\limits_0^\tau\int\limits_\Omega|D^\alpha u_\varepsilon|^p\,dxdt+\frac{2}{\varepsilon}\int\limits_0^\tau\int\limits_\Omega Mu_\varepsilon^2\,dxdt\leqslant$$

$$\leqslant C_3\int\limits_0^\tau\int\limits_\Omega\sum_{|\alpha|\leqslant m}|f^0_\alpha(x,t)|^{p'}\,dxdt, \tag{26}$$

where $C_3 > 0$ is a constant which doesn't depend on τ and ε. From (26) we have

$$\int\limits_{\Omega} [u_\varepsilon(x,t)]^2\,dx \leqslant C_4 \qquad \forall t \in [0,T], \tag{27}$$

$$\iint\limits_{G_0} |D^\alpha u_\varepsilon|^p\,dxdt \leqslant C_5 \quad \forall\alpha \quad (|\alpha| \leqslant m), \tag{28}$$

$$\iint\limits_{G_0} |a^0_\alpha(x,t,\delta u_\varepsilon)|^{p'}\,dxdt \leqslant C_6 \quad \forall\alpha \quad (|\alpha| \leqslant m), \tag{29}$$

$$\iint\limits_{G_0\setminus Q_0} u_\varepsilon^2\,dxdt \leqslant C_7\cdot\varepsilon, \tag{30}$$

where $C_4, C_5, C_6, C_7 > 0$ are the constants which don't depend on ε. Hence, it follows that a sequence $\{\varepsilon_j\}$, $\varepsilon_j \searrow 0$ when $j \to +\infty$, functions $u \in \overset{\circ}{W}{}^{m,0}_{\vec{p}}(G_0) \cap L_\infty((0,T); L_2(\Omega))$ and $\chi_\alpha \in L_{p'_\alpha}(G_0)$ $(|\alpha| \leqslant m)$ exist with the following conditions

$$u_{\varepsilon_j} \to u \quad *\text{-weakly in } L_\infty((0,T); L_2(\Omega)), \tag{31}$$

$$D^\alpha u_{\varepsilon_j} \to D^\alpha u \quad \text{weakly in } L_p(G_0) \quad (|\alpha| \leqslant m), \tag{32}$$

$$a^0_\alpha(\cdot,\cdot,\delta u_{\varepsilon_j}(\cdot,\cdot)) \to \chi_\alpha(\cdot,\cdot) \quad \text{weakly in } L_{p'}(G_0) \quad (|\alpha| \leqslant m), \tag{33}$$

$$u_{\varepsilon_j} \to 0 \quad \text{strongly in } L_2(G_0 \setminus Q_0), \tag{34}$$

$$u_{\varepsilon_j}(x,t) \to 0 \quad \text{for almost all} \quad (x,t) \in G_0 \setminus Q_0. \tag{35}$$

Thus, $u = 0$ on $G_0 \setminus Q_0$. Let us note that equality (23) is valid for any function ψ which belongs to the space $C_0^\infty(Q)$. From (23), (31)–(35), taking $\varepsilon = \varepsilon_j$ in (23), we get

$$-\iint\limits_{Q_0} u\psi_t\,dxdt + \iint\limits_{Q_0}\Big\{\sum_{|\alpha|\leqslant m}\chi_\alpha(x,t)D^\alpha\psi - \sum_{|\alpha|\leqslant m} f^0_\alpha(x,t)D^\alpha\psi\Big\}\,dxdt = 0 \tag{36}$$

for all $\psi \in C_0^\infty(Q)$.

Let us show that

$$\iint\limits_{Q_0}\sum_{|\alpha|\leqslant m}\chi_\alpha(x,t)D^\alpha\psi(x,t)\,dxdt = \iint\limits_{Q_0}\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u(x,t))D^\alpha\psi(x,t)\,dxdt \tag{37}$$

for all $\psi \in \overset{\circ}{W}{}^{m,0}_p(Q_0)$.

We consider the expression

$$M_{\varepsilon_j} = \iint\limits_{Q_0} g_\delta(t)\sum_{|\alpha|\leqslant m}(a_\alpha(x,t,\delta v) - a_\alpha(x,t,\delta u_{\varepsilon_j}))(D^\alpha v - D^\alpha u_{\varepsilon_j})\,dxdt \geqslant 0, \tag{38}$$

where $v \in \overset{\circ}{W}{}^{m,0}_p(Q_0)$ is now an arbitrary function, g_δ is partly-smooth function on $\mathbb{R}$ such that $g_\delta(t) = 1$, if $t \in (-\infty, T-\delta]$, $g_\delta(t) = \frac{T-t}{\delta}$, if $t \in (T-\delta, T]$, $g_\delta(t) = 0$, if $t \in (T; +\infty)$, where $\delta \in (0,T)$ is an arbitrary real number. Thus,

$$M_{\varepsilon_j} = \iint\limits_{Q_0} g_\delta(t)\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta v)(D^\alpha v - D^\alpha u_{\varepsilon_j})\,dxdt -$$

$$-\iint\limits_{Q_0} g_\delta(t)\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_{\varepsilon_j})D^\alpha v\,dxdt+\iint\limits_{Q_0} g_\delta(t)\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_{\varepsilon_j})D^\alpha u_{\varepsilon_j}\,dxdt\geqslant 0.$$

From (23) and [5, Lemma 2] we get

$$\iint\limits_{G_0} g_\delta(t)\Big\{\sum_{|\alpha|\leqslant m} a^0_\alpha(x,t,\delta u_{\varepsilon_j})D^\alpha u_{\varepsilon_j}+\frac{1}{\varepsilon_j}Mu^2_{\varepsilon_j}-\sum_{|\alpha|\leqslant m} f^0_\alpha D^\alpha u_{\varepsilon_j}\Big\}\,dxdt-\frac12\iint\limits_{G_0} g'_\delta u^2_{\varepsilon_j}\,dx\,dt=0.$$

Hence,

$$\iint\limits_{Q_0} g_\delta\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_{\varepsilon_j})D^\alpha u_{\varepsilon_j}\,dxdt\leqslant\iint\limits_{G_0} g_\delta\sum_{|\alpha|\leqslant m} a^0_\alpha(x,t,\delta u_{\varepsilon_j})D^\alpha u_{\varepsilon_j}\,dxdt=$$

$$=\iint\limits_{G_0} g_\delta\Big\{\sum_{|\alpha|\leqslant m} f^0_\alpha D^\alpha u_{\varepsilon_j}-\frac{1}{\varepsilon_j}Mu^2_{\varepsilon_j}\Big\}\,dxdt+\frac12\iint\limits_{G_0} g'_\delta u^2_{\varepsilon_j}\,dx\,dt\leqslant$$

$$\leqslant\iint\limits_{G_0} g_\delta\sum_{|\alpha|\leqslant m} f^0_\alpha D^\alpha u_{\varepsilon_j}\,dxdt-\frac{1}{2\delta}\int\limits_{T-\delta}^{T}\int\limits_{\Omega} u^2_{\varepsilon_j}\,dxdt.$$

Thus,

$$0\leqslant M_{\varepsilon_j}\leqslant\iint\limits_{Q_0} g_\delta\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta v)(D^\alpha v-D^\alpha u_{\varepsilon_j})\,dxdt-\iint\limits_{Q_0} g_\delta\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_{\varepsilon_j})D^\alpha v\,dxdt+$$

$$+\iint\limits_{G_0} g_\delta\sum_{|\alpha|\leqslant m} f^0_\alpha D^\alpha u_{\varepsilon_j}\,dxdt-\frac{1}{2\delta}\int\limits_{T-\delta}^{T}\int\limits_{\Omega} u^2_{\varepsilon_j}\,dxdt.$$

Since $u_{\varepsilon_j}\to u$ weakly in $L_2((T-\delta,T);L_2(\Omega))$, we obtain

$$-\iint\limits_{Q_0} g'_\delta u^2\,dxdt=\frac1\delta\int\limits_{T-\delta}^{T}\int\limits_{\Omega} u^2\,dxdt\leqslant\varliminf_{\varepsilon_j\to0}\frac1\delta\int\limits_{T-\delta}^{T}\int\limits_{\Omega} u^2_{\varepsilon_j}\,dxdt.$$

Hence and from (31)–(35) we have

$$0\leqslant\varliminf_{\varepsilon_j\to0} M_{\varepsilon_j}\leqslant\iint\limits_{Q_0} g_\delta\sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta v)(D^\alpha v-D^\alpha u)\,dxdt-$$

$$-\iint\limits_{Q_0} g_\delta\sum_{|\alpha|\leqslant m}\chi_\alpha D^\alpha v\,dxdt+\iint\limits_{Q_0} g_\delta\sum_{|\alpha|\leqslant m} f_\alpha D^\alpha u\,dxdt+\frac12\iint\limits_{Q_0} g'_\delta u^2\,dxdt. \tag{39}$$

Using (36) and Lemma 1 we obtain that $\|u(\cdot,t)\|_{L_2(\Omega_t)}\in C([0,T])$. ¿From (26) and (31) it follows that

$$\|u(\cdot,\tau)\|_{L_2(\Omega_\tau)}\leqslant C_3\cdot\iint\limits_{Q_{0,\tau}} f(x,t)\,dxdt,\quad \tau\in[0,T],$$

in other words, $\|u(\cdot,0)\|_{L_2(\Omega_0)}=0$. Using Lemma 1, from (36) we get

$$-\frac{1}{2}\iint\limits_{Q_0} g'_\delta u^2\,dxdt+\iint\limits_{Q_0} g_\delta\Big\{\sum_{|\alpha|\leqslant m}\chi_\alpha D^\alpha u-\sum_{|\alpha|\leqslant m} f_\alpha D^\alpha u\Big\}\,dxdt=0. \tag{40}$$

Thus, from (39) and (40) we have

$$\iint\limits_{Q_0} g_\delta(t)\sum_{|\alpha|\leqslant m}(a_\alpha(x,t,\delta v(x,t))-\chi_\alpha(x,t))(D^\alpha v(x,t)-D^\alpha u(x,t))\,dxdt\geqslant 0. \tag{41}$$

Putting $v=u+\lambda\psi$ in (41), where $\lambda>0$ is any real number and $\psi\in C_0^\infty(Q_0)$, we get

$$\lambda\iint\limits_{Q_0} g_\delta(t)\sum_{|\alpha|\leqslant m}(a_\alpha(x,t,\delta u(x,t)+\lambda\delta v(x,t))-\chi_\alpha(x,t))D^\alpha\psi(x,t)\,dxdt\geqslant 0. \tag{42}$$

Let us divide (42) by λ and take the limit as $\lambda\to 0$. Taking into consideration conditions (A1)–(A4) and Lebesgue theorem about passage to the limit, we get

$$\sum_{|\alpha|\leqslant m}\iint\limits_{Q_0} g_\delta(t)(a_\alpha(x,t,\delta u(x,t))-\chi_\alpha(x,t))D^\alpha\psi(x,t)\,dxdt\geqslant 0$$

for any $\psi\in C_0^\infty(Q_0)$ and $\delta\in(0,T)$. Hence we have (37). From (36) and (37) we get (19). The existence of a generalized solution of problem (16)–(18) is proved.

Let us prove that this solution is unique. Let u_1 and u_2 be two generalized solutions of problem (16)–(18). Subtracting from integral identity (19) for u_1 the same identity for u_2 and using Lemma 1 with $t_1=0$, $t_2=t_1\in(0,T]$, $\theta_\delta=1$, $v=u_1-u_2$, we get

$$\int\limits_{\Omega_\tau}[u_1(x,\tau)-u_2(x,\tau)]^2\,dx+$$

$$\int\limits_0^\tau\int\limits_{\Omega_\tau}\sum_{|\alpha|\leqslant m}(a_\alpha(x,t,\delta u_1(x,t))-a_\alpha(x,t,\delta u_2(x,t)))D^\alpha(u_1(x,t)-u_2(x,t))\,dxdt=0. \tag{43}$$

From the monotony of spacial part of the equation (16) we conclude that the second term of the left side of (43) is non-negative. Thus two terms of the left side of (43) are equal to zero. Hence we have the uniqueness of the generalized solution of Problem (16)–(18). Lemma 2 is proved. □

Lemma 3. *Let $\tilde u$ be a generalized solution of problem which differs from Problem (1), (2) only by $\tilde f_\alpha$ instead of $f_\alpha(|\alpha|\leqslant m)$ in the right side of equation (1). Suppose that all the conditions of Theorem hold.*

Then for arbitrary numbers t_1,t_2,δ such that $-\infty<t_1<t_2\leqslant T,\delta>0$ the following inequality holds:

$$\sup_{t\in[t_1,t_2]}\int\limits_{\Omega_t}|u(x,t)-\tilde u(x,t)|^2\,dx+\iint\limits_{Q_{t_1,t_2}}\sum_{|\alpha|=m}|D^\alpha u-D^\alpha\tilde u|^p\,dxdt\leqslant$$

$$\leqslant C_1\delta^{-\frac{p}{p-2}}\int\limits_{t_1-\delta}^{t_2}[\lambda_\gamma(t)]^{\frac{2p}{p-2}}\,dt+C_2\iint\limits_{Q_{t_1-\delta,t_2}}\varkappa_{m-|\alpha|}^{p'}(t)\sum_{|\alpha|\leqslant m}|f_\alpha-\tilde f_\alpha|^{p'_\alpha}\,dxdt, \tag{44}$$

where C_1,C_2 are constants depending only on n,m,γ,K_0,p.

Proof. Let us subtract from integral identity (3) for u the same integral identity for $\tilde{u}$. As a result, taking $w = u - \tilde{u}$, we obtain

$$\iint\limits_{Q} \Big\{ -w\psi_t + \sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\delta u) - a_\alpha(x,t,\delta\tilde{u}))D^\alpha\psi \Big\}\, dxdt = \iint\limits_{Q} \sum_{|\alpha|\leqslant m} (f_\alpha - \tilde{f}_\alpha)D^\alpha\psi\, dxdt \tag{45}$$

for arbitrary $\psi \in C_0^\infty(Q)$.

Let $\theta_1(t)$ be a function from the space $C^\infty(\mathbb{R}^1)$ with the following properties: $0 \leqslant \theta_1(t) \leqslant 1$, $\theta_1'(t) \geqslant 0$ on $\mathbb{R}^1$, $\theta_1(t) = 0$, if $t \in (-\infty, -1]$, $\theta_1(t) = \exp\{-1/(t+1)\}$, if $t \in (-1, -1/2]$, $\theta_1(t) \geqslant \exp\{-2\}$, if $t \in (-1/2, 0)$, $\theta_1(t) = 1$, if $t \in [0, +\infty)$.

It is clear that $\sup\limits_{t\in(-1;+\infty)} \theta_1'(t)\theta_1^{-\varkappa}(t) \leqslant C_0$, where $0 < \varkappa < 1$, $C_0 > 0$ is a constant depending only on $\varkappa$.

Let $\theta_{\delta,t_1}(t) = \theta_1(\frac{t-t_1}{\delta})$ where t_1, δ are arbitrary numbers such that $-\infty < t_1 \leqslant T, \delta > 0$. From Lemma 1 and identity (45) we obtain

$$\int\limits_{\Omega_s} w^2(x,s)\, dx - \iint\limits_{Q_{t_1-\delta,s}} w^2\theta_{\delta,t_1}'\, dxdt + 2\iint\limits_{Q_{t_1-\delta,s}} \theta_{\delta,t_1} \sum_{|\alpha|\leqslant m} (a_\alpha(x,t,\delta u) -$$
$$- a_\alpha(x,t,\delta\tilde{u}))D^\alpha w\, dxdt = 2\iint\limits_{Q_{t_1-\delta,s}} \theta_{\delta,t_1} \sum_{|\alpha|\leqslant m} (f_\alpha - \tilde{f}_\alpha)D^\alpha w\, dxdt, \tag{46}$$

where t_1, t_2, δ are arbitrary numbers such that $-\infty < t_1 < t_2 \leqslant T, \delta > 0$; s is an arbitrary number from the interval $[t_1, t_2]$.

Let us estimate the second term of the left side of equality (46). Taking into account the properties of the function θ_{δ,t_1}, (4) and the Young inequality we obtain

$$\begin{aligned} \iint\limits_{Q_{t_1-\delta,s}} w^2\theta_{\delta,t_1}'\, dxdt &= \int\limits_{t_1-\delta}^{s} \theta_{\delta,t_1}' \int\limits_{\Omega_t} w^2\, dxdt = \int\limits_{t_1-\delta}^{t_1} \|w\|^2_{L_2(\Omega_t)}\theta_{\delta,t_1}'\, dt \leqslant \\ &\leqslant \int\limits_{t_1-\delta}^{t_1} \lambda_\gamma^2(t)\|D^\gamma w\|^2_{L_p(\Omega_t)}\theta_{\delta,t_1}'\, dt = \int\limits_{t_1-\delta}^{t_1} \lambda_\gamma^2(t)\|D^\gamma w\|^2_{L_p(\Omega_t)} \frac{{\theta_{\delta,t_1}}^{2/p}}{{\theta_{\delta,t_1}}^{2/p}}\theta_{\delta,t_1}'\, dt \leqslant \\ &\leqslant \varepsilon_\gamma \int\limits_{t_1-\delta}^{s} \|D^\gamma w\|^p_{L_p(\Omega_t)}\theta_{\delta,t_1}\, dt + C_8\varepsilon_\gamma^{-\frac{2}{p-2}} \int\limits_{t_1-\delta}^{t_1} (\theta_{\delta,t_1}'{\theta_{\delta,t_1}}^{-2/p}\lambda_\gamma^2(t))^{\frac{p}{p-2}}\, dt \leqslant \\ &\leqslant \varepsilon_\gamma \iint\limits_{Q_{t_1-\delta,s}} |D^\gamma w|^p\theta_{\delta,t_1}\, dxdt + C_9\,{\varepsilon_\gamma}^{-\frac{2}{p-2}}\delta^{-\frac{p}{p-2}} \int\limits_{t_1-\delta}^{t_1} [\lambda_\gamma(t)]^{\frac{2p}{p-2}}\, dt, \end{aligned} \tag{47}$$

where $\varepsilon_\gamma > 0$ is an arbitrary number, $C_8, C_9 > 0$ are the constants depending only on γ, p.

Now we estimate the right side of (46), using the Hölder inequality, (4′) and the Young inequality

$$2\iint\limits_{Q_{t_1-\delta,s}} \theta_{\delta,t_1} \sum_{|\alpha|\leqslant m} (f_\alpha - \tilde{f}_\alpha)D^\alpha w\, dxdt \leqslant 2\sum_{|\alpha|\leqslant m} \int\limits_{t_1-\delta}^{s} \Big(\theta_{\delta,t_1} \int\limits_{\Omega_t} (f_\alpha - \tilde{f}_\alpha)D^\alpha w\, dx\Big)\, dt \leqslant$$

$$\leqslant 2\sum_{|\alpha|\leqslant m}\int\limits_{t_1}^{s}\theta_{\delta,t_1}\Big(\int\limits_{\Omega_t}|D^\alpha w|^p\,dx\Big)^{\frac{1}{p}}\Big(\int\limits_{\Omega_t}|f_\alpha-\tilde f_\alpha|^{p'}\,dx\Big)^{\frac{1}{p'}}\,dt\leqslant$$

$$\leqslant 2\sum_{|\alpha|\leqslant m}\int\limits_{t_1-\delta}^{s}\theta_{\delta,t_1}\varkappa_{m-|\alpha|}\Big(\sum_{|\alpha|\leqslant m}\int\limits_{\Omega_t}|D^\beta w|^p\,dx\Big)^{\frac{1}{p}}\Big(\int\limits_{\Omega_t}|f_\alpha-\tilde f_\alpha|^{p'}\,dx\Big)^{\frac{1}{p'}}\,dt\leqslant$$

$$\leqslant\sum_{|\alpha|\leqslant m}\Big(\varepsilon_\alpha\iint\limits_{Q_{t_1-\delta,s}}\theta_{\delta,t_1}\sum_{|\beta|=m}|D^\beta w|^p\,dxdt+C_\alpha\varepsilon_\alpha^{-1/(p-1)}\iint\limits_{Q_{t_1-\delta,s}}\theta_{\delta,t_1}\varkappa^{p'}_{m-|\alpha|}|f_\alpha-\tilde f_\alpha|^{p'}\,dxdt\Big),\quad(48)$$

where $\varepsilon_\alpha>0$ is an arbitrary number, $C_\alpha>0$ is a constant depending only on p ($|\alpha|\leqslant m$). From (4), (46)–(48), choosing ε_α ($|\alpha|\leqslant m$) as need, we obtain (44). Lemma 3 is proved. □

4. Proof of basic results.

Proof of Theorem. Uniqueness. Let u_1,u_2 be two generalized solutions of Problem (1), (2). From Lemma 3 we obtain

$$\sup_{t\in[t_1,t_2]}\int\limits_{\Omega_t}|u_1(x,t)-u_2(x,t)|^2\,dx\leqslant C_1\delta^{-\frac{p}{p-2}}\int\limits_{t_1-\delta}^{t_1}[\lambda_\gamma(t)]^{\frac{2p}{p-2}}\,dt,\qquad(49)$$

where δ is an arbitrary number and t_1,t_2 are any numbers such that $-\infty<t_1<t_2\leqslant T$.

From conditions of Theorem for arbitrary $\tau\in(-\infty,T]$ we have

$$\int\limits_{\tau-\delta}^{\tau}[\lambda_\gamma(t)]^{\frac{2p}{p-2}}\,dt\sim\delta^{\frac{2p\cdot r-2}{p-2}}\qquad(50)$$

when $\delta\to+\infty$.

Let us take the limit in (49) at $\delta\to+\infty$, when t_1,t_2 are fixed. Taking into account (50), independence C_1 on δ and that $r<\frac{1}{p}$ we obtain $u_1=u_2$ almost everywhere on Q_{t_1,t_2}.

Existence. Now we construct a sequence of functions approximating the generalized solution of Problem (1), (2) in some sense. Let $Q_\mu=Q_{T-\mu,T},\Sigma_\mu=\partial Q\cap\{(x,t):T-\mu<t<T\},\mu\in\mathbb{N}$. Let us consider the family of problems ($\mu\in\mathbb{N}$)

$$\hat u_{\mu t}+\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha a_\alpha(x,t,\delta\hat u_\mu)=\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha f_\alpha(x,t)\ \text{ in }\ Q_\mu,\qquad(1_\mu)$$

$$\frac{\partial^j\hat u_\mu}{\partial\nu^j}=0\quad\text{on}\quad\Sigma_\mu,\quad j=0,1,...,m-1,\qquad(2_\mu)$$

$$\hat u_\mu(x,T-\mu)=0.\qquad(3_\mu)$$

A function $\hat u_\mu\in\overset{\circ}{W}{}^{m,0}_{\vec p}(Q_\mu)$ is called a generalized solution of problem (1_μ)–(3_μ), if it satisfies the integral equality

$$\int\limits_{Q_\mu}\Big\{-\hat u_\mu\psi_t+\sum_{|\alpha|\leqslant m}a_\alpha(x,t,\delta\hat u_\mu)D^\alpha\psi\Big\}\,dxdt=\int\limits_{Q_\mu}\sum_{|\alpha|\leqslant m}f_\alpha D^\alpha\psi\,dxdt\qquad(4_\mu)$$

for arbitrary $\psi\in C_0^\infty(Q)$.

The existence and the uniqueness of a generalized solution $\hat{u}_\mu$ of Problem (1_μ)–(3_μ) follows from Lemma 2. Let us extend the function $\hat{u}_\mu$ by zero on $\overline{Q}\setminus\overline{Q}_\mu$ and denote this extension by u_μ. Put $f_{\alpha,\mu}(x,t)=f_\alpha(x,t)$ for $(x,t)\in Q_\mu$ and $f_{\alpha,\mu}(x,t)=0$ for $(x,t)\in Q\setminus Q_\mu(|\alpha|\leqslant m)$. It is evident that u_μ is a generalized solution of Fourier problem

$$u_{\mu t}+\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha a_\alpha(x,t,\delta u_\mu)=\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha f_{\alpha,\mu}(x,t) \text{ in } Q, \tag{5_μ}$$

$$\frac{\partial^j u_\mu}{\partial\nu^j}=0 \quad \text{on} \quad \Sigma, \quad j=0,1,...,m-1, \tag{6_μ}$$

i.e. $u_\mu \in \overset{\circ}{W}{}^{m,0}_p(Q)\subset \overset{\circ}{W}{}^{m,0}_{p,\mathrm{loc}}(Q)$ and satisfies the identity

$$\iint_Q \Big\{-u_\mu\psi_t+\sum_{|\alpha|\leqslant m}a_\alpha(x,t,\delta u_\mu)D^\alpha\psi-\sum_{|\alpha|\leqslant m}f_{\alpha,\mu}(x,t)D^\alpha\psi\Big\}\,dxdt=0 \tag{7_μ}$$

for arbitrary $\psi\in C_0^\infty(Q)$.

Let us show that the sequence of restrictions of $\{u_\mu\}$ on $Q_\varkappa$, where $\varkappa$ is an arbitrary natural number, is fundamental in the space $\overset{\circ}{W}{}^{m,0}_p(Q_\varkappa)\cap L_\infty([T-\varkappa,T];L_2(\Omega_t))$. Let k,l be arbitrary natural numbers such that $k>2\varkappa, l>2\varkappa$. Using Lemma 3 in the case of $u=u_k, \tilde{u}=u_l, f_\alpha=f_{\alpha,k}, \tilde{f}_\alpha=f_{\alpha,l}(|\alpha|\leqslant m), t_1=T-\varkappa, t_2=T$, $\delta=\min\{k-\varkappa,l-\varkappa\}$, we obtain

$$\sup_{[T-\varkappa,T]}\int_{\Omega_t}|u_k(x,t)-u_l(x,t)|^2\,dx+\iint_{Q_{T-\varkappa,T}}\sum_{|\alpha|=m}|D^\alpha u_k-D^\alpha u_l|^p\,dxdt\leqslant$$
$$\leqslant C_1\delta^{-\frac{p}{p-2}}\int_{T-\varkappa-\delta}^{T-\varkappa}[\lambda_\gamma(t)]^{\frac{2p}{p-2}}\,dt. \tag{51}$$

Since the right side of inequality (51) tends to zero at $\delta\to+\infty$ and C_1 does not depends on δ it follows that the sequence of restrictions $\{u_\mu\}$ on $Q_\varkappa$ is fundamental in the space $\overset{\circ}{W}{}^{m,0}_p(Q_\varkappa)\cap L_\infty((T-\varkappa,T);L_2(\Omega_t))$, where $\varkappa$ is an arbitrary natural number. Therefore, there exists a function $u\in\overset{\circ}{W}{}^{m,0}_{p,\mathrm{loc}}(\overline{Q})\cap L_\infty((-\infty,T];L_2(\Omega_t))$ such that

$$u_\mu\to u \quad \text{in} \quad L_{\infty,\mathrm{loc}}((-\infty,T];L_2(\Omega_t)), \tag{52}$$

$$D^\alpha u_\mu\to D^\alpha u \quad \text{strong in} \quad L_{p,\mathrm{loc}}(\overline{Q}) \quad (|\alpha|\leqslant m). \tag{53}$$

From (53) and condition $(A2)$ it follows that there exist a subsequence $\{u_{\mu_i}\}\subset\{u_\mu\}$ and functions $\tilde{\chi}_\alpha\in L_{p',\mathrm{loc}}(\overline{Q})(|\alpha|\leqslant m)$, such that

$$a_\alpha(\cdot,\cdot,\delta u_{\mu_i}(\cdot,\cdot))\to\tilde{\chi}_\alpha(\cdot,\cdot) \quad \text{weakly in} \quad L_{p'},\mathrm{loc}(\overline{Q}) \quad (|\alpha|\leqslant m), \tag{54}$$

$$D^\alpha u_{\mu_i}\to D^\alpha u \quad \text{nearly everywhere on} \quad Q \quad (|\alpha|\leqslant m). \tag{55}$$

From (54), (55), condition $(A1)$ and Lemma 1.3 from [8;p.25] we obtain

$$a_\alpha(\cdot,\cdot,\delta u_{\mu_i}(\cdot,\cdot))\to a_\alpha(\cdot,\cdot,\delta u(\cdot,\cdot)) \quad \text{weakly in} \quad L_{p',\mathrm{loc}}(\overline{Q}) \quad (|\alpha|\leqslant m). \tag{56}$$

Let us show that u is a generalized solution of Problem (1),(2). Let $\psi \in C_0^\infty(Q)$ be an arbitrary fixed function. From (7_μ) we obtain

$$\iint\limits_Q \Big\{ -u_{\mu_i}\psi_t + \sum_{|\alpha|\leqslant m} a_\alpha(x,t,\delta u_{\mu_i}) D^\alpha \psi \Big\}\, dxdt = \iint\limits_Q \sum_{|\alpha|\leqslant m} f_{\alpha,\mu_i} D^\alpha \psi \, dxdt. \tag{57}$$

Let us take the limit at $i \to +\infty$ in (57). Taking into account (52), (53), (56) and the definition of f_{α,μ_i}, we obtain identity (3).

Continuous dependence on the data-in. Let $\{f_{\alpha,k}\}_{k=1}^\infty \subset L_{p',\mathrm{loc}}(\overline{Q})(|\alpha| \leqslant m)$ be sequences of functions such that $f_{\alpha,k} \to f_\alpha$ in $L_{p',\mathrm{loc}}(\overline{Q})(|\alpha| \leqslant m)$, and $\{u_k\}$ be a sequence of generalized solutions of problems which differ from Problem (1), (2) only by $f_{\alpha,k}$ instead of f_α in the right side of the equation (1). Let us show that $u_k \to u$ in $\overset{\circ}{W}{}^{m,0}_{p,\mathrm{loc}}(\overline{Q}) \cap L_{\infty,\mathrm{loc}}((-\infty,T];L_2(\Omega_t))$.

Indeed, let $\varepsilon > 0$ be an arbitrary number, t_1, t_2 are any numbers such that $-\infty < t_1 < t_2 \leqslant T$. ¿From Lemma 3 we obtain

$$\sup_{[t_1,t_2]} \int\limits_{\Omega_t} |u_k(x,t) - u(x,t)|^2\, dx + \iint\limits_{Q_{t_1,t_2}} \sum_{|\alpha|\leqslant m} |D^\alpha u_k - D^\alpha u|^p \, dxdt \leqslant$$

$$\leqslant C_1 \delta^{-\frac{p}{p-2}} \int\limits_{t_1-\delta}^{t_1} [\lambda_\gamma(t)]^{\frac{2p}{p-2}} dt + C_2 \iint\limits_{Q_{t_1-\delta,t_2}} \sum_{|\alpha|\leqslant m} |f_{\alpha,k} - f_\alpha|^{p'}\, dxdt, \tag{58}$$

where $\delta > 0$ is an arbitrary number, C_1, C_2 are the constants which do not depend on δ.

¿From conditions of the theorem it follows that we can choose $\delta > 0$ such that

$$C_1 \delta^{-\frac{p}{p-2}} \int\limits_{t_1-\delta}^{t_1} [\lambda_\gamma(t)]^{\frac{p}{p-2}}\, dt < \frac{\varepsilon}{2}. \tag{59}$$

Since $f_{\alpha,k} \to f_\alpha$ in $L_{p'}(Q_{t_1-\delta,t_2})(|\alpha| \leqslant m)$, we obtain the existence of a number $k_0 \in \mathbb{N}$ such that for any $k \geqslant k_0$

$$C_2 \iint\limits_{Q_{t_1-\delta,t_2}} \sum_{|\alpha|\leqslant m} |f_{\alpha,k} - f_\alpha|^{p'}\, dxdt \leqslant \frac{\varepsilon}{2}. \tag{59}$$

Taking into account (59), (60), we conclude that the right side of (58) is less than ε for any $k \geqslant k_0$. The theorem is proved. □

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