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ARTINIAN MODULES OVER GROUPS OF RANK 2

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The problem of description of Artinian modules over the ring ZG, where G is a free abelian group of rank ≤ 2, is "wild"from the viewpoint of the representation theory. But it is possible to describe some types of such modules. The study of Artinian modules over groups of rank 2 is initiated in this paper. In particular, we describe the structure of the injective hull of Artinian modules over the group ring FG, where F is a field and G is a free abelian group of rank 2.

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Задача описания артриновых модулей над кольцом ZG, где G — свободная абелевая группа ранга ≤ 2, является "дикой" с точки зрения теории представлений. Но можно описать некоторые типы таких модулей. В статье начинается изучение артиновых модулей над группами ранга 2. В частности, описана структура инъективной оболочки артиновых модулей над кольцом FG, где F — поле, а G — свободная абелевая группа ранга 2.

The problem of description of Artinian and Noetherian modules over various rings is one of the oldest classical problems in algebra. The study of Artinian modules over some types of group rings is important for many problems of group theory. For example, the investigation of artinian modules over the group ring ZG, where G is an abelian Černikov group, is necessary for study of metabelian groups with minimal condition for normal subgroups. These modules have been studied by Hartly and McDougall [1]. In particular, such modules decompose into direct sum of monolithic modules and if A is a monolithic ZG-module, then $G/C_G(A)$ is a group of (special) rank 1. On the other hand, if G is a free abelian group of rank 2 and A is an Artinian module over the ring ZG then the problem of description of module A is "wild" from the point of view of the representation theory. This does not mean that the study of Artinian modules over finitely generated abelian groups of rank 2 is hopeless. So, Musson in [3] studied the injective hull of Artinian modules over the ring ZG, where G is a polycyclic group. Besides, from Theorem A of [4] it follows that Artinian modules over the ring ZG can be described up to subgroups of finite index. So, there is a natural question here: which Artinian modules over the ring ZG, where G is a finitely generated abelian group, can be described? The first natural step here is the study of Artinian modules over the ring ZG, where G is a free abelian group of rank 2. If A is an Artinian ZG-module, $S = \mathrm{Soc}_{ZG}(A)$, then the submodule S is finite, i.e. the index $|G : H|$ is finite, where $H = C_G(S)$. By

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Theorem A of [4], which have been mentioned above, A is a ZH-hypercentral module. So, it suffices to study ZH-hypercentral modules. The additive group A is periodic (see, e.g., [5, Lemma 2.4]), i.e. $A = \bigoplus_{p\in\Pi(G)} A_p$, where A_p is a p-component of A. In other words, the following reduction is a reduction to the situation where A is a p-module, p is a prime number. To study p-module it is naturally to study the structure of its lower layer, thus we have the reduction to F_pH-module (F_p is a field of characteristic p).

The study of Artinian hypercentral modules over the ring FG, where F is a field and G is a free abelian group of rank 2, is initiated in this paper. The main result is the description of the injective hull of such modules. The structure of the injective hull of Artinian modules over the ring ZG, where G is an abelian group of finite rank, was studied by Kurdachenko in [5]. In this paper this structure is described with particular defining correlation.

Let F be a field, G an abelian group, A an FG-module. We say that A is FG-hypercentral if A possesses an ascending series of submodules

$$\langle 0\rangle \leq A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots \leq A_\gamma \leq A$$

such that $A_{\alpha+1} \leq A_\alpha$ for any $g \in G$ and for any $\alpha < \gamma$.

Lemma 1. *Let F be a field, G an abelian group, A an FG-module. If A is an FG-hypercentral module, then for any element $g \in G$ the length of upper $F\langle g\rangle$-central series of A is $\leq \omega$.*

Proof. Since A is an FG-hypercentral module, the natural semidirect product $A\lambda G$ is a hypercentral group. Let $a \in A$, then $\langle a, g\rangle$ (as a $A\lambda G$-subgroup) is nilpotent. Let $A_1 = \langle a\rangle\langle g\rangle$, then $\langle a, g\rangle = A_1\lambda\langle g\rangle$ and A_1 are finitely generated. Hence the F-subspace $A_2 = A_1F$ has finite dimension. Obviously, $A_2 = aF\langle g\rangle$, i.e. A_2 is a $F\langle g\rangle$-submodule of A. Since A is an FG-hypercentral module, we see that $A_2 \cap F\langle g\rangle(A) = A_3\langle 0\rangle$. In particular, $\dim_F(A_2 + \zeta_{F\langle g\rangle}(A)/\zeta_{F\langle g\rangle}(A)) = \dim_F(A_2/A_3) < \dim_F A_2$. Using now a common induction, it is not difficult to prove that some FG-hypercentre of finite number includes A_2. In other words, any element of the module A is contained in some FG-hypercentre of finite number. It means that the length of upper $F\langle g\rangle$-central series of A is $\leq \omega$. □

Let F be a field, G an abelian group, A an FG-module. Put $\Omega_{g-1,n}(A) = \{a \in A | a(g-1)^n = 0\}, n \in \mathbb{N}$. Obviously, $\Omega_{g-1,n}(A)$ is an FG-submodule for any $n \in \mathbb{N}$. We say that $\Omega_{g-1,n}(A)$ is n-th $(g-1)$-layer of module A. Obviously, $\Omega_{g-1,n}(A) \leq \Omega_{g-1,n+1}(A)$ for any $n \leq \mathbb{N}$ and by Lemma 1 $A = \bigcup_{n\in\mathbb{N}} \Omega_{g-1,n}(A)$. If $a \in A$ then there exists $n \in \mathbb{N}$ such that $a \in \Omega_{g-1,n}(A) \setminus \Omega_{g-1,n-1}(A)$. Put $O_g(a) = n$.

Let B be an $F\langle g\rangle$-submodule of A, and $a \in B$.

Consider the equation $x(g-1)^n = a$, $n \in \mathbb{N}$. If this equation has a solution in the module B for any $n \in \mathbb{N}$, then we put $h_{B,g}(a) = \infty$. If there exists a number $m \in \mathbb{N}$, such that the equation $x(g-1)^m = a$ is solvable in B but the equation $x(g-1)^{m+1} = a$ already has no solution in B then we put $h_{B,g}(a) = m$.

Obviously, $h_{B,g}(a) \leq h_{A,g}(a)$. We say that $F\langle g\rangle$-submodule B is $(g-1)$-pure if $h_{B,g}(a) = h_{A,g}(a)$ for any nonzero element $a \in B$.

In other words, B is $(g-1)$-pure if and only if for any nonzero element $a \in B$ and for any $n \in \mathbb{N}$ from solvability of the equation $x(g-1)^n = a$ in module A it follows that this equation is solvable in the module B.

Let F be a field, A an F-vector space with countable dimension, $\{a_n | n \in \mathbb{N}\}$ a basis of A, $\langle g \rangle$ an infinite cyclic group. A can be considered as an $F\langle g \rangle$-module by means of the action defined by: $a_1 g = a_1$, $a_{n+1} g = a_{n+1} + a_n$, $n \in \mathbb{N}$, or $a_1(g-1) = 0$, $a_{n+1}(g-1) = a_n$, $n \in \mathbb{N}$.

We call this module a Prufer $(g-1)$-module and denote it by $C_{(g-1)^\infty}$.

From the definition it follows that $A(g-1) = A$. A is an $F\langle g \rangle$-hypercentral module, any its proper submodule coincides with $A_n = a_1 F \oplus a_n F$, $A_n = a_n F\langle g \rangle$ and $A_1 \leq A_2 \leq \dots \leq A_n \leq \dots$, $\bigcup_{n \in \mathbb{N}} A_n = A$. In particular, $a_1 F$ is the least submodule of A, i.e. $a_1 F$ is a monolith of A.

Thereby, characteristics of this module are similar to those of Prufer p-groups.

Lemma 2. *Let F be a field, $\langle g \rangle$ an infinite cyclic group, A an $F\langle g \rangle$-module. A is $F\langle g \rangle$-hypercentral and monolithic if and only if A is isomorphic to a Prufer $(g-1)$-module or to some its submodule.*

Proof. Let $A_n = \Omega_{g-1,n}(A)$, $n \in \mathbb{N}$, then $A_1 = \zeta_{F\langle g \rangle}(A)$. Since A is monolithic, $\dim_F A_1 = 1$, i.e. $A_1 = a_1 F$. The map $\phi : A_2 \to A_1$ defined by $a\phi = a(g-1)$, $a \in A_2$ is a module homomorphism, hence $\operatorname{Im}\phi$ and $\operatorname{Ker}\phi$ are $F\langle g \rangle$-submodules, moreover, $\operatorname{Im}\phi \leq A_1$, $\operatorname{Ker}\phi = A_1$. If $A_2 = A_1$, then $A = A_1$, i.e. $A = a_1 F$ and the lemma is proved. If $A_2 \neq A_1$, then $\operatorname{Im}\phi \neq \langle 0 \rangle$, i.e. $\operatorname{Im}\phi = a_1 F$ and $\operatorname{Im}\phi \cong A_2/\operatorname{Ker}\phi = A_2/A_1$, in particular, $\dim_F(A_2/A_1) = 1$. Let a_2 be one of prototypes of the element a_1, $a_2(g-1) = a_1$. Since $a_2 \notin A_1$, we have $a_2 F\langle g \rangle + A_1 = A_2$. However $A_1 = a_1 F \leq a_2 F\langle g \rangle$, so $A_2 \leq a_2 F\langle g \rangle$.

Using similar arguments we can choose elements a_n with characteristics $a_n F\langle g \rangle = A_n$, $a_{n+1}(g-1) = a_n$. If $A_n \neq A_{n+1}$ for any $n \in \mathbb{N}$, then A is a Prufer $(g-1)$-module. If $A = A_n$ for some $n \in \mathbb{N}$, then A is a submodule of a Prufer $(g-1)$-module (its n-th $(g-1)$-layer). □

Lemma 3. *Let F be a field, $\langle g \rangle$ an infinite cyclic group, A an $F\langle g \rangle$-module. Let A be an $F\langle g \rangle$-hypercentral and B is an $F\langle g \rangle$-submodule of A such that $B(g-1) = B$. If C is maximal among $F\langle g \rangle$-submodules of A, which have zero intersection with B, then $A = B \oplus C$.*

Proof. Assuming the contrary, let $A \neq B \oplus C$. Since A is $F\langle g \rangle$-hypercentral, $A/(B \oplus C)$ is also $F\langle g \rangle$-hypercentral, in particular, $\zeta_{F\langle g \rangle}(A/(B \oplus C))$ is a nonzero submodule. Hence, there exists an element $a \notin B \oplus C$, such that $a(g-1) \in B \oplus C$. So, $a(g-1) = b + c$, $b \in B$, $c \in C$. Since $B = B(g-1)$, we get $b = b_1(g-1)$ for some $b_1 \in B$. Put $a_1 = a - b_1$, then $a_1(g-1) = a(g-1) - b_1(g-1) = b + c - b = c \in C$. Besides $a_1 \notin B \oplus C$, i.e. $a_1 F \cap (B \oplus C) = \langle 0 \rangle$. Put $E = C \oplus a_1 F$. If $e \in E$, then $e = c_1 + \alpha a_1$, where $c_1 \in C$, $\alpha \in F$, so $eg = c_1 g + \alpha(a_1 g) = c_1 g + \alpha c + \alpha a_1 \in E$. So, E is a $F\langle g \rangle$-submodule. Obviously, $E \cap B = \langle 0 \rangle$. Since $a_1 \notin C$, we have $E \neq C$. We get a contradiction with the choice of C. This contradiction proves the equality $A = B \oplus C$. □

Let R be a ring, A and B modules over the ring R. The fact that A is isomorphic to B or to some submodule of B will be denoted by $A \widetilde{<} B$

Lemma 4. *Let F be a field, $G = \langle g \rangle$ an infinite cyclic group, A a monolithic Artinian FG-module, $A_n = \Omega_{(g-1),n}(A)$, $n \in \mathbb{N}$. If A is FG-hypercentral, then the factor A_{n+1}/A_n for $n \in \mathbb{N}$, is isomorphic to the factor A_n/A_{n-1} or to some its submodule.*

Proof. The map $\phi_2 : A_2 \to A_1$ defined by $a\phi_2 = a(g-1)$, $a \in A_2$, is a module homomorphism, hence $\operatorname{Im}\phi_2$ and $\operatorname{Ker}\phi_2$ are $F\langle g \rangle$-submodules, moreover $\operatorname{Im}\phi_2 \leq A_1$, $\operatorname{Ker}\phi_2 = A_1$. So, $\operatorname{Im}\phi_2 \cong A_2/\operatorname{Ker}\phi_2 = A_2/A_1$. It means that A_2/A_1 is isomorphic to A_1 or to some its submodule.

So, when $n = 1$ this statement is proved. Let now $n > 1$. Use induction by the number n. Assume that A_{n+1}/A_n is isomorphic to the factor A_n/A_{n-1} or to some its submodule.

Consider the factor A/A_n. Obviously, $\Omega_{(g-1),1}(A/A_n) = A_{n+1}/A_n$ and $\Omega_{(g-1),2}(A/A_n) = A_{n+2}/A_n$. As we proved above $A_{n+2}/A_n\Big/A_{n+1}/A_n \widetilde{<} A_{n+1}/A_n$, and since $A_{n+2}/A_n\Big/A_{n+1}/A_n \widetilde{<} A_{n+2}/A_{n+1}$, we conclude $A_{n+2}/A_{n+1} \widetilde{<} A_{n+1}/A_n$, which completes the proof. □

Consider modules over groups of rank 2.

Corollary 1. *Let F be a field, $G = \langle g_1\rangle \times \langle g_2\rangle$ a free abelian group of rank 2, A a monolithic Artinian FG-module, $A_n = \Omega_{(g_2-1),n}(A)$, $n \in \mathbb{N}$. If A is FG-hypercentral, then for every $n \in \mathbb{N}$ the factor A_{n+1}/A_n is isomorphic to a Prufer $(g_1 - 1)$-module.*

Proof. By Lemma 2 A_1 is isomorphic to a Prufer $(g_1 - 1)$-module or to some its submodule and by Lemma 4 $A_{n+1}/A_n \widetilde{<} A_n/A_{n-1} \widetilde{<} \cdots \widetilde{<} A_2/A_1 \widetilde{<} A_1$. □

Proposition 1. *Let F be a field, $G = \langle g_1\rangle \times \langle g_2\rangle$ a free abelian group of rank 2, A a monolithic Artinian FG-module, $A_n = \Omega_{(g_2-1),n}(A)$, $n \in \mathbb{N}$. If A is FG-hypercentral then $A_n = P_1 \oplus \cdots \oplus P_n$ for any $n \in \mathbb{N}$, where $P_i = \bigoplus_{n\in\mathbb{N}} a_{i,n}F$ is a Prufer $(g_1 - 1)$-module, i.e. $a_{i,1}(g_1-1) = 0$, $a_{i,n+1}(g_1-1) = a_{i,n}$, $n \in \mathbb{N}$. Besides, $a_{1,n}(g_2-1) = 0$, $a_{j+1,n}(g_2-1) = a_{j,n}$, $j, n \in \mathbb{N}$. If $A \neq A_n$ for any $n \in \mathbb{N}$ then $A = P_1 \oplus P_2 \oplus \cdots \oplus P_n \cdots$.*

Proof. By Lemma 2, A_1 is a Prufer $(g_1 - 1)$-module, i.e. $A_1 = P_1 = \bigoplus_{n\in\mathbb{N}} a_{1n}F$, where $a_{11}(g_1 - 1) = 0$, $a_{1,n+1}(g_1 - 1) = a_{1n}$, $n \in \mathbb{N}$.

Since A_2/A_1 is (g_1-1)-divisible, it is infinite. By Corollary 1, A_2/A_1 is a Prufer (g_1-1)-module and by Lemma 3, A_2 decomposes in direct sum $A_2 = A_1 \oplus T_1$, where T_1 is a Prufer $(g_1 - 1)$-module, i.e. $B_1 = \bigoplus_{n\in\mathbb{N}} b_nF$, where $b_1(g_1 - 1) = 0$, $b_{n+1}(g_1 - 1) = b_n$, $n \in \mathbb{N}$. Put $b_1(g_2 - 1) = c_1 \in A_1$, then $b_1(g_1 - 1) = b_1(g_1 - 1)(g_2 - 1) = 0$, i.e. $c_1 = \alpha a_{11}$. If $\alpha = 0$, then $b_1 \in \zeta_{FG}(A)$, i.e. $\dim_F(\zeta_{FG}(A)) \geq 2$. But this is impossible, because A is a monolithic module. So, $\alpha \neq 0$. Put $a_{21} = b_1\alpha^{-1}$, $\bar{b}_2 = b_2\alpha^{-1}$, $\bar{b}_n = b_n\alpha^{-1}$, $n \geq 2$. Then $a_{21}(g_2 - 1) = (a_{11}\alpha)\alpha^{-1} = a_{11}$, $\bar{b}_2(g_1 - 1) = a_{21}$, $\bar{b}_{n+1}(g_1 - 1) = \bar{b}_n$, $n \in \mathbb{N}$.

Let now $\bar{b}_2(g_2-1) = c_2 \in A_1$. Again $c_2(g_1-1) = \bar{b}_2(g_2-1)(g_1-1) = \bar{b}_2(g_1-1)(g_2-1) = a_{21}(g_2 - 1) = a_{11}$. It follows that $c_2 \in a_{11}F \oplus a_{12}F$, i.e. $c_2 = a_{11}\alpha_1 + a_{12}\alpha_2$. Since $a_{11} = c_2(g_1-1) = a_{11}\alpha_2$, we get $\alpha_2 = 1$. Put $a_{22} = \bar{b}_2 - a_{21}\alpha_1$, then $a_{22}(g_1 - 1) = \bar{b}_2(g_1 - 1) - a_{21}\alpha_1(g_1 - 1) = a_{21}$ and $a_{22}(g_2 - 1) = \bar{b}_2(g_2 - 1) - a_{21}\alpha_1(g_2 - 1) = a_{11}\alpha_1 + a_{12} - a_{11}\alpha_1 = a_{12}$. Let now $\bar{\bar{b}}_3 = \bar{b}_3 - \bar{b}_2\alpha_1$, $\bar{\bar{b}}_{n+1} = \bar{b}_{n+1} - \bar{b}_n\alpha_1$, then $\bar{\bar{b}}_3(g_1 - 1) = \bar{b}_3(g_1 - 1) - \bar{b}_2(g_1 - 1)\alpha_1 = \bar{b}_2 - a_{21}\alpha_1 = a_{22}$, $\bar{\bar{b}}_{n+1}(g_1 - 1) = \bar{b}_{n+1}(g_1 - 1) - \bar{b}_n(g_1 - 1)\alpha_1 = \bar{b}_n - \bar{b}_{n-1}\alpha_1 = \bar{\bar{b}}_n$, $n \geq 3$.

Consider the element $\bar{\bar{b}}_3(g_2 - 1) = c_3$. We have now $c_3(g_1 - 1) = \bar{\bar{b}}_3(g_2 - 1)(g_1 - 1) = \bar{\bar{b}}_3(g_2 - 1)(g_1 - 1) = a_{22}(g_2 - 1) = a_{12}$. It follows that $c_3 \in a_{11}F \oplus a_{12}F \oplus a_{13}F$, i.e. $c_3 = a_{11}\beta_1 + a_{12}\beta_2 + a_{13}\beta_3$. Since $a_{12} = c_3(g_1 - 1) = a_{11}\beta_2 + a_{12}\beta_3$, we get $\beta_3 = 1$, $\beta_2 = 0$, i.e. $c_3 = a_{11}\beta_1 + a_{13}$. Put now $a_{23} = \bar{\bar{b}}_3 - a_{21}\beta_1$, then $a_{23}(g_2 - 1) = \bar{\bar{b}}_3(g_2 - 1) - a_{21}(g_2 - 1)\beta_1 = a_{11}\beta_1 + a_{13} - a_{11}\beta_1 = a_{13}$ and $a_{23}(g_1 - 1) = \bar{\bar{b}}_3(g_1 - 1) - a_{21}(g_1 - 1)\beta_1 = a_{22}$.

Using similar arguments by induction we can choose other elements $a_{2,n}$ such that $a_{2,n}(g_2 - 1) = a_{1,n}$, $a_{2,n+1}(g_1 - 1) = a_{2,n}$, $n \in \mathbb{N}$. Put $P_2 = \bigoplus_{n\in\mathbb{N}} a_{2,n}F$.

So, when $n = 2$ the statement is proved. Let now $n > 2$. Use the induction by number n. Assume that the equality $A_n = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ has been proved. Consider the $(n+1)$-th $(g_2 - 1)$-layer A_{n+1}. A_{n+1}/A_n is $(g_1 - 1)$-divisible and by Corollary 1 A_{n+1}/A_n is a Prufer

(g_1-1)-module. By Lemma 3 $A_{n+1}=A_n\oplus D$, where $D=\bigoplus_{n\in\mathbb{N}}d_nF$ is a Prufer (g_1-1)-module, i.e. $d_1(g_1-1)=0$, $d_{n+1}(g_1-1)=d_n$, $n\in\mathbb{N}$. Using inductive assumptions for the factor-module A_{n+1}/A_n, we can consider that $d_i(g_2-1)+A_1=a_{n,i}+A_1$.

Consider the element $d_1(g_2-1)=e_1\in A_n$. We have that $e_1(g_1-1)=d_1(g_2-1)(g_1-1)=d_1(g_1-1)(g_2-1)=0$. Since $d_1(g_2-1)\in a_{n1}+A_1$, it follows that $e_1=a_{11}\gamma+a_{n,1}$. Put $a_{n+1,1}=d_1-a_{21}\gamma$, then $a_{n+1,1}(g_2-1)=d_1(g_2-1)-a_{21}\gamma(g_2-1)=a_{11}\gamma+a_{n,1}-a_{11}\gamma=a_{n,1}$. Let now $\overline{d}_2=d_2-a_{22}\gamma$, $\overline{d}_n=d_n-a_{2,n}\gamma$, $n\geq 2$, then $\overline{d}_2(g_1-1)=d_2(g_1-1)-a_{22}\gamma(g_1-1)=d_1-a_{2,1}\gamma=a_{n+1,1}$, $\overline{d}_{n+1}(g_1-1)=d_{n+1}(g_1-1)-a_{2,n+1}\gamma(g_1-1)=d_n-a_{2,n}\gamma=\overline{d}_n$, $n\geq 2$.

Next consider the element $\overline{d}_2(g_2-1)=e_2\in A_n$. Again $e_2(g_1-1)=\overline{d}_2(g_2-1)(g_1-1)=\overline{d}_2(g_1-1)(g_2-1)=a_{n+1,1}(g_2-1)=a_{n,1}$. Besides, $\overline{d}_2(g_2-1)\in a_{n,2}+A_1$, where $e_2=a_{11}\gamma_1+a_{12}\gamma_2+a_{n,2}$. Since $a_{n,1}=e_2(g_1-1)=a_{11}\gamma_2+a_{n1}$ we get $\gamma_2=0$, i.e. $e_2=a_{11}\gamma_1+a_{n,2}$.

Put now $a_{n+1,2}=\overline{d}_2-a_{21}\gamma_1$, then $a_{n+1,2}(g_2-1)=\overline{d}_2(g_2-1)-a_{21}(g_2-1)\gamma_1=a_{11}\gamma_1+a_{n,2}-a_{11}\gamma_1$ and $a_{n+1,2}(g_1-1)=\overline{d}_2(g_1-1)-a_{21}(g_1-1)\gamma_1=a_{n+1,1}$.

Using similar arguments by induction we can choose other elements $a_{n+1,i}$, $i\geq 3$. □

Let J be a principal ideal domain, A a J-module, $a\in A$. If $\mathrm{Ann}_J(a)\neq\langle 0\rangle$ then the element a is called J-periodic. The set $t_J(A)$ of all J-periodic elements (J-periodic part of A) is, obviously, a J-submodule of A. If $A=t_J(A)$, then the module A is called J-periodic; if $t_J(A)=\langle 0\rangle$ then A is called J-torsion-free.

Let P be a maximal ideal of J. Then $A_P=\{a\in A|\,\mathrm{Ann}_J(a)=P^n \text{ for some } n\in\mathbb{N}\}$ is, obviously, a J-submodule of A. Since J is a principal ideal domain, we have $P=Jy$ for some prime element $y\in J$. So we denote A_P also by A_y. Submodules $A_P=A_y$ are called p-component of the module A or its y-component. Let $\mathrm{Spec}(J)=\{P\mid P \text{ is a non-zero maximal ideal of } A\}$. If a is a J-periodic element of A, then $\mathrm{Ann}_J(a)=P_1^{k_1}\cdots P_s^{k_s}$ for some $P_i\in\mathrm{Spec}(J)$, $k_i\in\mathbb{N}$, $1\leq i\leq s$. Put $\Pi_J(a)=\{P_1,\ldots,P_s\}$ and $\Pi_J(A)=\bigcup_{a\in t_j(A)}\Pi_J(a)$. We say that A is a π-periodic module if A is a J-periodic module, $\pi\subseteq\mathrm{Spec}(J)$ and $\Pi_J(A)\subseteq\pi$.

At last, put $\Omega_{P,n}(A)=\Omega_{y,n}(A)=\{aA|aP^n=\langle 0\rangle\}=\{a\in A|ay^n=0\}$. Obviously, $\Omega_{P,n}(A)$ is a J-submodule of A for any $n\in\mathbb{N}$ and $\Omega_{P,n}(A)\leq\Omega_{P,n+1}(A)$ and $\bigcup_{n\in\mathbb{N}}\Omega_{P,n}(A)=A_P$.

Note also that $t_J(A)=\bigoplus_{P\in\Pi_J(A)}A_P$ (it follows from Theorem 10.30 of [6]).

Lemma 5. *Let F be a field, $G=\langle g_1\rangle\times\langle g_2\rangle$ a free abelian group of rank 2, A a monolithic Artinian FG-module, E an FG-injective hull of A. If A is FG-hypercentral then E is an Artinian FG-hypercentral module.*

Proof. Since the ring FG is Noetherian (see, e.g., Corollary to Lemma 5.35 of [7]) E is an Artinian FG-module that follows from a result of [8].

Put $J_i=F\langle g_i\rangle$ then J_i is a principal ideal domain, $i=1,2$. $FG=J_1\langle g_2\rangle=J_2\langle g_1\rangle$. Put $T_i=t_J(E)$, $P_i=J_i(g_i-1)$, $i=1,2$. Obviously, $A\leq T_1$, $A\leq T_2$. Then, $T_i=E_{P_i}\oplus D_i$, where $D_i=\bigoplus_{Q\in\Pi_{J_i}(A),Q\neq P_i}EQ$. Since G is an abelian group, E_{P_i} and D_i are FG-submodules. Since A is monolithic and hypercentral, $A\leq E_{P_1}$ and $A\leq E_{P_2}$, hence $T_i=E_{P_i}$.

Assume that $E\neq T_i$. Then there exists an element $e\in E$ such that $\mathrm{Ann}_{J_i}(e)=\langle 0\rangle$. Let $i=1$. In this case $FG=J_1\langle g_2\rangle$. Let $E_1=eFG=eJ_1\langle g_2\rangle$. By Theorem of Ph. Holl (see, e.g., Corollary 1 to Lemma 9.53 of [9]) E_1 contains a free J_1-submodule $U=\bigoplus_{n\in\mathbb{N}}U_n$, $U_n\cong J_1$ such that E_1/U is a J_1-periodic module with finite set $\Pi_{J_1}(E_1/U)$. Since $\mathrm{Spec}(J_1)$ is infinite, there exists the maximal ideal $P\notin\Pi_{J_1}(E_1/U)$. From the choice of e we get that $U\neq\langle 0\rangle$. If $l\in\mathbb{N}$ then $UP^l=\bigoplus_{n\in\mathbb{N}}U_nP^l$, where $U_nP^l\cong P^l$, in particular, U/UP^l is a p-component of E_1/UP^l. Thence we get that $E_1/UP^l=U/UP^l\oplus R_l/UP^l$, where $R_l/UP^l\cong E_1/U$. In this case $(E_1/UP^l)P^l = R_l/UP^l$. On the other hand $(E_1/UP^l)P^l = ((E_1P^l + UP^l)/UP^l = E_1P^l/UP^l$. From here $E_1P^l = R_l$ and $E_1/E_1P^l \cong R_l/UP^l$. Hence $E_1 \neq E_1P$, $E_1P^l \neq E_1P^{l+1}$ for any $l \in \mathbb{N}$. In other words, an ascending series $E_1 > E_1P > \cdots > E_1P^l > \cdots$ is infinite. Obviously, E_1P^l is a FG-submodule for any $l \in \mathbb{N}$, i.e. we get a contradiction that E is Artinian. This contradiction proves the equalities $E = T_1 = E_{P_1}$, $E = T_2 = E_{P_2}$, which mean that E is an $F\langle g_i\rangle$-hypercentral module, $i = 1, 2$. Thus it follows that E is FG-hypercentral. □

Lemma 6. *Let F be a field, $G = \langle g_1\rangle \times \langle g_2\rangle$ a free abelian group of rank 2, A a monolithic Artinian FG-module, E an FG-injective hull of A, $E_n = \Omega_{(g_2-1),n}(E)$, $n \in \mathbb{N}$. If A is an FG-hypercentral module then $E_n(g_1 - 1) = E_n$ for any $n \in \mathbb{N}$.*

Proof. Assume the contrary and choose the least n such that $E_n(g_1 - 1) \neq E_n$. By Lemma 5 E is an FG-hypercentral module, so $E = \bigcup_{k\in\mathbb{N}} E_k$.

By Proposition 1, $E_{n-1} = P_1 \oplus \cdots \oplus P_{n-1}$, where P_i is a Prufer $(g_1 - 1)$-module, $P_i = \bigoplus_{k\in\mathbb{N}} a_{i,k}F$, i.e. $a_{i,1}(g_1 - 1) = 0$, $a_{i,k+1}(g_1 - 1) = a_{i,k}$, $1 \leq i \leq n - 1$. Besides $a_{i+1,k}(g_2 - 1) = a_{i,k}$, $1 \leq i \leq n - 2$, $k \in \mathbb{N}$. Using arguments from Proposition 1 we can show that $E_n = E_{n-1} \oplus B$, where $B = b_1F \oplus \cdots \oplus b_tF$, $b_1(g_2 - 1) = a_{n-1,1}, \ldots, b_t(g_2 - 1) = a_{n-1,t}$, $b_1(g_1 - 1) = 0$, $b_2(g_1 - 1) = b_1, \ldots, b_t(g_1 - 1) = b_{t-1}$.

Let now $P_n = \bigoplus_{k\in\mathbb{N}} a_{n,k}F$ is a Prufer (g_1-1)-module, i.e. $a_{n,1}(g_1-1) = 0$, $a_{n,k+1}(g_1-1) = a_{n,k}$, $k \in \mathbb{N}$. Put $R = E_{n-1} \oplus P_n$ and define an action g_2 on R by $a_{n,k}(g_2 - 1) = a_{n-1,k}$, $k \in \mathbb{N}$. Let $S = E_{n-1} \oplus a_{n,1}F \oplus \cdots \oplus a_{n,t}F$ then FG-modules E_{n-1} and S are isomorphic.

Let $\phi\colon E_{n-1} \to S$ be an isomorphism. Consider the FG-module $E \times R$. Let $C = \{a - a\phi | a \in E_{n-1}\}$. Since $(a - a\phi)x = ax - (a\phi)x = ax - (ax)\phi$ for any $x \in G$, and $(a-a\phi)-(a_1-a_1\phi) = (a-a_1)-(a\phi-a_1\phi) = (a-a_1)-(a-a_1)\phi$, $a, a_1 \in E_{n-1}$, C is an FG-submodule of $E \times R$. Obviously, $C \cap E = \langle 0\rangle = C \cap R$, hence $E \cong E/(E\cap C) = (E+C)/C$ and $R \cong R/(R\cap C) = (R+C)/C$. Put $\overline{E} = (E+C)/C$, $\overline{E}_1 = (E+R)/C$, $\overline{R} = (R+C)/C$, $\overline{E}_{n-1} = (E_{n-1} + R)/C$. From the choice of C it follows that $\overline{E}_{n-1} \leq \overline{R}$. According to our admissions, $\overline{E} \neq \overline{E}_1$. As $\overline{E} \cong E$, $\overline{E}$ is an injective FG-module. In this case $\overline{E}_1 = \overline{E} \oplus \overline{V}$ for some FG-submodule $\overline{V} \neq \langle 0\rangle$ (see, e.g., Theorem 2.15 of [10]). It follows from the construction of $\overline{E}_1$ that $\overline{E}_1$ is monolithic. This contradiction proves the lemma. □

Theorem 1. *Let F be a field, $G = \langle g_1\rangle \times \langle g_2\rangle$ a free abelian group of rank 2, A a monolithic Artinian FG-module, E an FG-injective hull of A. If A is FG-hypercentral, then $E = P_1 \oplus \cdots \oplus P_n \oplus \cdots$, where $P_n = \bigoplus_{k\in\mathbb{N}} a_{n,k}F$ is a Prufer $(g_1 - 1)$-module, i.e. $a_{n,1}(g_1 - 1) = 0$, $a_{n,k+1}(g_1 - 1) = a_{n,k}$; besides $P_1(g_2 - 1) = \langle 0\rangle$, $a_{n+1,k}(g_2 - 1) = a_{n,k}$, $n, k \in \mathbb{N}$.*

Proof. By Lemma 5, E is an FG-hypercentral module. Put $E_n = \Omega_{(g_2-1),n(E)}$, $n \in \mathbb{N}$. By Lemma 6 $E_n(g_1 - 1) = E_n$ for any $n \in \mathbb{N}$. Since E is an essential expansion of the module A (see, e.g., Proposition 2.20 [10]) E is a monolithic module. Now it remains to use Proposition 1. This completes the proof. □

We see that the condition of $(g_1 - 1)$-divisibility of every $(g_2 - 1)$-layer is a very strong condition. It ensures the complementation of every $(g_2 - 1)$-layer. In turn, presence of such complementation draws the $(g_1 - 1)$-purity of every layer. Consider now the modules with such condition.

Lemma 7. *Let F be a field, $G = \langle g_1\rangle \times \langle g_2\rangle$ a free abelian group of rank 2, A a monolithic Artinian FG-module, $A_n = \Omega_{(g_2-1),n}(A)$, $n \in \mathbb{N}$. If A is FG-hypercentral and A_n is $(g_1 - 1)$-pure, then $A_n = A_{n-1} \oplus B_n$ for some $F\langle g_1\rangle$-submodule T_n, $n \in \mathbb{N}$.*

Proof. Use the induction by number n. Consider at first A_2. Let T_2 be a maximal $F\langle g_1\rangle$-submodule of A_2 for which $A_1 \cap T_2 = \langle 0\rangle$. Assume that $A_1 \oplus T_2 \neq A_2$. Since the module A is FG-hypercentral, there exists an element $a \in A_2 \setminus (A_1 \oplus T_2)$ such that $a(g_1 - 1) = a_1 + b \in A_1 \oplus B_2$, i.e. $a_1 \in A_1$, $b \in B_2$. If $a_1 \in A_1(g_1 - 1)$ then $a_1 = a_2(g_1 - 1)$ for some $a_2 \in A_1$. In this case $(a - a_2)(g_1 - 1) = a(g_1 - 1) - a_2(g_1 - 1) = a_1 + b - a_1 = b$ and $a - a_1 \notin T_2 \oplus A_1$. But then $((a - a_2)F \oplus B_2) \cap A_1 = \langle 0\rangle$ that contradicts to the choice of T_2. So, consider now the case when $a_1 \notin A_1(g_1 - 1)$. In particular, it means that $A_1 \neq A_1(g_1 - 1)$. By Lemma 2 $A_1 = a_{11}F \oplus \cdots \oplus a_{1,k}F$, $a_{11}(g_1 - 1) = 0$, $a_{12}(g_1 - 1) = a_{11}, \ldots, a_{1,k}(g_1 - 1) = a_{1,k-1}$. Since $a \in A_2 \setminus A_1$, we get $a(g_2 - 1) \neq 0$ and $a(g_2 - 1) \in A_1$, so $a(g_2 - 1) = a_{11}\alpha_1 + \cdots + a_{1,k}\alpha_k$ for some $\alpha_1, \ldots, \alpha_k \in F$. We see that $0 = a(g_2 - 1)(g_1 - 1)^k = a(g_1 - 1)^k(g_2 - 1)$, so $a(g_1 - 1)^k \in A_1$. Now we have $a(g_1 - 1)^k = a_1(g_1 - 1)^{k-1} + b(g_1 - 1)^{k-1}$.

Since $a_1 \notin A_1(g_1 - 1)$, we obtain $a_1(g_1 - 1)^{k-1} = a_{11}\gamma$, where $\gamma \neq 0$. Hence $b(g_1 - 1)^{k-1} = a(g_1 - 1)^k - a_{11}\gamma \in A_1$, i.e. $b(g_1 - 1)^{k-1} \in A_1 \cap T_2 = \langle 0\rangle$. So, $a(g_1 - 1)^k = a_{11}\gamma$. It means that $h_{A,g_1}(a_{11}) \geq k$, while $h_{A,g_1}(a_{11}) = k - 1$. We again get a contradiction, which proves the equality $A_2 = A_1 \oplus T_2$.

Let now $n > 2$ and assume that the equality $A_n = A_{n-1} \oplus B_n$ has been already proved. Choose an $F\langle g_1\rangle$-submodule B_{n+1} in the module A_{n+1}, which is maximal among submodules that have zero intersection with A_n. Assume again that $A_n \oplus T_{n+1} \neq A_{n+1}$. Since A is FG-hypercentral, there exists an element $a \in A_{n+2} \setminus (A_n \oplus T_{n+1})$ such that $a(g_1 - 1) = d + c$, where $d \in A_n$, $c \in B_{n+1}$. If $d \in A_n(g_1 - 1)$, then, like in the previous case, we get a contradiction. So, assume that $d \notin A_n(g_1 - 1)$. From what is proved earlier we get a decomposition $A_n = B_1 \oplus \cdots \oplus B_n$, where $B_1 = A_1$ and B_i is a monolithic $F\langle g_1\rangle$-module. Hence $d = d_1 + d_2 + \cdots + d_n$, $d_i \in B_i$. Since $d \notin A_n(g_1 - 1)$, there exists an index j for which $d_j \notin B_j(g_1 - 1)$. Let m be a maximal among such indexes, so $d_m \notin B_m(g_1 - 1)$, but $d_{m+1} = u_{m+1}(g_1 - 1), \cdots, d_n = u_n(g_1 - 1)$ for some elements $u_i \in B_i$, $m + 1 \leq i \leq n$.

Consider the element $a^* = a - u_{m+1} - \cdots - u_n$. Then $a^*(g_1 - 1) = a(g_1 - 1) - u_{m+1}(g_1 - 1) - \cdots - u_n(g_1 - 1) = d + c - d_{m+1} - \cdots - d_n = c + d_1 + \cdots + d_m$.

By Lemma 2 $B_m = b_{m,1}F \oplus \cdots \oplus b_{m,k}F$, moreover, $b_{m,1}(g_1 - 1) = 0$, $b_{m,2}(g_1 - 1) = b_{m,1}, \cdots, b_{m,k}(g_1 - 1) = b_{m,k-1}$. We see that $a^*(g_2 - 1)^{n+m-1} \in A_m$, i.e. $a^*(g_2 - 1)^{n+m-1} = b_{m,1}\alpha_1 + \cdots + b_{m,k}\alpha_k + v$, where $v \in A_{m-1}$, $\alpha_i \in F$, $1 \leq i \leq k$. Hence $a^*(g_2 - 1)^{n+m-1}(g_1 - 1)^k \in A_{m-1}$ but, on the other hand, $a^*(g_2 - 1)^k = (c + d_1 + \cdots + d_m)(g_2 - 1)^{k-1} = c(g_2 - 1)^{k-1} + d_1(g_2 - 1)^{k-1} + \cdots + d_m(g_2 - 1)^{k-1} = c(g_2 - 1)^{k-1} + d_1(g_2 - 1)^{k-1} + \cdots + d_{m-1}(g_2 - 1)k - 1 + b_{m,1}\beta$, where $\beta \neq 0$. Since $a^*(g_2 - 1)^{n+m-1}(g_1 - 1)^k \in A_{m-1}$, we get $a^*(g_1 - 1)^k \in A_n$. Thence we get that $c(g_1 - 1)^{k-1} \in A_n \cap T_{n+1} = \langle 0\rangle$, i.e. $c(g_1 - 1)^{k-1} = 0$. Put $w = (d_1 + \cdots + d_{m-1})(g_1 - 1)^{k-1} + b_{m,1}\beta$, then $w = a^*(g_1 - 1)^k$, i.e. $h_{A,g_1}(w) \geq k$, on the other hand, $h_{A,g_1}(w) = k - 1$. We again get a contradiction, which proves the equality $A_{n+1} = A_n \oplus B_{n+1}$. □

Theorem 2. *Let F be a field, $G = \langle g_1\rangle \times \langle g_2\rangle$a free abelian group of rank 2, A a monolithic Artinian FG-module, $A_n = \Omega_{(g_2-1),n}(A)$, $n \in \mathbb{N}$. If A is FG-hypercentral and A_n is $(g_1 - 1)$-pure for any $n \in \mathbb{N}$ then $A_n = P_1 \oplus \cdots \oplus P_n$ for $n \leq k$ and $A_n = P_1 \oplus \cdots \oplus P_n \oplus B_{k+1} \oplus \cdots \oplus B_n$ for $n > k$, where k is a maximal index, for which $A_k(g_1 - 1) = A_k$, $P_i = \bigoplus_{n\in\mathbb{N}} a_{i,n}F$ is a Prufer $(g_1 - 1)$-module, $T_i = a_{i,1}F \oplus \cdots \oplus a_{i,s_i}F$, $s_i \in \mathbb{N}$, is a some submodule of Prufer $(g_1 - 1)$-module, i.e. $a_{i,1}(g_1 - 1) = 0$, $a_{i,n+1}(g_1 - 1) = a_{i,n}$, $n \in \mathbb{N}$. Besides, $a_{1,n}(g_2 - 1) = 0$, $a_{j+1,n}(g_2 - 1) = a_{j,n}$, $j, n \in \mathbb{N}$.*

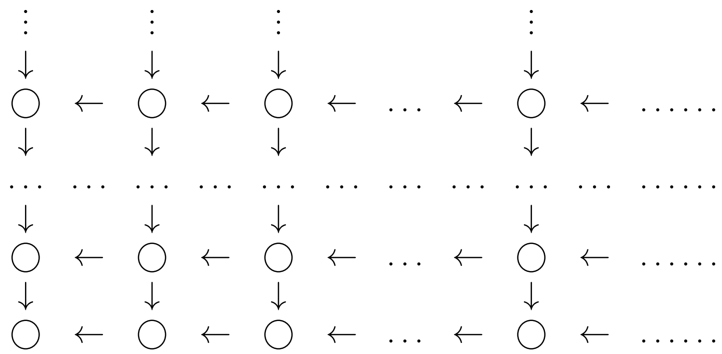
If $A \neq A_n$ for any $n \in \mathbb{N}$ and $k \neq \infty$, then $A = P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus B_{k+1} \oplus \cdots \oplus B_n \oplus \cdots$.

If $n = \infty$, then $A = P_1 \oplus P_2 \oplus \cdots \oplus P_k \cdots$.

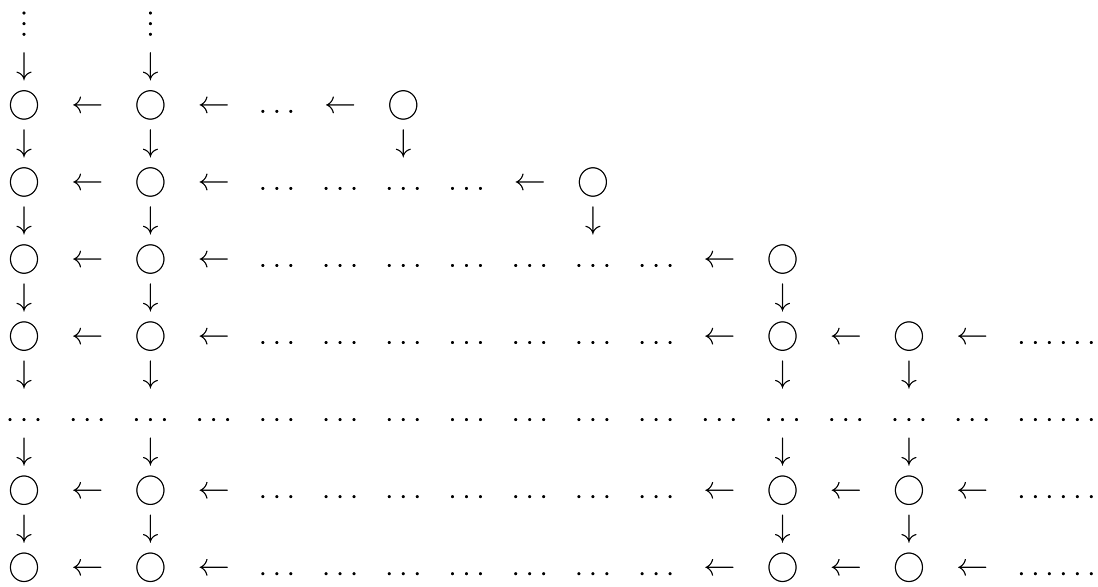
Proof. The theorem can be proved using Lemma 7 and arguments of the proof of Proposition 1. □

So, the condition of (g_1-1)-purity of every (g_2-1)-layer also, either as condition of (g_1-1)-divisibility of every (g_2-1)-layer draws the complementation of every (g_2-1)-layer.

The structures of such modules can be represented graphically, as one can see on Pictures 1 and 2. On these pictures circumferences are elements of basis of a module as a vector space over the field F. The arrow downwards from basis element shows where this element goes when it is acted by (g_2-1). The arrow to the left shows where this element goes when it is acted by (g_1-1). If arrow does not come out from element in some direction, then this elements goes to zero.



Pict.1 (condition of (g_1-1)-divisibility of every (g_2-1)-layer)



Pict.2 (condition of (g_1-1)-purity of every (g_2-1)-layer)

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