УДК 517.927.25

N. O. Babych, Yu. D. Golovaty

COMPLETE WKB ASYMPTOTICS OF HIGH FREQUENCY VIBRATIONS IN A STIFF PROBLEM

N. O. Babych, Yu. D. Golovaty *Complete WKB asymptotics of high frequency vibrations in a stiff problem*, Matematychni Studii, **14** (2000) 59–72.

We study an asymptotic behavior of eigenvalues and eigenfunctions of a stiff problem for an ordinary differential operator of the forth order. The stiffness of a system depends on a small parameter ε and vanishes as $\varepsilon \to 0$ in a prescribed region. Such system possesses of low frequency and high frequency proper vibrations. The low frequency vibrations are described by power series on ε. But this method is not applicable to description of the high frequencies. The asymptotics on ε of the high frequency vibrations was constructed based on the WKB method.

Н. О. Бабыч, Ю. Д. Головатый. *Полная ВКБ-асимптотика высокочастотных собственных колебаний в жесткой задаче* // Математичні Студії. – 2000. – Т.14, №1. – С.59–72.

Исследовано асимптотическое поведение собственных значений и собственных функций одномерной жесткой задачи для дифференциального оператора четвертого порядка. Коэффициент жесткости колебательной системы зависит от малого параметра ε и стремится к нулю на части интервала. Такая задача обладает двумя типами собственных колебаний — низкочастотными и высокочастотными. С использованием метода ВКБ-приближений найдена полная асимптотика высокочастотных колебаний жесткой системы. Асимптотические разложения построены и обоснованы на последовательностях “допустимых"значений малого параметра $\varepsilon_s \to 0$, что согласуется с дискретным характером этого явления.

Stiff vibrating systems belong to the class of systems with singular perturbed potential energy. Stiff problems are differential equation problems with very different values of coefficients in different parts of a domain. They model vibrations of elastic systems consisting of two materials (or more), one of them is very stiff with respect to the other. Stiff problems originated in [1]. J.L. Lions has investigated the boundary value problems.

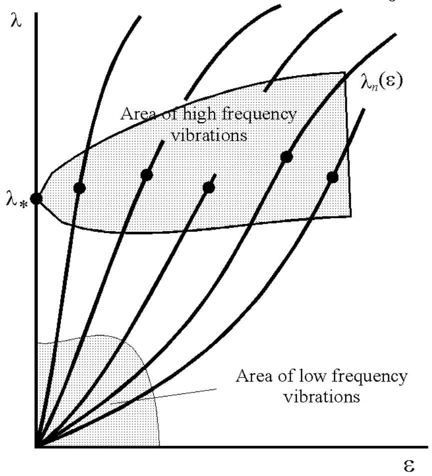
On the other hand, it is of interest to describe the asymptotic behavior of spectra of the stiff problems. Such system possesses of two kinds of vibrations. There are so-called low frequency eigenvibrations and high frequency ones. From a physical viewpoint we can postulate that two kinds of eigenvibrations can appear: one for the stiffer structure and the other for the less stiff structure. Different aspects of the spectral stiff problems are considered in [2]–[10]. The best general reference here is [8]. The asymptotic behavior of the low frequency vibrations has been widely studied with different techniques [2],[3],[7] and [9].

In this paper, following [10] we consider the phenomenon of high frequency vibrations. The leading terms of high frequency vibrations for different problems were constructed in

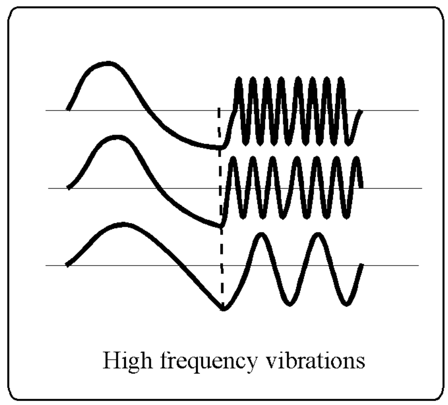
2000 *Mathematics Subject Classification:* 34E20; 74K10.

[7-9]. Information on the behavior of high frequency vibrations was also provided in [10]. We study the stiff problem for the forth order differential operator. Using the WKB technique we construct the complete asymptotic expansions of such vibrations.

Why do high frequency vibrations appear? On the one hand, the spectrum $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ of the stiff problem is asymptotically dense in $[0,\infty)$ as $\varepsilon\to 0$ (see [10]). On the other hand, for fixed i the asymptotic expansions of an eigenvalue λ_i^ε and the corresponding eigenfunction u_i^ε are nonuniform with respect to i. By the above sequences $u_{i(\varepsilon)}^\varepsilon$ with $i(\varepsilon)\to\infty$ can support stable forms of vibration as $\varepsilon\to 0$. In addition, the corresponding sequences of eigenvalues λ_i^ε converge to some positive limit points (see pic. 1, 2). It is also shown that the approximation of the limit form of vibrations by sequence $\{u_i^\varepsilon\}$ has the discrete character. Therefore we construct and justify the asymptotics for a discrete set of small parameter.



Pic. 1



Pic. 2

Moreover, such approximation is ambiguously determined. Then we introduce a deformation parameter δ in the asymptotics. Consequently, for each δ we construct expansions of u^ε, though they are asymptotically equivalent when $\varepsilon\to 0$.

1. Statement of the problem. Let (a,b) be an interval in $\mathbb{R}$ containing the origin. Let us consider the eigenvalue problem

$$\frac{d^2}{dx^2}\left(k_\varepsilon(x)\frac{d^2u_\varepsilon}{dx^2}\right)-\lambda_\varepsilon u_\varepsilon(x)=0,\qquad x\in(a,b), \tag{1}$$

$$u_\varepsilon(a)=u_\varepsilon'(a)=0,\qquad u_\varepsilon(b)=u_\varepsilon'(b)=0, \tag{2}$$

$$u_\varepsilon(-0)=u_\varepsilon(+0),\qquad u_\varepsilon'(-0)=u_\varepsilon'(+0), \tag{3}$$

$$(k_0(x)u_\varepsilon'')(-0)=\varepsilon^4(k_1(x)u_\varepsilon'')(+0),\quad (k_0(x)u_\varepsilon'')'(-0)=\varepsilon^4(k_1(x)u_\varepsilon'')'(+0), \tag{4}$$

Here ε is a small positive parameter and the function

$$k_\varepsilon(x)=\begin{cases}k_0(x), & a<x<0,\\ \varepsilon^4k_1(x), & 0<x<b\end{cases}$$

is smooth strictly positive in $(a,0)\cup(0,b)$. We study the asymptotic behavior of eigenvalues λ_ε and eigenfunctions u_ε of (1)–(4) as $\varepsilon\to 0$.

The problem models eigenvibrations of a nonhomogeneous rod. The rod consists of two components with the same density of mass and strongly different elastic properties. The forth power of ε in the definition of k_ε is suitable for the next consideration (see Section 4).

Let us introduce the Sobolev space $H_0^2(a,b)$ as the closure of set $C_0^\infty(a,b)$ with respect to the norm

$$\|u\| = \left(\int_a^b |u''|^2\, dx \right)^{1/2},$$

and bilinear forms in $H_0^2(a,b)$

$$a_0(\varphi,\psi) = \int_a^0 k_0 \varphi''\psi''\, dx, \quad a_1(\varphi,\psi) = \int_0^b k_1 \varphi''\psi''\, dx, \qquad a_\varepsilon = a_0 + \varepsilon^4 a_1.$$

For each $\varepsilon > 0$ denote by $\|\cdot\|_\varepsilon$ the norm in $H_0^2(a,b)$ associated with the form $a_\varepsilon(\cdot,\cdot)$. It is so-called energetic norm associated with the elastic-like energy.

Find λ_ε and $u_\varepsilon \in H_0^2(a,b)$ such that

$$a_\varepsilon(u_\varepsilon,\varphi) - \lambda_\varepsilon (u_\varepsilon,\varphi)_{L_2(a,b)} = 0, \quad \varphi \in H_0^2(a,b). \tag{5}$$

Problem (5) is a standard eigenvalue problem with real discrete spectrum. For each fixed ε let us consider

$$0 < \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \dots \le \lambda_i^\varepsilon \le \dots, \quad \lambda_i^\varepsilon \to \infty, \quad i \to \infty,$$

the sequence of eigenvalues. Note that each eigenvalue is simple. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions which are assumed to be an orthonormal basis in $L_2(a,b)$.

2. Asymptotic expansions of low frequency vibrations. We wish to investigate the asymptotic behavior of eigenvalues and eigenfunctions for a fixed number m.

Lemma 1. *Every eigenvalues λ_i^ε is a continuous function with respect to ε, $\varepsilon \in (0,1)$ and*

$$\lambda_i^\varepsilon \le C_i \varepsilon^4,$$

where C_i is a constant independent of ε.

Proof. The continuity of eigenvalues follows from the variational principle

$$\lambda_i^\varepsilon = \sup_P \inf_{f \in P^\perp \setminus \{0\}} \frac{a_0(f,f) + \varepsilon^4 a_1(f,f)}{\|f\|^2_{L_2(a,b)}}, \tag{6}$$

where P is a $(i-1)$-subspace of $H_0^2(a,b)$ and $P^\perp$ is the orthonormal complement of P.

Suppose that the subspace P_i extremalizes the supremum in (6). Since the P_i is finite dimensional, we can choose a function $f_i \in P_i^\perp \setminus \{0\}$ that vanishes in $(a,0)$. Then

$$\lambda_i^\varepsilon = \inf_{f \in P_i^\perp \setminus \{0\}} \frac{a_0(f,f) + \varepsilon^4 a_1(f,f)}{\|f\|^2_{L_2(a,b)}} \le \frac{a_0(f_i,f_i) + \varepsilon^4 a_1(f_i,f_i)}{\|f_i\|^2_{L_2(a,b)}} = \varepsilon^4 \frac{a_1(f_i,f_i)}{\|f_i\|^2_{L_2(0,b)}} = C_i \varepsilon^4,$$

becuase $a_1(f_i,f_i) \ne 0$. $\square$

We postulate expansions of the eigenvalue $\lambda_\varepsilon = \lambda_i^\varepsilon$ and the eigenfunction $u_\varepsilon = u_i^\varepsilon$ for given $i \in \mathbb{N}$:

$$\begin{aligned} &\lambda_\varepsilon \sim \varepsilon^4(\lambda_0 + \varepsilon^4 \lambda_1 + \dots), \\ &u_\varepsilon(x) \sim \begin{cases} \varepsilon^4(u_0(x) + \varepsilon^4 u_1(x) + \dots), & x \in (a,0), \\ v_0(x) + \varepsilon^4 v_1(x) + \dots, & x \in (0,b). \end{cases} \end{aligned} \tag{7}$$

Substituting (7) into (1)-(4) we deduce that λ_0 is an eigenvalue and v_0 is an eigenfunction of the problem

$$\begin{gathered}(k_1 v_0'')''(x) - \lambda_0 v_0(x) = 0, \quad x \in (0,b),\\ v_0(0) = v_0'(0) = 0, \quad v_0(b) = v_0'(b) = 0.\end{gathered} \tag{8}$$

Note that all eigenvalues of the problem are simple.

The function u_0 is a solution of the boundary value problem

$$\begin{gathered}(k_0 u_0'')''(x) = 0, \quad x \in (a,0),\\ u_0(a) = u_0'(a) = 0, \quad k_0(0)u_0''(0) = k_1(0)v_0''(0), \quad (k_0u_0'')'(0) = (k_1 v_0'')'(0).\end{gathered}$$

Let us define next terms of expansions (7). The function v_1 satisfies

$$\begin{gathered}(k_1 v_1'')''(x) - \lambda_0 v_1(x) = \lambda_1 v_0(x), \quad x \in (0,b),\\ v_1(0) = u_0(0), \quad v_1'(0) = u_0(0), \quad v_1(b) = v_1'(b) = 0.\end{gathered}$$

Since λ_0 is an eigenvalue of (8), such problem has a solution under the condition

$$\lambda_1 = \|v_0\|^{-1}_{L_2(0,b)}((k_1 v_0'')'u_0 - k_1 v_0'' u_0')|_{x=0}.$$

Further, the u_1 is a solution of

$$\begin{gathered}(k_0 u_1'')''(x) = \lambda_0 u_0(x), \quad x \in (a,0),\\ u_1(a) = u_1'(a) = 0, \quad k_0(0)u_1''(0) = k_1(0)v_1''(0), \quad (k_0u_1'')'(0) = (k_1 v_1'')'(0).\end{gathered}$$

We can find general terms of (7) in the same way. The asymptotics are justified by classical methods [14, 2].

The vibrations described above are corresponded to the low levels of potential energy, since $a_\varepsilon(u_\varepsilon, u_\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$. Naturally we have vibrations of the soft part of a system only. The soft part is rigid at $x = 0$ by the immovable stiffer structure. Note that the leading terms of expansions (7) are determined by the soft part. Since every eigenfunction u_i^ε converges to an eigenfunction v_0 of (8) extended by zero to (a,b), set $\{u_i^\varepsilon\}_{i=1}^\infty$ is not a basis in $L_2(a,b)$ for $\varepsilon = 0$. In other words, the "low frequency" region does not provide a good insight on the vibration problem all over (a,b). Therefore we consider here another kind of vibrations, namely, vibrations with "finite energy".

3. High frequency vibrations. On the one hand, the asymptotics of λ_i^ε and u_i^ε are nonuniform with respect to i, on the other hand, the spectrum of (1)–(4) is asymptotically dense in the positive spectral semiaxis (see Lemma 2). On account of the above remarks, we can find stable vibrations $u^\varepsilon_{i(\varepsilon)}$ associated with sequence of eigenvalue $\lambda^\varepsilon_{i(\varepsilon)}$, $i(\varepsilon) \to \infty$.

We will denote by E the set of all pairs $(\varepsilon, \lambda(\varepsilon))$, where $\lambda(\varepsilon)$ is an eigenvalue of (1)–(4) for some $\varepsilon \in (0,1)$.

Lemma 2. *The closure of E includes semiaxis $\{(\varepsilon,\lambda) : \varepsilon = 0, \lambda \geq 0\}$. That is to say each positive λ can be approximated by sequence of eigenvalue $\lambda(\varepsilon)$.*

Proof. Suppose there exists a neighborhood U of a point $(0,\lambda)$ for some $\lambda > 0$ such that $U \cap E = \varnothing$. Choose $\varepsilon^* > 0$ such that $(\varepsilon^*, \lambda) \in U$. Recall that $\lambda_i(\varepsilon)$ is a continuous function with respect to ε and $\lambda_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ (see pic.1). Since the U doesn't contain the points $(\varepsilon^*, \lambda_i(\varepsilon^*))$, we see that $\lambda_i(\varepsilon^*) \leq \lambda$ for all $i \in \mathbb{N}$, which is impossible. □

Let us consider a sequence of pairs $(\varepsilon_i, \lambda_i) \in E$ that converges in $\mathbb{R}^2$ to $(0, \omega^4)$ with $\omega > 0$ as $i \to \infty$. Let u_i be a normalized in $L_2(a,b)$ eigenfunction of (1)-(4) associated with λ_i.

Definition 1. *We say that a sequence $\{u_i\}_{i=1}^\infty$ supports high frequency vibrations as $i \to \infty$, if the sequence of restrictions $u_i|_{(a,0)}$ has a nonzero limit v in $H^2(a,0)$. We call ω the limit frequency and v the limit form of such vibrations.*

Remark 1. It will be shown that functions u_i have strongly oscillatory character in $(0,b)$. Moreover, the vibrations u_i have "energy" closed to ω^4 as $i \to \infty$.

Theorem 1. *If a sequence $\{u_i\}_{i=1}^\infty$ supports the high frequency vibrations with the limit frequency ω and the limit form v, then ω^4 is an eigenvalue and v is an eigenfunction of the problem*

$$\begin{gathered}(k_0(x)v'')'' - \omega^4 v = 0, \quad x \in (a,0), \\ v(a) = v'(a) = 0, \quad v''(0) = v'''(0) = 0.\end{gathered} \tag{9}$$

and the restrictions $u_i|_{(0,b)}$ converge to zero in weak topology of $L_2(0,b)$.

Proof. From (5) we have $\|u_i\|_\varepsilon = \lambda_i \|u_i\|_{L_2(a,b)} = \lambda_i$. Then

$$\varepsilon_i^2 a_1(u_i, \varphi) \le \varepsilon_i^2 a_1^{1/2}(u_i,u_i) a_1^{1/2}(\varphi,\varphi) \le \|u_i\|_{\varepsilon_i} a_1^{1/2}(\varphi,\varphi) \le \lambda_i a_1^{1/2}(\varphi,\varphi) \tag{10}$$

for any $\varphi \in H_0^2(a,b)$. Let us consider identity (5) for test functions $\varphi \in C_0^\infty(0,b)$

$$(u_i, \varphi)_{L_2(0,b)} = \varepsilon_i^4 \lambda_i^{-1} a_1(u_i, \varphi).$$

Using (10) we get

$$|(u_i, \varphi)_{L_2(0,b)}| \le \varepsilon_i^2 a_1^{1/2}(\varphi, \varphi).$$

It follows immediately that $u_i \to 0$ in $L_2(0,b)$ weakly.

Throughout the proof, $\mathcal{H}_a$ denotes the space $\{v \in H^2(a,0) : v(a) = v'(a) = 0\}$. Passing to the limit as $e_i \to 0$ in (5) we obtain

$$a_0(v, \varphi) - \lambda(v, \varphi)_{L_2(a,0)} = 0, \quad \varphi \in H_0^2(a,b). \tag{11}$$

Since the restriction $\varphi|_{(0,a)}$ is an element of $\mathcal{H}_a$, identity (11) corresponds to the eigenvalue problem (9). This finishes the proof, because v is a nonzero function. □

Whence, the limit frequencies ω are generated by means of stiffer part of a vibrating system.

4. Asymptotic expansions of high frequency vibrations. The leading terms. By ω_ε we denote an eigenfrequency $\lambda_\varepsilon^{1/4}$ of vibrating system (1)–(4). We search for ω_ε with the asymptotic expansion:

$$\omega_\varepsilon \sim \sum_{k=0}^\infty \varepsilon^k \omega_k, \qquad \omega_0 \ne 0, \tag{12}$$

Consider the one-parameter family of vector-functions

$$N(\varkappa, x) = \left(\cos \varkappa S(x), \sin \varkappa S(x), e^{-\varkappa S(x)}, e^{\varkappa(S(x)-S(b))}\right), \qquad x \in (0,b), \quad \varkappa \in \mathbb{R},$$

where

$$S(x) = \omega_0 \int_0^x k_1^{-1/4}(t)\,dt.$$

Let $\langle\cdot,\cdot\rangle$ be the inner product in $\mathbb{R}^4$. We postulate expansions of u_ε in the form:

$$u_\varepsilon(x) \sim \begin{cases} \sum\limits_{k=0}^{\infty} \varepsilon^k v_k(x), & x \in (a,0), \\ \sum\limits_{k=0}^{\infty} \varepsilon^k < f_k(x), N\left(\varepsilon^{-1}, x\right) >, & x \in (0,b), \end{cases} \tag{13}$$

where $f_k : (0,b) \to \mathbb{R}^4$.

Remark 2. In $(0,b)$ we have the equation

$$\varepsilon^4 \frac{d^2}{dx^2}\left(k_1 \frac{d^2 u_\varepsilon}{dx^2}\right) - \omega_\varepsilon^4 u_\varepsilon = 0$$

with a small parameter. The construction of a expansion (13) in $(0,b)$ was motivated by [13]. We can find a solution u_ε in the form $e^{-\frac{S(x)}{\varepsilon}} \sum_{k=0}^{\infty} \varepsilon^k a_k(x)$, where the function $S(x)$ has to satisfy the eikonal equation $k_1(x) S'(x)^4 - \omega_0^4 = 0$. It is the method of WKB-approximations or short-wave approximations. Since there exist 4 different solutions of the eikonal equation with respect to S', we have introduced the function $N(\varkappa,\cdot)$.

Substituting (13) for u_ε in conditions (4), we obtain

$$k_0(0)\, v_0''(0) + \cdots \sim \varepsilon^2\, k_1(0)\, S'(0)^2 \langle f_0(0), T^2 N\left(\varepsilon^{-1},0\right)\rangle + \ldots, \tag{14}$$

$$(k_0 v_0'')'(0) + \cdots \sim \varepsilon\, k_1(0)\, S'(0)^3 \langle f_0(0), T^3 N\left(\varepsilon^{-1},0\right)\rangle + \ldots . \tag{15}$$

Above we take into account the equality

$$\frac{d}{dx} N\left(\varepsilon^{-1}, x\right) = \varepsilon^{-1} S'(x)\, T N\left(\varepsilon^{-1}, x\right)$$

with orthonormal matrix

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{where} \quad T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By (14) and (15), we get $v_0''(0) = 0$ and $v_0'''(0) = 0$. Substituting expansions (12)–(13) for ω_ε and u_ε in equation (1) and boundary value conditions (2), we get the eigenvalue problem

$$\begin{gathered} \frac{d^2}{dx^2}\left(k_0(x) \frac{d^2 v_0}{dx^2}\right) - \omega_0^4\, v_0(x) = 0, \quad x \in (a,0), \\ v_0(a) = v_0'(a) = 0, \quad v_0''(0) = v'''(0) = 0. \end{gathered} \tag{16}$$

Let ω_0 and v_0 be an eigenfrequency and an eigenfunction of the problem (compare with Th.1). Note that every eigenvalue ω_0 is simple. Suppose the function v_0 satisfies the condition $\|v_0\|_{L_2(a,0)} = 1$.

To find the leading term f_0 of expansion (13), we substitute the second series (13) in equation (1). In particular, we obtain

$$(k_1 S'^4 - \omega_0^4) f_0 = 0, \tag{17}$$

$$4k_1 S'^3 T f_0' + (2k_1' S'^3 T + 6k_1 S'^2 S'' T - 4\omega_0^3 \omega_1 I) f_0 = -(k_1 S'^4 - \omega_0^4) f_1. \tag{18}$$

Since the function S satisfies eikonal equation $k_1 S'^4 - \omega_0^4 = 0$, (17) holds. Then (18) is a homogeneous system of linear differential equations with respect to f_0.

Taking into account conditions (3) and boundary value conditions (2) at $x = b$, we can write

$$\begin{aligned} f_0' &= A(x) f_0, \quad x \in (0,b), \\ \langle f_0(0),\, N(\varepsilon^{-1},0)\rangle = v_0(0), &\quad \langle f_0(0),\, TN(\varepsilon^{-1},0)\rangle = 0, \\ \langle f_0(b),\, N(\varepsilon^{-1},b)\rangle = 0, &\quad \langle f_0(b),\, TN(\varepsilon^{-1},b)\rangle = 0. \end{aligned} \tag{19}$$

with matrix

$$A = \begin{pmatrix} -\frac{k_1'}{8k_1} & \frac{\omega_1}{\sqrt[4]{k_1}} & 0 & 0 \\ -\frac{\omega_1}{\sqrt[4]{k_1}} & -\frac{k_1'}{8k_1} & 0 & 0 \\ 0 & 0 & -\frac{k_1'}{8k_1} - \frac{\omega_1}{\sqrt[4]{k_1}} & 0 \\ 0 & 0 & 0 & -\frac{k_1'}{8k_1} + \frac{\omega_1}{\sqrt[4]{k_1}} \end{pmatrix}.$$

Note that the A depends on a parameter ω_1. We shall define ω_1 below.

On the one hand, problem (19) depends on ε by means of the boundary value conditions, on the other hand, one is an ill-posed problem. To resolve both these problems, we consider a discrete set of small parameter ε. We shall choose below the sequence $\varepsilon_p \to 0$ of a small parameter and asymptotic expansions will have a discrete character with respect to ε. That agrees with a discrete phenomenon of high frequency vibrations.

Lemma 3. *Let $w : (0,b) \to \mathbb{R}^4$ be a smooth vector-function and σ be a vector in $\mathbb{R}^4$. There exists a infinitely small sequence $\{\varepsilon_p\}_{p=1}^{\infty}$ such that the problem*

$$y'(\varepsilon,x) = A(x) y(\varepsilon,x) + w(x), \quad x \in (0,b), \tag{20}$$

$$\langle y(\varepsilon,0),\, N(\varepsilon^{-1},0)\rangle = \sigma_1, \quad \langle y(\varepsilon,0),\, TN(\varepsilon^{-1},0)\rangle = \sigma_2, \tag{21}$$

$$\langle y(\varepsilon,b),\, N(\varepsilon^{-1},b)\rangle = \sigma_3, \quad \langle y(\varepsilon,b), TN(\varepsilon^{-1},b)\rangle = \sigma_4 \tag{22}$$

has a unique solution $y(\varepsilon_p,\cdot)$ for $p \in \mathbb{N}$. The family of solutions $\{y(\varepsilon_p,\cdot)\}_{p=1}^{\infty}$ holds the inequality

$$\|y(\varepsilon_p,\cdot) - y_*\|_{C^1} \le C e^{-\frac{M}{\varepsilon_p}}$$

for a smooth function $y_* : [0,b] \to \mathbb{R}^4$. Here C and M are constants independent of ε.

Proof. The fundamental matrix of (20) has the form

$$\Phi(x) = k_1^{-1/8}(x) \begin{pmatrix} \cos\frac{\omega_1}{\omega_0} S(x) & \sin\frac{\omega_1}{\omega_0} S(x) & 0 & 0 \\ -\sin\frac{\omega_1}{\omega_0} S(x) & \cos\frac{\omega_1}{\omega_0} S(x) & 0 & 0 \\ 0 & 0 & e^{-\frac{\omega_1}{\omega_0} S(x)} & 0 \\ 0 & 0 & 0 & e^{\frac{\omega_1}{\omega_0}(S(x)-S(b))} \end{pmatrix}.$$

Therefore we have a representation of the general solution

$$y(x) = \Phi(x)(\beta + h(x)),$$

where β is a constant vector and $h(x) = \int_0^x \Phi^{-1}(t) w(t)\, dt$. Suppose the vector-function $y(\varepsilon,x) = \Phi(x)(\beta_\varepsilon + h(x))$ is a solution of (20)–(22). Set $\gamma_\varepsilon = \varepsilon^{-1} + \omega_1\omega_0^{-1}$. It is easy to check that

$$\Phi^t(x) N(\varepsilon^{-1},x) = k_1^{-1/8}(x) N(\gamma_\varepsilon, x), \tag{23}$$

where Φ^t is the transposed matrix. Then

$$\langle y(\varepsilon,x), N(\varepsilon^{-1},x)\rangle = k_1^{-1/8}(x)\langle \beta_\varepsilon + h(x), N(\gamma_\varepsilon,x)\rangle$$

and we can write (21), (22) in the form

$$\langle \beta_\varepsilon, \quad N(\gamma_\varepsilon,0)\rangle = m_0\sigma_1 \tag{24}$$
$$\langle \beta_\varepsilon, TN(\gamma_\varepsilon,0)\rangle = m_0\sigma_2 \tag{25}$$
$$\langle \beta_\varepsilon, \quad N(\gamma_\varepsilon,b)\rangle = m_1\sigma_3 - \langle h(b),\, N(\gamma_\varepsilon,b))\rangle \tag{26}$$
$$\langle \beta_\varepsilon, TN(\gamma_\varepsilon,b)\rangle = m_1\sigma_4 - \langle h(b),\, TN(\gamma_\varepsilon,b))\rangle, \tag{27}$$

where $m_0 = k_1^{1/8}(0)$, $m_1 = k_1^{1/8}(b)$. Note that the matrix Φ^t commutes with T and $h(0) = 0$.

Whence, the vector β_ε is a solution of the linear system with the matrix

$$G(\gamma_\varepsilon) = \begin{pmatrix} 1 & 0 & 1 & e^{-\gamma_\varepsilon S(b)} \\ 0 & 1 & -1 & e^{-\gamma_\varepsilon S(b)} \\ \cos\gamma_\varepsilon S(b) & \sin\gamma_\varepsilon S(b) & e^{-\gamma_\varepsilon S(b)} & 1 \\ -\sin\gamma_\varepsilon S(b) & \cos\gamma_\varepsilon S(b) & -e^{-\gamma_\varepsilon S(b)} & 1 \end{pmatrix}.$$

Since $S(b) \neq 0$, the determinant

$$\det G(\gamma_\varepsilon) = -2\cos\gamma_\varepsilon S(b) + 2e^{-\gamma_\varepsilon S(b)}\big(2 - e^{-\gamma_\varepsilon S(b)}\cos\gamma_\varepsilon S(b)\big)$$

doesn't vanish for all $\varepsilon > 0$.

Fix $\delta \in [0, 2\pi)$. Choose a sequence ε_p from set of conditions $\gamma_{\varepsilon_p} S(b) = \delta + 2\pi p$ for $p = 1, 2, \dots$, namely, let

$$\varepsilon_p = \frac{\omega_0 S(b)}{\omega_0(\delta + 2\pi p) - \omega_1 S(b)} \tag{28}$$

for all $p \geq p_0$. Here p_0 is the smallest natural number such that the denominator of (25) is positive. Without restriction of generality we set $p_0 = 1$. Let $\gamma_p = \gamma_{\varepsilon_p}$. Since $\gamma_p \to \infty$ as $\varepsilon_p \to 0$, the matrix $G(\gamma_p)$ is an exponentially small perturbation of the matrix

$$G_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \cos\delta & \sin\delta & 0 & 1 \\ -\sin\delta & \cos\delta & 0 & 1 \end{pmatrix}.$$

Since $N(\gamma_p, b) = \big(\cos\delta, \sin\delta, e^{-\gamma_p S(b)}, 1\big)$, the right-hand side of (24) is an exponentially small perturbation of the vector

$$g = (m_0\sigma_1,\, m_0\sigma_2,\, m_1\sigma_3 - \langle h(b),\, N_\delta\rangle,\, m_1\sigma_4 - \langle h(b),\, TN_\delta\rangle),$$

where N_δ differs from the vector $N(\gamma_p, b)$ by the third component only.

Suppose δ isn't equal to $\pi/2$ and $3\pi/2$. Then the matrix G_0 is nondegenerated. From the theory of finite-dimensional perturbations we obtain

$$\|\beta_{\varepsilon_p} - \beta_*\|_{\mathbb{R}^4} \leq Ce^{-\gamma_p S(b)},$$

where β_* is a solution of $G_0\beta = g$. Let $y_*(x) = \Phi(x)(\beta_* + h(x))$, then

$$\|y(\varepsilon_p, \cdot) - y_*\|_{C^1} \leq \|\Phi\|_{C^1}\|\beta_{\varepsilon_p} - \beta_*\|_{\mathbb{R}^4},$$

where $\|\Phi\|_{C^1} = \max\limits_{x\in(0,b)}(\|\Phi(x)\| + \|\Phi'(x)\|)$. Note that $\gamma_\varepsilon \geq c_o\varepsilon^{-1}$ with a positive constant c_0, which proves the lemma. □

Remark. From now on, we shall say that y_* is a solution of (20)–(22) with neglect of exponentially small terms. However the choice of sequence ε_p is nonunique and depends on δ and ω_1. We shall define ω_1 in the next step. On the other hand, we shall keep the dependence on δ. It is a so-called deformation parameter. Approximation of the limit function v_0 by eigenfunctions $u^{\varepsilon}_{n(\varepsilon)}$ is ambiguously determined. Hence we cannot define δ uniquely. Consequently, for each δ we construct the expansions of u^{ε}, though they are asymptotically equivalent as $\varepsilon \to 0$.

Return to study of problem (19), that is a subcase of (20)–(22) with the right-hand side $w = 0$ and $\sigma = (v_0(0), 0, 0, 0)$. Since (19) is a homogeneous system, we obtain $f_0 = \Phi\beta_0$, where $\beta_0 = \frac{1}{2}k_1^{1/8}(0)v_0(0)\left(1 - tg\,\delta, 1 + tg\,\delta, 1 + tg\,\delta, -\frac{1}{\cos\delta}\right)$ is a solution of the corresponding system with the matrix G_0.

To calculate a first-order correction ω_1, we consider the problem for v_1

$$\begin{gathered}\frac{d^2}{dx^2}\left(k_0(x)\frac{d^2 v_1}{dx^2}\right) - \omega_0^4 v_1 = 4\omega_1\omega_0^3 v_0, \quad x \in (a, 0), \\ v_1(a) = v_1'(a) = 0, \quad v_1''(0) = 0, \quad v_1'''(0) = -\frac{k_1^{1/4}(0)}{k_0(0)}\omega_0^3 v_0(0)(1 + tg\,\delta).\end{gathered} \tag{29}$$

Since ω_0^4 is a simple eigenvalue of (16), there exists a solution of (27) under the condition $\omega_1 = -\frac{1}{4}k_1^{1/4}(0)v_0^2(0)(1 + tg\,\delta)$. We choose a solution v_1 such that $(v_1, v_0)_{L_2(a,0)} = 0$.

Now by (25), it follows that

$$\varepsilon_p(\delta) = \frac{4\omega_0 S(b)}{4\omega_0(\delta + 2\pi p) + k_1^{1/4}(0)v_0^2(0)S(b)(1 + tg\,\delta)}, \tag{30}$$

where $\delta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}$. Denote by $\mathcal{E}_\delta$ the sequence (28) for fixed δ. We shall construct the asymptotics for $\varepsilon \in \mathcal{E}_\delta$. Whence, we found ω_0, v_0, f_0, corrections ω_1, v_1 and sequence $\mathcal{E}_\delta$. Recall that f_0, ω_1 and v_1 depend on δ.

5. Complete asymptotics of high frequency vibrations. Let us find general terms f_k, ω_{k+1} and v_{k+1} of expansions (12), (13) for $k \geq 1$. In the same way as for vector f_0, we can obtain the boundary value problem for f_k. With neglect of exponentially small terms in boundary conditions we can write the problem in the form

$$\begin{gathered}f_k' = A(x)f_k + w_k, \quad x \in (0, b), \\ \langle f_k(0), N(\varepsilon^{-1}, 0)\rangle = v_k(0), \qquad \langle f_k(b), N(\varepsilon^{-1}, b)\rangle = 0, \\ \langle f_k(0), TN(\varepsilon^{-1}, 0)\rangle = \omega_0^{-1}(k_1^{1/4}(0)v_{k-1}'(0) - k_1^{1/8}(0)\langle\Phi^{-1}(0)f_{k-1}'(0), N_0\rangle), \\ \langle f_k(b), TN(\varepsilon^{-1}, b)\rangle = -\omega_0^{-1}k_1^{1/8}(b)\langle\Phi^{-1}(b)f_{k-1}'(b), N_\delta\rangle,\end{gathered} \tag{31}$$

where $N_0 = (1, 0, 1, 0)$ and the N_δ is defined in the proof of Lemma 3.

For each k the right-hand side w_k of (29) is a smooth function, namely,

$$\begin{aligned}w_k = \frac{1}{4k_1 S'^3}\Big(\sum_{m=0}^{k-1}\lambda_{k-m}T^3 f_m - S'T^2 P_k - \frac{d}{dx}(k_1'(S')^2 T f_{k-1} +& \\ +3k_1 S'S''T f_{k-1} + 3k_1(S')^2 T f_{k-1}' + T^3 P_{k-1})\Big),&\end{aligned}$$

where $P_l = k_1 S'T^3 f_{l-1}'' + \frac{d}{dx}(2k_1 S'T^3 f_{l-1}' + k_1 S''T^3 f_{l-1} + k_1 f_{l-2}'')$. The proof is by induction on k. According to Lemma 3 there exists a solution of (29) for $\varepsilon \in \mathcal{E}_\delta$.

Now we can find ω_{k+1} and v_{k+1}:

$$
\begin{gathered}
\frac{d^2}{dx^2}\left(k_0(x)\frac{d^2 v_{k+1}}{dx^2}\right) - \omega_0^4 v_{k+1} = \sum_{m=1}^{k+1}\lambda_m v_{k-m+1}, \quad x\in(a,0),\\
v_{k+1}(a)=0, \qquad v'_{k+1}(a)=0, \\
(k_0 v''_{k+1})(0) = \langle \Phi^{-1}(\omega_0^2 k_1^{3/8}T^2 f_{k-1} + k_1^{-1/8}Q_{k-2}), N_0\rangle|_{x=0},\\
(k_0 v''_{k+1})'(0) = \langle \Phi^{-1}(\omega_0^3 k_1^{1/8}T f_k + k_1^{-1/8}R_{k-1}), N_0\rangle|_{x=0}.
\end{gathered}
\tag{32}
$$

Here $Q_k = k_1\frac{d}{dx}(S'T^3 f_k + f'_{k-1}) + S'T^3 f'_k$, $R_k = S'T^3 Q_{k-1} + \frac{d}{dx}(\omega_0^2 k_1^{1/2} T^2 f_k + Q_{k-1})$ and $\lambda_m = \sum \omega_i\omega_j\omega_l\omega_s$, where $i+j+l+s=m$. Since ω_0^4 is an eigenvalue of (16), boundary value problem (30) hasn't a solution for an arbitrary right-hand side. We can write the existence condition in the form

$$
\begin{aligned}
\omega_{k+1} = \frac{1}{4\omega_0^3}\langle v_0\Phi^{-1}(\omega_0^2 k_1^{3/8}T^2 f_{k-1} + k_1^{-1/8}Q_{k-2}) - \\
- v'_0\Phi^{-1}(\omega_0^3 k_1^{1/8}T f_k + k_1^{-1/8}R_{k-1}), N_0\rangle|_{x=0}.
\end{aligned}
$$

Choose a solution v_{k+1} of (30) such that $(v_{k+1}, v_0)_{L_2(a,0)} = 0$.

Hence, the algorithm scheme of asymptotics (12), (13) is

$$
\omega_0 \to v_0 \to f_0 \to \omega_1 \to \mathcal{E}_\delta \to v_1 \to \dots \to f_k \to \omega_{k+1} \to v_{k+1} \to \dots.
$$

Note that all terms, except for ω_0 and v_0, depend on parameter δ.

6. Justification of asymptotics. The nonstandard object of investigations, namely, the sequences of eigenfunctions like $u^\varepsilon_{i(\varepsilon)}$, changes the classical scheme of justification. Note that in Sections 4 and 5 we constructed series (12), (13), however we didn't define the object that is approximated by ones.

Only using the formal series (12) we shall define a sequence $u^\varepsilon_{k(\varepsilon)}$ such that supports the high frequency vibrations with the limit frequency ω_0 and the limit form v_0.

Fix $n\in\mathbb{N}$. Let us introduce a sequence of real numbers $\{\lambda^{(n)}_\varepsilon\}_{\varepsilon\in\mathcal{E}_\delta}$ and sequence of functions $\{u^{(n)}_\varepsilon\}_{\varepsilon\in\mathcal{E}_\delta}$ in $H^2(a,b)$. Namely, for each $\varepsilon\in\mathcal{E}_\delta$ we set

$$
\begin{gathered}
\lambda^{(n)}_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \dots \varepsilon^n\lambda_n, \qquad \lambda_s = \sum_{i+j+k+l=s}\omega_i\omega_j\omega_k\omega_l,\\
u^{(n)}_\varepsilon(x) = \begin{cases} v_0(x) + \varepsilon v_1(x) + \dots + \varepsilon^n v_n(x), & x\in(a,0),\\ \sum_{i=0}^n \varepsilon^i\langle f_i(x), N(\varepsilon^{-1},x)\rangle, & x\in(0,b),\end{cases}
\end{gathered}
\tag{33}
$$

where ω_k, functions v_k, vectors f_k and set $\mathcal{E}_\delta$ are defined in Sections 4 and 5.

Lemma 4. *There exists a sequence $\{\lambda^\varepsilon\}_{\varepsilon\in\mathcal{E}_\delta}$ of eigenvalues of (1)–(4) such that*

$$
|\lambda^{(n)}_\varepsilon - \lambda^\varepsilon| \le C_n\varepsilon^{n+1}, \quad \varepsilon\in\mathcal{E}_\delta. \tag{34}
$$

Proof. Denote by $\mathcal{H}_\varepsilon$ the space $H^2_0(a,b)$ with the inner product $a_\varepsilon(\cdot,\cdot)$, Let us introduce the operator A_ε:

$$
a_\varepsilon(A_\varepsilon u, v) = (u,v)_{L_2(a,b)}, \qquad v\in\mathcal{H}_\varepsilon,
$$

that is self-adjoint and compact for all $\varepsilon > 0$. Then we can write problem (1)–(4) in the form

$$A_\varepsilon u_\varepsilon - (\lambda^\varepsilon)^{-1} u_\varepsilon = 0.$$

Substituting sequences (31),(32) in problem (1)-(4), we obtain

$$\begin{gathered}
\frac{d^2}{dx^2}\left(k_\varepsilon(x)\frac{d^2}{dx^2}u_\varepsilon^{(n)}\right) - \lambda_\varepsilon^{(n)} u_\varepsilon^{(n)} = F_n(\varepsilon, x), \quad x \in (a,b), \\
u_\varepsilon^{(n)}(a) = 0, \quad \frac{d}{dx}u_\varepsilon^{(n)}(a) = 0, \quad u_\varepsilon^{(n)}(b) = g_n^{(1)}(\varepsilon), \quad \frac{d}{dx}u_\varepsilon^{(n)}(b) = g_n^{(2)}(\varepsilon), \\
[u_\varepsilon^{(n)}]_0 = g_n^{(3)}(\varepsilon), \quad \left[\frac{d}{dx}u_\varepsilon^{(n)}\right]_0 = h_n(\varepsilon), \\
\left[k_\varepsilon \frac{d^2}{dx^2}u_\varepsilon^{(n)}\right]_0 = z_n^{(1)}(\varepsilon), \quad \left[\frac{d}{dx}(k_\varepsilon \frac{d^2}{dx^2}u_\varepsilon^{(n)})\right]_0 = z_n^{(2)}(\varepsilon),
\end{gathered} \tag{35}$$

where $[f]_0$ is a jump of function f at $x = 0$. The right-hand sides of problem (34) satisfy the inequalities

$$\begin{gathered}
\|F_n(\varepsilon, x)\|_{C^0(a,b)} \le C_n \varepsilon^{n+1}, \quad |g_n^{(i)}(\varepsilon)| \le C_n e^{-\frac{M}{\varepsilon}}, \quad i = 1,2,3, \\
|h_n(\varepsilon)| \le C_n \varepsilon^n, \quad |z_n^{(i)}(\varepsilon)| \le C_n \varepsilon^{n+1}, \quad i = 1,2.
\end{gathered} \tag{36}$$

The function $u_\varepsilon^{(n)}$ doesn't belong to space $\mathcal{H}_\varepsilon$, because it has the point of discontinuity at $x = 0$. Taking into account (35) we can choose function $\varphi_\varepsilon^{(n)}$ such that $u_\varepsilon^{(n)} + \varphi_\varepsilon^{(n)} \in \mathcal{H}_\varepsilon$ and

$$\max_{x\in(a,b)}\left(\left|\varphi_\varepsilon^{(n)}\right| + \left|\frac{d}{dx}\varphi_\varepsilon^{(n)}\right| + \left|k_\varepsilon^{1/2}\frac{d^2}{dx^2}\varphi_\varepsilon^{(n)}\right|\right) \le C_n \varepsilon^n. \tag{37}$$

Let $V_\varepsilon^{(n)} = \varkappa_\varepsilon(u_\varepsilon^{(n)} + \varphi_\varepsilon^{(n)})$, where $\varkappa_\varepsilon$ is a normalizing constant such that $\|V_\varepsilon^{(n)}\|_{\mathcal{H}_\varepsilon} = 1$. It is easy to check that $\varkappa_\varepsilon \ge \varkappa_0 > 0$. It follows from (35), (36) that

$$\left\|\left(A_\varepsilon - (\lambda_\varepsilon^{(n)})^{-1} I\right) V_\varepsilon^{(n)}\right\|_\varepsilon \le K_n \varepsilon^n, \quad \varepsilon \in \mathcal{E}_\delta, \tag{38}$$

where K_n is a constant independent of ε. Hence, according to the Vishik-Lusternik lemma [14] there exists the eigenvalue $(\lambda^\varepsilon)^{-1}$ of operator A_ε such that

$$\left|\frac{1}{\lambda^\varepsilon} - \frac{1}{\lambda_\varepsilon^{(n)}}\right| \le K_n \varepsilon^n.$$

Applying this inequality for value $n+1$, we obtain

$$|\lambda^\varepsilon - \lambda_\varepsilon^{(n)}| \le C_n \varepsilon^{n+1}, \qquad \varepsilon \in \mathcal{E}_\delta.$$

□

Lemma 5. *There exists positive d such that for each $\varepsilon \in \mathcal{E}_\delta$ the interval $I_\varepsilon = (\lambda_\varepsilon^{(n)} - d\varepsilon, \lambda_\varepsilon^{(n)} + d\varepsilon)$ contains exactly one eigenvalue $\lambda_{k(\varepsilon)}^\varepsilon$ of problem (1)–(4). Moreover, the number $k(\varepsilon)$ of eigenvalue satisfies the inequality*

$$a_0 \varepsilon^{-1} \le k(\varepsilon) \le a_1 \varepsilon^{-1}, \tag{39}$$

where a_0, a_1 are constants independent of ε.

Proof. According to Section 2, we have

$$\lambda_m^\varepsilon = \varepsilon^4 \mu_m(1+\alpha_m(\varepsilon)), \quad \varepsilon \to 0,$$

where μ_m is an eigenvalue of (8) and $\alpha_m(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Moreover, the asymptotics hold

$$\mu_m = m^4(c_0 + \tau(m)), \quad \text{where} \quad \tau(m) = o(1), \quad m \to \infty.$$

Then

$$\lambda_m^\varepsilon = (\varepsilon m)^4(c_0 + \tau(m))(1+\alpha_m(\varepsilon)). \tag{40}$$

By Lemma 4, for each $\varepsilon \in \mathcal{E}_\delta$ there exists number $k = k(\varepsilon)$ such that

$$\lambda_0 - b_0\varepsilon \le \lambda_{k(\varepsilon)}^\varepsilon \le \lambda_0 + b_0\varepsilon$$

with $b_0 > 0$. We concluded from (39) that

$$\lambda_0 - b_0\varepsilon \le (\varepsilon k)^4(c_0+\tau(k))(1+\alpha_k(\varepsilon)) \le \lambda_0 + b_0\varepsilon,$$

and finally that (38) holds. According to (39) we have

$$\lambda_{k(\varepsilon)}^\varepsilon = c_0\varepsilon^4 k(\varepsilon)^4 + o(\varepsilon^4), \qquad \mathcal{E}_\delta \ni \varepsilon \to 0.$$

Then the distance between two neighboring eigenvalues close to the point λ_0 has asymptotics

$$|\lambda_{k(\varepsilon)+1}^\varepsilon - \lambda_{k(\varepsilon)}^\varepsilon| = 4c_0\varepsilon^4 k(\varepsilon)^3 + o(\varepsilon^4), \qquad \mathcal{E}_\delta \ni \varepsilon \to 0. \tag{41}$$

Taking into account behavior of $k(\varepsilon)$, we obtaing that there exists an interval I_ε of length $2d\varepsilon$ such that exactly one eigenvalue λ^ε of problem (1)–(4) belongs to one. The case that point $\lambda_\varepsilon^{(n)}$ is a midpoint of I_ε is impossible. This contradicts (33) and (40). □

Now we can introduce the object that is approximated by formal series (13) constructed in Sections 4 and 5. Let $\{\lambda_{k(\varepsilon)}^\varepsilon\}_{\varepsilon\in\mathcal{E}_\delta}$ be the sequence which is defined in Lemma 5, where $\lambda_{k(\varepsilon)}^\varepsilon$ is the nearest eigenvalue by value $\lambda_\varepsilon^{(n)}$. Let $\{u_{k(\varepsilon)}^\varepsilon\}_{\varepsilon\in\mathcal{E}_\delta}$ be the sequence of corresponding eigenfunctions, $\|u_{k(\varepsilon)}^\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$.

Theorem 2. *The sequence $\{u_{k(\varepsilon)}^\varepsilon\}_{\varepsilon\in\mathcal{E}_\delta}$ supports as $\mathcal{E}_\delta \ni \varepsilon \to 0$ the high frequency vibrations with limit frequency ω_0 and limit form v_0. Moreover,*

$$\|u_{k(\varepsilon)}^\varepsilon - V_\varepsilon^{(n)}\|_\varepsilon \le C_n\varepsilon^{n+1}, \qquad \varepsilon \in \mathcal{E}_\delta \tag{42}$$

for $n = 0, 1, \dots$. In particular, the inequalities hold

$$\begin{gathered}\Big\|u_{k(\varepsilon)}^\varepsilon(x) - \varkappa_\varepsilon \sum\nolimits_{k=0}^n v_k(x)\varepsilon^k\Big\|_{C^1(a,0)} \le C_n\varepsilon^{n+1},\\ \Big\|u_{k(\varepsilon)}^\varepsilon(x) - \varkappa_\varepsilon \sum\nolimits_{k=0}^n \langle f_k(x), N(\varepsilon^{-1},x)\rangle\varepsilon^k\Big\|_{C^1(0,b)} \le C_n\varepsilon^{n-1},\end{gathered} \tag{43}$$

where $\varkappa_\varepsilon = \|V_\varepsilon^{(n)}\|_\varepsilon^{-1}$ is a normalizing constant and $\varkappa_\varepsilon \ge \varkappa_0 > 0$.

Proof. By Lemma 7, we can choose $d_0 > 0$ such that $d_0\varepsilon$-neighborhood of the point $(\lambda_\varepsilon^{(n)})^{-1}$ contains exactly one eigenvalue $(\lambda_{k(\varepsilon)}^\varepsilon)^{-1}$ of operator A_ε. From inequality (37), we obtain (see, [14])

$$\|u_\varepsilon - V_\varepsilon^{(n)}\|_\varepsilon \le K_n d_0^{-1} \varepsilon^{n-1}, \tag{44}$$

where u_ε is a normalized eigenfunction of A_ε associated with eigenvalue $(\lambda_{k(\varepsilon)}^\varepsilon)^{-1}$. Hence, $u_\varepsilon = \pm u_{k(\varepsilon)}^\varepsilon$. There is no loss of generality in assuming that (43) holds for the function $u_{k(\varepsilon)}^\varepsilon$. Taking into account that terms of series (13) are bounded in $\mathcal{H}_\varepsilon$-norm from (43) for number $n+2$, we obtain (41).

In addition, by (36) and the definition of norm, we get

$$\|u_{k(\varepsilon)}^\varepsilon - \varkappa_\varepsilon u_\varepsilon^{(n)}\|_{H^2(a,0)} \le C_n \varepsilon^{n+1}, \qquad \|u_{k(\varepsilon)}^\varepsilon - \varkappa_\varepsilon u_\varepsilon^{(n)}\|_{H^2(0,b)} \le C_n \varepsilon^{n-1}.$$

Estimate (42) follows from the Sobolev embedding theorem.

It is easy to check that the constant $\varkappa_\varepsilon$ has a nonzero limit as $\varepsilon \to 0$. Then sequence $\{u_{k(\varepsilon)}^\varepsilon\}_{\varepsilon \in \mathcal{E}_\delta}$ converges to the eigenfunction $\varkappa v_0$ of problem (8) in $(a,0)$, namely, this sequence supports the high frequency vibrations. □

REFERENCES

1. Lions J. L. *Perturbations singulières dans les problèmes aus limites et en contrôle optima,* Lect. Notes in Math, Vol.323, Springer, 1973.

2. Панасенко Г. П. *Асимптотика решений и собственных значений эллиптических уравнений с сильно изменяющимися коэффициентами,* ДАН СССР, **252** (1980), 1320–1324.

3. Gibert P. *Less basses et les moyennes fréquencies dans des structures fortement hétérogenes,* C. R. Acad. Sci. Paris, Ser. II, **295** (1982), 951–954.

4. Geymonat G., Sanchez-Palencia E. *Spectral properties of certain stiff problems in elasticity and acoustics,* Math. Meth. Appl. Sci. **4** (1982), 291–306.

5. Geymonat G., Lobo-Hidalgo M., Sanchez-Palencia E. *Spectral properties of certain stiff problems in elasticity and acoustics, Part II,* Proceedings of the Centre for the Mathematical Analysis, Australian National Univ. **5** (1984), 15–38.

6. Санчез-Паленсия Е. *Неоднородные среды и теория колебаний,* М.: Мир, 1984.

7. Lobo-Hidalgo M., Sanchez-Palencia E. *Low and high frequency vibration in stiff problems,* in De Giorgi 60th Birthday, Partial Differential Equations and the Calculus of Variations, Birkhaüser, 1990, Vol.2. P.729–742.

8. Sanchez-Hubert J., Sanchez-Palencia E. *Vibration and coupling of continuous systems. Asymptotic methods,* Springer-Verlag, 1989, 421 pp.

9. Sanchez-Palencia E. *Asymptotic and spectral properties of a class of singular–stiff problems,* J. Math. Pures, Appl. **71** (1992), 379–406.

10. Lobo M., Perez E. *High frequency vibrations in a stiff problem,* Math. Methods. Appl. Sci. **7** (1997), no. 2, 291–311.

11. Маслов В. П. *Теория возмущений и асимптотические методы,* М: Изд-во МГУ, 1965, 554 с.

12. Федорюк М. В. *Уравнения с быстро осциллирующими решениями,* "Современные проблемы математики. Фундаментальные направления. Т. 34 (Итоги науки и техн. ВИНИТИ АН СССР)". М., 1988. С. 5–56.

13. Федорюк М. В. *Асимптотические методы для линейных обыкновенных дифференциальных уравнений,* М.: Наука, 1983, 352 с.

14. Вишик М. И., Люстерник А. А. *Регулярное вырождение и пограничный слой для линейных дифференциальных уравнений с малым параметром,* УМН **12** (1957), №5, 3–122.

Lviv National University, Faculty of Mechanics and Mathematics,
Lviv, 79000, Ukraine

Institute of Mathematics, Cracow University of Technology,
Warszawska 24, 31-155 Cracow, Poland

Received 1.07.1999