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THE BEST APPROXIMATION ON $[0,1)$ OF ANALYTIC IN THE UNIT DISK FUNCTIONS WITH NONNEGATIVE TAYLOR COEFFICIENTS

Dedicated to the 70th anniversary of Prof. A. A. Gol'dberg

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The lower estimates of the best approximation on $[0,1)$ of analytic in the unit disk functions with nonnegative Taylor coefficients by the algebraic polynomials with real coefficients is investigated.

Р. Д. Боднар. *Наилучшее приближение на* $[0,1)$ *аналитических в единичном круге функций с неотрицательными тейлоровскими коэффициентами*// Математичні Студії. – 2000. – Т.14, №1. – С.85–88.

Исследуются оценки снизу наилучшего приближения на $[0,1)$ аналитических в единичном круге функций с неотрицательными тейлоровскими коэффициентами алгебраическими многочленами с действительными коэффициентами.

Let

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k, \quad a_k \geq 0, \tag{1}$$

be an analytic in the disk $D = \{z : |z| < 1\}$ function. For $0 \leq r < 1$ set $M_f(r) = \max\{|f(z)| : |z| = r\}$ and assume that

$$0 < \omega = \varliminf_{r\uparrow 1} (1-r)^{\rho} \ln M_f(r) \leq \varlimsup_{r\uparrow 1} (1-r)^{\rho} \ln M_f(r) = \tau < +\infty, \quad 0 < \rho < +\infty, \tag{2}$$

i. e. f has order ρ, lower type ω and type τ. Let Π_n be the class of all algebraic polynomials of degree at most n with real coefficients and set

$$\lambda_{0,n}(f) = \inf\left\{\sup\left\{\left|\frac{1}{f(x)} - \frac{1}{p(x)}\right| : 0 \leq x < 1\right\} : p \in \Pi_n\right\}.$$

It is proved in [1] that

$$\varliminf_{n\to\infty} \left(\frac{\tau\rho}{n}\right)^{\rho/(\rho+1)} \ln\frac{1}{\lambda_{0,n}(f)} \geq \left(\frac{\rho}{\rho+1}\right)^{\rho} \omega,$$

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from which we have $\overline{\lim\limits_{n\to\infty}}\, \lambda_{0,n}^{1/n}(f) \le 1$. We will show in this paper that

$$\underline{\lim\limits_{n\to\infty}}\, \lambda_{0,n}^{1/n}(f) \ge \exp\left\{ -\frac{1}{\rho}\frac{\omega\delta_0^\rho}{\tau}\sqrt{\frac{\delta_0}{\delta_0-1}} \right\}, \tag{3}$$

where $\delta_0 > \left(\frac{\tau}{\omega}\right)^{1/\rho} \ge 1$ solves the equation

$$\frac{1}{\ln(\sqrt{\delta}+\sqrt{\delta-1})}\sqrt{\frac{\delta}{\delta-1}}\left(\frac{\omega\delta^\rho}{\tau}-1\right) = 2\rho. \tag{4}$$

We will get this statement from a more general theorem. First, we need two lemmas.

Lemma 1 [2, p. 231]. *Let $P_n(x)$ be any algebraic polynomial of degree at most n. If this polynomial is bounded by M on an interval $[a,b]$ then at any point t outside this interval we have*

$$|P_n(t)| \le M\left|T_n\left(\frac{2t-a-b}{b-a}\right)\right|, \tag{5}$$

where

$$T_n(t) = \frac{1}{2}\left\{(t+\sqrt{t^2-1})^n + (t-\sqrt{t^2-1})^n\right\}, \quad |t| \ge 1, \tag{6}$$

is the Chebyshev polynomial.

For $\delta > 1$, $c > 1$ and $0 < \varepsilon_2 < \varepsilon < 1$ set $d(\delta) = \frac{\omega}{\tau}\delta^\rho$, $d_\varepsilon(\delta) = \frac{1-\varepsilon}{1+\varepsilon}d(\delta)$, $\gamma = \gamma(\delta,c) = \frac{2\ln(\sqrt{\delta}+\sqrt{\delta-1})}{\ln c}$ and

$$\varphi = \varphi(\delta,c,\varepsilon_2) = \frac{1}{2}\left(\left\{(1-\varepsilon-\gamma(\delta,c))^2 + 4(\gamma(\delta,c)+\varepsilon_2)\right\}^{1/2} + 1 - \varepsilon - \gamma(\delta,c)\right).$$

Lemma 2. *If δ, c, ε and ε_2 are such that for all $r_0 \le r < 1$ the following inequality*

$$f(r^{1/\delta}) \ge f(r)^{\varphi+\gamma+\varepsilon}, \tag{7}$$

holds then for any $\varepsilon_0 \in (0,1)$ and all $n \ge n_0$

$$\lambda_{0,n}(f) \ge (1-\varepsilon_0)\left\{(\sqrt{\delta}+\sqrt{\delta-1})^{2n}\right\}^{-(\varphi+\gamma)/\gamma} = (1-\varepsilon_0)c^{-n(\varphi+\gamma)}. \tag{8}$$

Proof. Conversly, suppose that there exist $\varepsilon_0 \in (0,1)$, an increasing sequence (n_k) of natural numbers and a sequence of polynomials $Q_{n_k} \in \Pi_{n_k}$ such that

$$\sup\left\{\left|\frac{1}{f(x)} - \frac{1}{Q_{n_k}(x)}\right| : x \in [0,1)\right\} < (1-\varepsilon_0)c^{-n_k(\varphi+\gamma)} < c^{-n_k(\varphi+\gamma)}. \tag{9}$$

It follows from (2) that $f(r) \uparrow +\infty$ $(r \uparrow 1)$ and therefore there exists a sequence $(r_k) \uparrow 1$ such that

$$f(r_k) = c^{\varphi n_k}. \tag{10}$$

We obtain for this sequence (r_k) from (9) that

$$|Q_{n_k}| \le \frac{f(x)c^{(\varphi+\gamma)n_k}}{c^{(\varphi+\gamma)n_k} - f(x)} \le \frac{f(r_k)c^{(\varphi+\gamma)n_k}}{c^{(\varphi+\gamma)n_k} - f(r_k)}, \quad 0 \le x \le r_k,$$

and therefore
$$\max\{|Q_{n_k}(x)| : 0 \le x \le r_k\} \le \frac{c^{\varphi n_k}}{1 - c^{-\gamma n_k}}. \tag{11}$$
If we put now $x_k = r_k^{1/\delta}$ then
$$f(x_k) \ge f(r_k)^{\varphi+\gamma+\varepsilon} = c^{(\varphi+\gamma+\varepsilon)\varphi n_k} \tag{12}$$
follows from (7) and (10).

On the other hand, using (5), (6), (11) and the inequality
$$\sqrt{r_k^{(1-\delta)/\delta}} + \sqrt{r_k^{(1-\delta)/\delta} - 1} \le \sqrt{\delta} + \sqrt{\delta - 1}, \quad k \ge k_0,$$
we will get
$$|Q_{n_k}(x_k)| = |Q_{n_k}(r_k^{1/\delta})| \le \left| T_n\left(\frac{2r_k^{1/\delta} - r_k}{r_k}\right)\right| \frac{c^{\varphi n_k}}{1 - c^{-\gamma n_k}} \le$$
$$\le \left(\sqrt{r_k^{(1-\delta)/\delta}} + \sqrt{r_k^{(1-\delta)/\delta} - 1}\right)^{2n_k} \frac{c^{\varphi n_k}}{1 - c^{-\gamma n_k}} \le (\sqrt{\delta} + \sqrt{\delta - 1})^{2n_k} \frac{c^{\varphi n_k}}{1 - c^{-\gamma n_k}}. \tag{13}$$
It follows from inequalities (12) and (13) that
$$\left|\frac{1}{Q_{n_k}(x_k)} - \frac{1}{f(x_k)}\right| \ge \frac{1}{|Q_{n_k}(x_k)|} - \frac{1}{f(x_k)} \ge$$
$$\ge \frac{1 - c^{-\gamma n_k}}{(\sqrt{\delta} + \sqrt{\delta - 1})^{2n_k} c^{\varphi n_k}} - \frac{1}{c^{(\varphi+\gamma+\varepsilon)\varphi n_k}} = \frac{1}{c^{(\varphi+\gamma)n_k}}\left(1 - \frac{1}{c^{\gamma n_k}}\right) - \frac{1}{c^{(\varphi+\gamma+\varepsilon)\varphi n_k}} =$$
$$= \frac{1}{c^{(\varphi+\gamma)n_k}}\left\{1 - \frac{1}{(\sqrt{\delta} + \sqrt{\delta - 1})^{2n_k}} - \frac{1}{c^{\varepsilon_2 n_k}}\right\} > \frac{1 - \varepsilon_0}{c^{(\varphi+\gamma)n_k}}, \quad k \ge k_0,$$
which contradicts (9). Lemma 2 is proved. □

Inequality (3) is a simple corollary of the following theorem.

Theorem. *For any $\varepsilon_0 \in (0,1)$, $\varepsilon_1 \in (0,1)$, $\varepsilon > 0$, $\varepsilon_2 \in (0, \varepsilon)$ and all sufficiently large n the following inequality holds*
$$\lambda_{0,n}(f) \ge (1 - \varepsilon_0)c^{-n}, \tag{14}$$
where
$$c = c(\varepsilon, \varepsilon_1, \varepsilon_2) =$$
$$= \exp\left\{2 \min\left\{\frac{d_{\varepsilon_1}(\delta)(d_{\varepsilon_1}(\delta) - \varepsilon)}{(d_{\varepsilon_1}(\delta) - 1)(d_{\varepsilon_1}(\delta) - \varepsilon) - \varepsilon_2} \ln(\sqrt{\delta} + \sqrt{\delta - 1}) : d_{\varepsilon_1}(\delta) > (1 + \varepsilon)\right\}\right\} =$$
$$= \exp\left\{\frac{1}{\rho}\sqrt{\frac{\delta_0}{\delta_0 - 1}} \frac{d_{\varepsilon_1}(\delta_0)(d_{\varepsilon_1}(\delta_0) - \varepsilon)^2}{(d_{\varepsilon_1}(\delta_0) - \varepsilon)^2 + \varepsilon_2(2d_{\varepsilon_1}(\delta_0) - \varepsilon)}\right\}$$
and δ_0 is a solution of the equation
$$\frac{\sqrt{\delta/(\delta - 1)}}{\ln(\sqrt{\delta} + \sqrt{\delta - 1})} \frac{(d_{\varepsilon_1}(\delta) - \varepsilon)\{(d_{\varepsilon_1}(\delta) - 1)(d_{\varepsilon_1}(\delta) - \varepsilon) - \varepsilon_2\}}{(d_{\varepsilon_1}(\delta) - \varepsilon)^2 + \varepsilon_2(2d_{\varepsilon_1}(\delta) - \varepsilon)} = 2\rho. \tag{15}$$

Proof. Let $d_{\varepsilon_1}(\delta) > (1+\varepsilon)$. It follows from (2) that

$$\frac{(1+o(1))\omega}{(1-r)^\rho} \le \ln f(r) = \ln M_f(r) \le \frac{(1+o(1))\tau}{(1-r)^\rho}, \quad r \uparrow 1,$$

and, since $\delta(1-r^{1/\delta}) = (1+o(1))(1-r)$ as $r \to 1$

$$\ln f(r^{1/\delta}) \ge \frac{(1+o(1))\omega}{(1-r^{1/\delta})^\rho} = \frac{(1+o(1))\omega\delta^\rho}{(1-r)^\rho} \ge (1+o(1))\frac{\omega\delta^\rho}{\tau}\ln f(r), \quad r \uparrow 1.$$

Therefore, for all $r \in [r_0(\varepsilon_1), 1)$ we have $f(r^{1/\delta}) \ge f(r)^{d_{\varepsilon_1}(\delta)}$, that is (7) holds with $\varphi+\gamma+\varepsilon = d_{\varepsilon_1}(\delta) > 1+\varepsilon$.

If we now put $\gamma = d_{\varepsilon_1}(\delta) - \varphi - \varepsilon$ in the definition of φ then we will get the equalities $\varphi = 1 - \frac{\varepsilon-\varepsilon_2}{d_{\varepsilon_1}(\delta)}$ and $\gamma = d_{\varepsilon_1}(\delta) - 1 + \frac{\varepsilon-\varepsilon_2}{d_{\varepsilon_1}(\delta)} - \varepsilon$. Therefore by Lemma 2

$$\lambda_{0,n}(f) \ge (1-\varepsilon_0)\exp\left\{-2n\frac{\varphi+\gamma}{\gamma}\ln(\sqrt{\delta}+\sqrt{\delta-1})\right\} =$$
$$= (1-\varepsilon_0)\exp\left\{-2n\frac{d_{\varepsilon_1}(\delta)(d_{\varepsilon_1}(\delta)-\varepsilon)}{(d_{\varepsilon_1}(\delta)-1)(d_{\varepsilon_1}(\delta)-\varepsilon)-\varepsilon_2}\ln(\sqrt{\delta}+\sqrt{\delta-1})\right\}. \tag{16}$$

It is easy to show by elementary methods that the function

$$F(\delta) = \frac{d_{\varepsilon_1}(\delta)(d_{\varepsilon_1}(\delta)-\varepsilon)}{(d_{\varepsilon_1}(\delta)-1)(d_{\varepsilon_1}(\delta)-\varepsilon)-\varepsilon_2}\ln(\sqrt{\delta}+\sqrt{\delta-1})$$

attains its minimum at the point δ_0 which satisfies (15) and

$$F(\delta_0) = \frac{d_{\varepsilon_1}(\delta_0)(d_{\varepsilon_1}(\delta_0)-\varepsilon)^2}{(d_{\varepsilon_1}(\delta_0)-\varepsilon)^2+\varepsilon_2(2d_{\varepsilon_1}(\delta_0)-\varepsilon)}\frac{1}{2\rho}\sqrt{\frac{\delta_0}{\delta_0-1}}.$$

So (14) follows from (16) and this completes the proof. □

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