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ON A FUNCTORIAL ISOMORPHISM IN THE DERIVED CATEGORY OF ℓ-ADIC SHEAVES

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For a vector bundle $h\colon E \to B$ of dimension d over the algebraic closure of a finite field we prove that the functor $\bar{R}h_! \circ h^*\colon D^b(B,\mathbb{Q}_\ell) \to D^b(B,\mathbb{Q}_\ell)$ is isomorphic to the (twisted) shift functor $[-2d](-d)$.

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Для векторного расслоения $h\colon E \to B$ размерности d над алгебраическим замыканием конечного поля доказано, что функтор $\bar{R}h_! \circ h^*\colon D^b(B,\mathbb{Q}_\ell) \to D^b(B,\mathbb{Q}_\ell)$ изоморфен (скрученному) функтору сдвига $[-2d](-d)$.

Let $\mathbb{F}$ be the algebraic closure of the field $\mathbb{F}_p$ of p elements. Denote by $\mathbb{Q}_\ell$ the field of ℓ-adic numbers, ℓ is a prime, different from the prime p. The bounded derived category of ℓ-adic sheaves on a scheme X is denoted $D^b(X,\mathbb{Q}_\ell)$. We are going to discuss a functor in derived category, which is expected to be a kind of braiding in a monoidal 2-category.

Theorem 1. *Let B be a quasicompact scheme over $\mathbb{F}$. Let $h\colon E \to B$ be a vector bundle of dimension d over $\mathbb{F}$. Then there is an isomorphism of functors*

$$\big[D^b(B,\mathbb{Q}_\ell) \xrightarrow{h^*} D^b(E,\mathbb{Q}_\ell) \xrightarrow{\bar{R}h_!} D^b(B,\mathbb{Q}_\ell)\big] \simeq \big[D^b(B,\mathbb{Q}_\ell) \xrightarrow{[-2d](-d)} D^b(B,\mathbb{Q}_\ell)\big].$$

Proof. Let $\mathcal{E}$ be a locally free $\mathcal{O}_B$-module of rank d such that $E \simeq \mathbf{Spec}\, S_{\mathcal{O}_B}(\mathcal{E})$, where $S_{\mathcal{O}_B}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ [4, Exercise II.5.18]. Considering this symmetric algebra as a graded algebra sheaf, we get $q\colon Q \to B$, $Q = \mathbf{Proj}\, S^\bullet_{\mathcal{O}_B}(\mathcal{E})$, the bundle of projective spaces associated with the vector bundle E. Denote by $\mathbb{1}$ the trivial line bundle on B, $\mathbb{1} = \mathbb{F} \times B \to B$. Then $p\colon P \to B$, $P = \mathbf{Proj}\, S^\bullet_{\mathcal{O}_B}(\mathcal{O}_B \oplus \mathcal{E})$ is the bundle of projective spaces associated with the vector bundle $\mathbb{1} \oplus E$.

There is an open embedding $j\colon E \hookrightarrow P$ and a closed embedding $i\colon Q \hookrightarrow P$, given in local coordinates by $j(e) = (1,e)$ and $i(q) = (0,q)$. In other words, j identifies E with the open subscheme $D_+(\xi) \subset P$ for the section $\xi = 1 \oplus 0 \in \mathcal{O}_B(B) \oplus \mathcal{E}(B)$, and i makes Q into a closed subscheme of P determined by the ideal $(\xi) \subset S^\bullet_{\mathcal{O}_B}(\mathcal{O}_B \oplus \mathcal{E})$ spanned by ξ. The set

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of all points of the scheme P is the disjoint union of sets of points of E and Q. Clearly, the following diagram of scheme morphisms commutes.

$$\begin{array}{ccccc} E & \xrightarrow{j} & P & \xleftarrow{i} & Q \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle q} & \\ & & B & & \end{array}$$

The exact sequence of etale sheaves on P [5, Remark II.3.13]

$$0 \to j_! j^* F \xrightarrow{a} F \xrightarrow{b} i_* i^* F \to 0$$

uses the exact functors $j_!$, j^*, i_*, i^*. We denote by the same symbol their extension to the derived category. For $F \in D^b(P, \mathbb{Q}_\ell)$ we have a distinguished triangle

$$j_! j^* F \xrightarrow{a} F \xrightarrow{b} i_* i^* F \xrightarrow{c} \quad .$$

In particular, for $S \in D^b(B, \mathbb{Q}_\ell)$ we have a triangle

$$j_! h^* S \xrightarrow{a} p^* S \xrightarrow{b} i_* q^* S \xrightarrow{c} \quad .$$

Applying Rp_* we get a distinguished triangle

$$\bar{R}h_! h^* S \xrightarrow{a'} Rp_* p^* S \xrightarrow{b'} Rq_* q^* S \xrightarrow{c'} \quad , \tag{1}$$

where $\bar{R}h_!$ denotes the functor $Rp_* \circ j_!$, obtained via the compactification $E \xrightarrow{j} P \xrightarrow{p} B$ of h. This version of the derived functor has been defined by

Deligne [3, Definition 5.1.9].

Let us consider the Chern class $\eta_Q \colon \mathbb{Q}_\ell \to \boldsymbol{\mu}[2] \in D^b(Q, \mathbb{Q}_\ell)$ of $\mathcal{O}_Q(1)$, in other terms, an element $\eta_Q \in H^2(Q, \boldsymbol{\mu})$. For an arbitrary $K \in D^b(Q, \mathbb{Q}_\ell)$ it defines a morphism, also denoted by η_Q,

$$K \xrightarrow{\sim} \mathbb{Q}_\ell \overset{L}{\to} \otimes K \xrightarrow{\eta_Q \otimes 1} \boldsymbol{\mu}[2] \overset{L}{\to} \otimes K \xrightarrow{\sim} K[2](1).$$

For $S \in D^b(B, \mathbb{Q}_\ell)$ we define

$$\alpha_i \colon S \xrightarrow{\mathrm{adj}_q} Rq_* q^* S \xrightarrow{(Rq_* \eta_Q)^i} Rq_* q^* S[2i](i).$$

The sum of those is an isomorphism

$$\sum \alpha_i : \bigoplus_{i=0}^{d-1} S[-2i](-i) \xrightarrow{\sim} Rq_* q^* S \tag{2}$$

by a version of Lemma 5.4.12 [1] over $\mathbb{Q}_\ell$.

Let S be a sheaf on B. Let us prove that the morphism b' from distinguished triangle (1) induces an isomorphism in r^{th} cohomology for all r, except for $r = 2d$. This follows from an equation for $S \in D^b(B, \mathbb{Q}_\ell)$

$$\alpha_i^q = \big(S[-2i](-i) \xrightarrow{\alpha_i^p} Rp_* p^* S \xrightarrow{b'} Rq_* q^* S\big),$$

which holds for all $i \geq 0$. Notice that for $0 \leq i < d$ the induced morphisms $\alpha_i \colon S(-i) \to R^{2i}p_*p^*S$ and $\alpha_i \colon S(-i) \to R^{2i}q_*q^*S$ are isomorphisms.

For $i = 0$ this equation is standard:

$$\mathrm{adj}_q = \bigl(S \xrightarrow{\mathrm{adj}_p} Rp_*p^*S \xrightarrow{Rp_*\mathrm{adj}_i} Rp_*i_*i^*p^* \xrightarrow{\sim} Rq_*q^*S\bigr).$$

To prove it for $i > 0$ it suffices to prove the equality

$$\bigl(Rp_*p^*S \xrightarrow{b'} 2)Rq_*q^*S \xrightarrow{Rq_*\eta_Q} Rq_*q^*S[2](1)\bigr) = \\ = \bigl(Rp_*p^*S \xrightarrow{Rp_*\eta_P} Rp_*p^*S[2](1) \xrightarrow{b'[2](1)} Rq_*q^*S[2](1)\bigr).$$

Since b' is Rp_*b composed with a canonical isomorphism, the above equality follows from

$$\bigl(p^*S \xrightarrow{b} i_*i^*p^*S \xrightarrow{i_*\eta_Q} i_*i^*p^*S[2](1)\bigr) = \bigl(p^*S \xrightarrow{\eta_P} p^*S[2](1) \xrightarrow{b=b[2](1)} i_*i^*p^*S[2](1)\bigr),$$

where $b = \mathrm{adj}_i$ is the adjunction unit. The above equality holds not only for $F = p^*S$, but also for an arbitrary $F \in D^b(P, \mathbb{Q}_\ell)$. To prove it, notice that in the diagram

$$\begin{array}{ccccccc}
\mathbb{Q}_\ell \otimes F & \xrightarrow{\mathrm{adj}_i \otimes 1} & i_*i^*\mathbb{Q}_\ell \otimes F & \xrightarrow{\sim} & i_*(i^*\mathbb{Q}_\ell \otimes i^*F) & \xrightarrow{\sim} & i_*i^*(\mathbb{Q}_\ell \otimes F) \\
{\scriptstyle \eta_P \otimes 1}\downarrow & & {\scriptstyle i_*\eta_Q \otimes 1}\downarrow & = & \downarrow{\scriptstyle i_*(\eta_Q \otimes 1)} & = & \downarrow{\scriptstyle i_*i^*(\eta_Q \otimes 1)} \\
\boldsymbol{\mu}[2] \otimes F & \xrightarrow{\mathrm{adj}_i \otimes 1} & i_*i^*\boldsymbol{\mu}[2] \otimes F & \xrightarrow{\sim} & i_*(i^*\boldsymbol{\mu}[2] \otimes i^*F) & \xrightarrow{\sim} & i_*i^*(\boldsymbol{\mu}[2] \otimes F)
\end{array}$$

the rows are equal to adj_i.

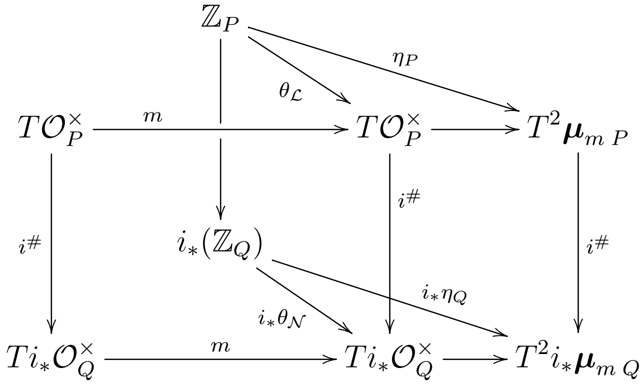
More generally, for bounded above complexes G, F on P the commutativity of the exterior of the left diagram implies the right one:

$$\begin{array}{ccc}
i^*(G \otimes F) & \xrightarrow{i^*(\mathrm{adj}_i \otimes 1)} & i^*(i_*i^*G \otimes F) \\
{\scriptstyle \nu}\downarrow\wr & = & \wr\downarrow{\scriptstyle \nu} \\
i^*G \otimes i^*F & \xrightarrow{i^*\mathrm{adj}_i \otimes 1} & i^*i_*i^*G \otimes i^*F \\
 & \searrow{=} & \downarrow{\scriptstyle \mathrm{adj}'_i \otimes 1} \\
 & & i^*G \otimes i^*F
\end{array}
\quad\Longrightarrow\quad
\begin{array}{ccc}
G \otimes F & \xrightarrow{\mathrm{adj}_i \otimes 1} & i_*i^*G \otimes F \\
{\scriptstyle \mathrm{adj}_i}\downarrow & = & \downarrow\wr \\
i_*i^*(G \otimes F) & \xrightarrow[\sim]{i_*\nu} & i_*(i^*G \otimes i^*F)
\end{array}$$

So it remains to prove the particular case $S = \mathbb{Q}_\ell$, that is, the equality

$$\bigl(\mathbb{Z}/m_P \longrightarrow i_*(\mathbb{Z}/m_Q) \xrightarrow{i_*\eta_Q} i_*\boldsymbol{\mu}_m[2]_Q\bigr) = \bigl(\mathbb{Z}/m_P \xrightarrow{\eta_P} \boldsymbol{\mu}_m[2]_P \xrightarrow{i^\#} i_*\boldsymbol{\mu}_m[2]_Q\bigr)$$

for $m = l^n$ or any $m > 0$ not divisible by p. Instead of working with 2-cocycles, we reduce the question to 1-cocycles via Kummer sequences and associated triangles:



Here $\mathcal{L} = \mathcal{O}_P(1)$, $\mathcal{N} = i^*\mathcal{L} \otimes_{i^*\mathcal{O}_P} \mathcal{O}_Q \simeq \mathcal{O}_Q(1)$, and $\theta_{\mathcal{L}}$, $\theta_{\mathcal{N}}$ are corresponding 1-cocycles. We apply the following lemma.

Lemma 2. *Let $f\colon X \to Y$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on Y, it induces an invertible sheaf $\mathcal{N} = f^*\mathcal{L} \otimes_{f^*\mathcal{O}_Y} \mathcal{O}_X$ on X. The class of $\mathcal{L}$ in the Picard group $\mathrm{Pic}Y \xrightarrow{\sim} H^1(Y_{et}, \mathcal{O}^\times)$ is denoted by $\theta_{\mathcal{L}}\colon \mathbb{Z}_Y \to T\mathcal{O}_Y^\times$, similarly for $\theta_{\mathcal{N}}$. Then the diagram*

$$\begin{array}{ccc} f^*\mathbb{Z}_Y & \xrightarrow{f^*\theta_{\mathcal{L}}} & Tf^*\mathcal{O}_Y^\times \\ \wr\downarrow & & \downarrow Tf^\# \\ \mathbb{Z}_X & \xrightarrow{\theta_{\mathcal{N}}} & T\mathcal{O}_X^\times \end{array}$$

is commutative.

Proof. The class $\theta_{\mathcal{L}}$ is the image of $[\mathcal{L}]$ under the sequence of isomorphisms

$$\mathrm{Pic}Y \to \check{H}^1(Y_{Zar}, \mathcal{O}^\times) \to H^1(Y_{Zar}, \mathcal{O}^\times) \to H^1(Y_{et}, \mathcal{O}^\times).$$

Let $\mathcal{U} = (\mathcal{U}_i)_{i\in I}$ be an affine Zariski covering of Y and $\phi_i : \mathcal{O}_Y\big|_{U_i} \to \mathcal{L}\big|_{U_i}$ are isomorphisms. Then

$$\mathcal{O}_Y\big|_{U_i\cap U_j} \xrightarrow{\phi_j} \mathcal{L}\big|_{U_i\cap U_j} \xrightarrow{\phi_i^{-1}} \mathcal{O}_Y\big|_{U_i\cap U_j}$$

is the action of a section $s_{ij}^{\mathcal{L}} \in \mathcal{O}_Y^\times(U_i \cap U_j)$. The collection $s^{\mathcal{L}} = (s_{ij}^{\mathcal{L}})_{i<j}$ is a Čech 1-cocycle. The corresponding morphism $\theta_{\mathcal{L}}\colon \mathbb{Z}_Y \to T\mathcal{O}_Y^\times$ in $D^b(Y_{Zar})$ (resp. $D^b(Y_{et})$) is constructed via the Čech resolution of $F = \mathcal{O}_Y^\times$

$$0 \to \mathcal{C}^0(\mathcal{U}, F) \to \mathcal{C}^1(\mathcal{U}, F) \to \mathcal{C}^2(\mathcal{U}, F) \to \mathcal{C}^p(\mathcal{U}, F) = \prod_{i_0<\dots<i_p} j_{i_0\dots i_p *} j^*_{i_0\dots i_p} F,$$

where $j_{i_0\dots i_p} : U_{i_0} \cap \dots \cap U_{i_p} \hookrightarrow Y$ is the embedding. In particular,

$$\mathcal{C}^p(\mathcal{U}, F)(V) = \prod_{i_0<\dots<i_p} F(V \cap U_{i_0} \cap \dots \cap U_{i_p}).$$

As a morphism in the derived category, $\theta_{\mathcal{L}}$ can be written as $(\mathbb{Z}_Y \xrightarrow{s^{\mathcal{L}}} T\mathcal{C}(\mathcal{U}, \mathcal{O}_Y^\times) \xleftarrow{\epsilon} T\mathcal{O}_Y^\times)$, where the quasi-isomorphism ϵ is the product of restriction maps.

Let us denote by f^{-1} the open covering $(f^{-1}U_i)_{i\in I}$ of X. The morphisms of Zariski sheaves on X

$$f^*\mathcal{O}_Y^\times = f^*\mathcal{O}_Y^\times \xrightarrow{f^\#} \mathcal{O}_X^\times$$

extend to chain maps of resolutions

$$f^*\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_Y^\times) \xrightarrow{\tau} \mathcal{C}^\bullet(f^{-1}\mathcal{U}, f^*\mathcal{O}_Y^\times) \xrightarrow{\mathcal{C}^\bullet(f^{-1}\mathcal{U}, f^\#)} \mathcal{C}^\bullet(f^{-1}\mathcal{U}, \mathcal{O}_X^\times).$$

The map τ corresponds to the map $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_Y^\times) \to f_*\mathcal{C}^\bullet(f^{-1}\mathcal{U}, f^*\mathcal{O}_Y^\times)$, constructed from the components

$$j_{i_0\dots i_p *} j^*_{i_0\dots i_p} F \to j_{i_0\dots i_p *} f_{i_0\dots i_p *} f^*_{i_0\dots i_p} j^*_{i_0\dots i_p} F \xrightarrow{\sim} f_* k_{i_0\dots i_p *} k^*_{i_0\dots i_p} f^* F,$$

where the maps are taken from

$$\begin{array}{ccc}
f^{-1}(U_{i_0} \cap \cdots \cap U_{i_p}) & \xhookrightarrow{k_{i_0 \dots i_p}} & X \\
{\scriptstyle f_{i_0 \dots i_p} = f} \Big\downarrow & \lrcorner & \Big\downarrow {\scriptstyle f} \\
U_{i_0} \cap \cdots \cap U_{i_p} & \xhookrightarrow{j_{i_0 \dots i_p}} & Y
\end{array}$$

The required diagram follows from the commutativity of the right rectangle in

$$\begin{array}{ccccccc}
f^*\mathcal{O}_Y^\times & \xrightarrow{f^*\epsilon} & f^*\mathcal{C}^0(\mathcal{U}, \mathcal{O}_Y^\times) & \longrightarrow & f^*\mathcal{C}^1(\mathcal{U}, \mathcal{O}_Y^\times) & \xleftarrow{f^*s^{\mathcal{L}}} & f^*\mathbb{Z}_Y \\
\Big\| & & {\scriptstyle \tau^0} \Big\downarrow & & {\scriptstyle \tau^1} \Big\downarrow & & \\
f^*\mathcal{O}_Y^\times & \xrightarrow{\epsilon} & \mathcal{C}^0(f^{-1}\mathcal{U}, f^*\mathcal{O}_Y^\times) & \longrightarrow & \mathcal{C}^1(f^{-1}\mathcal{U}, f^*\mathcal{O}_Y^\times) & (3) & \Big\downarrow \wr \\
{\scriptstyle f^\#} \Big\downarrow & & {\scriptstyle \mathcal{C}^0(f^{-1}\mathcal{U}, f^\#)} \Big\downarrow & & {\scriptstyle \mathcal{C}^1(f^{-1}\mathcal{U}, f^\#)} \Big\downarrow & & \\
\mathcal{O}_X^\times & \xrightarrow{\epsilon} & \mathcal{C}^0(f^{-1}\mathcal{U}, \mathcal{O}_X^\times) & \longrightarrow & \mathcal{C}^1(f^{-1}\mathcal{U}, \mathcal{O}_X^\times) & \xleftarrow{s^{\mathcal{N}}} & \mathbb{Z}_X
\end{array}$$

The explicit computation of the cocycle $s^{\mathcal{N}}$ shows that it is obtained from $s^{\mathcal{L}}$ via the algebra sheaf homomorphisms

$$\mathcal{O}_Y \xrightarrow{\mathrm{adj}_f} f_*f^*\mathcal{O}_Y \xrightarrow{f_*f^\#} f_*\mathcal{O}_X$$

restricted to $U_i \cap U_j$:

$$\mathcal{O}_Y(U_i \cap U_j) \longrightarrow f^*\mathcal{O}_Y(f^{-1}(U_i \cap U_j)) \xrightarrow{f^\#} \mathcal{O}_X(f^{-1}U_i \cap f^{-1}U_j), \quad s_{ij}^{\mathcal{L}} \mapsto s_{ij}^{\mathcal{N}}.$$

This is precisely the commutativity of rectangle (3). □

From the long exact sequence associated with (1) we deduce that for a sheaf S the derived sheaf $\bar{R}^j h_! h^* S$ vanishes for $j \neq 2d$ and $\bar{R}^{2d} h_! h^* S \simeq S[-2d](-d)$. Furthermore, the morphism a' composed with the canonical projection gives the functorial morphism for $S \in D^b(B, \mathbb{Q}_\ell)$

$$\bar{R}h_! h^* S \xrightarrow{a'} Rp_* p^* S \to S[-2d](-d),$$

which is an isomorphism for sheaves S. By devissage, we deduce that it is an isomorphism for all $S \in D^b(B, \mathbb{Q}_\ell)$. □

Problem 3. *Prove Theorem 1 for sheaves of $\mathbb{C}$-vector spaces on schemes over $\mathbb{C}$.*

This would have an equivariant analogue: assuming that a complex algebraic group acts equivariantly on a complex vector bundle $h \colon E \to B$, we would get an isomorphism for a vector bundle of dimension d

$$\big[D^b_G(B, \mathbb{C}) \xrightarrow{h^*} D^b_G(E, \mathbb{C}) \xrightarrow{Rh_!} D^b_G(B, \mathbb{C})\big] \simeq \big[D^b_G(B, \mathbb{C}) \xrightarrow{[-2d]} D^b_G(B, \mathbb{C})\big],$$

where $D^b_G(X, \mathbb{C})$ is the equivariant derived category of Bernstein and Lunts [2].

Problem 4. *Define an equivariant derived category of ℓ-adic sheaves on a G-scheme over $\mathbb{F}$, so that Theorem 1 had an equivariant version.*

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