УДК 517.5

Z. M. Sheremeta

ON ENTIRE SOLUTIONS OF A DIFFERENTIAL EQUATION

Dedicated to the 70th anniversary of Prof. A. A. Gol'dberg

Z. M. Sheremeta. *On entire solutions of a differential equation*, Matematychni Studii, **14** (2000) 54–58.

We study conditions on constant coefficients of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0,$$

under which an entire solution f of this equation and all its derivatives $f', f'', \dots$ are close-to-convex in the unit disk.

З. М. Шеремета. *О целых решениях одного дифференциального уравнения* // Математичні Студії. – 2000. – Т.14, №1. – С.54–58.

Исследуются условия на постоянные коэффициенты дифференциального уравнения

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0,$$

при выполнении которых целое решение f этого уравнения и все его производные f', f'', ... являются близкими к выпуклым в единичном круге.

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said be convex if $f(\mathbb{D})$ is a convex domain. The function f is said be close-to-convex in $\mathbb{D}$ if there exists a convex in $\mathbb{D}$ function Φ such that $\operatorname{Re} \frac{f'(z)}{\Phi'(z)} > 0\,(z \in \mathbb{D})$. Every close-to-convex in $\mathbb{D}$ function is univalent in $\mathbb{D}$ and therefore $f_1 \neq 0$.

S. M. Shah [1] investigated conditions on constant coefficients of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{2}$$

under which an entire solution f of this equation and all its derivatives $f', f'', \dots$ are close-to-convex in $\mathbb{D}$.

It is easy to show that f is a solution of (2) if and only if

$$\gamma_2 f_0 = 0, \quad (\beta_1 + \gamma_2)f_1 + \gamma_1 f_0 = 0 \tag{3}$$

2000 *Mathematics Subject Classification:* 34M05.

and

$$(n(n+\beta_1-1)+\gamma_2)f_n+(\beta_0(n-1)+\gamma_1)f_{n-1}+\gamma_0 f_{n-2}=0 \quad (n\geq 2). \tag{4}$$

If $n(n+\beta_1-1)+\gamma_2\neq 0$ then the latter equalities can be rewritten in the form

$$f_n=-\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)+\gamma_2}f_{n-1}-\frac{\gamma_0}{n(n+\beta_1-1)+\gamma_2}f_{n-2} \quad (n\geq 2). \tag{5}$$

If $\gamma_0=0$ then (5) implies

$$f_n=-\frac{\beta_0(n-1)+\gamma_1}{n(n+\beta_1-1)+\gamma_2}f_{n-1} \quad (n\geq 2). \tag{6}$$

Using recurrent formula (6), S. M. Shah [1] proved that under some conditions on other coefficients of equation (2) there exists an entire solution f of (2) such that $f, f', f'', \ldots$ are close-to-convex in $\mathbb{D}$ and $\ln M_a(r)=(1+o(1))|\beta_0|r$ $(r\to\infty)$, where $M_f(r)=\max\{|f(z)|:|z|=r\}$.

More complicated case when $\gamma_0\neq 0$ is considered in [2], where the following theorem is proved.

Theorem 0. *Suppose that the coefficients of differential equation (2) satisfy one of the following conditions:*

1) $\beta_0=-1$, $\beta_1=\gamma_1=\gamma_2=0$, $-1\leq\gamma_0<0$;

2) $\beta_0=-1$, $\beta_1>0$, $-\frac{6+\beta_1}{6+3\beta_1}\leq\gamma_0<0$, $-\beta_1\leq\gamma_1\leq-\beta_1/2$, $\gamma_2=-\beta_1$;

3) $-1<\beta_0<0$, $-2(1+\beta_0)<\beta_1\leq 0$, $\beta_0\leq\gamma_0<0$, $-(1+\beta_1/2+\beta_0)<\gamma_1\leq 0$, $\gamma_2=-\beta_1$;

4) $-1<\beta_0<0$, $\beta_1>0$, $-\frac{6+\beta_1}{6+3\beta_1}\leq\gamma_0<0$, $-(1+\beta_1/2+\beta_0)<\gamma_1\leq 0$, $\gamma_2=-\beta_1$.

Then there exists an entire solution

$$a(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{7}$$

of (2) such that $a, a', a'', \ldots$ *are close-to-convex in* $\mathbb{D}$ *and*

$$\ln M_a(r)=\frac{(1+o(1))}{2}(|\beta_0|+\sqrt{|\beta_0|^2+4|\gamma_0|})r, \quad r\to+\infty. \tag{8}$$

We remark that every condition 1)–4) implies $\gamma_2=-\beta_1$ and $\beta_1>-2$. Here we consider the case $\beta_1=-2$ and find a solution of (2) in the following form

$$a(z)=z+\frac{z^2}{2}+\sum_{n=3}^{\infty}a_n z^n. \tag{9}$$

Then from (4) with $n=2$ it follows that $\beta_0+\gamma_1=0$ and we can rewrite formula (5) in the following form

$$a_n=-\frac{\beta_0}{(n-1)}a_{n-1}-\frac{\gamma_0}{(n-1)(n-2)}a_{n-2} \quad (n\geq 3) \tag{10}$$

Theorem 1. *Suppose that* $\beta_1=-\gamma_2=-2$, $-\frac{2}{3}\leq\beta_0=-\gamma_1<0$ *and* $\frac{5}{6}\beta_0\leq\gamma_0\leq 0$. *Then there exists an entire solution (9) of (2) such that* $a, a', a'', \ldots$ *are close-to-convex in* $\mathbb{D}$ *and (8) holds.*

For the proof of Theorem 1 we need two lemmas.

Lemma 1 [3, p. 9]. *If coefficients of an analytic in $\mathbb{D}$ function (7) satisfy the condition*

$$1 \geq 2a_2 \geq 3a_3 \geq \dots \geq na_n \geq (n+1)a_{n+1} \geq \dots > 0, \tag{11}$$

then a is close-to-convex in $\mathbb{D}$.

Lemma 2 [2]. *If $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = \xi_n a_n + \eta_n a_{n-1}\,(n \geq 1)$, where ξ_n, $\eta_n > 0\,(n \geq 1)$ and*

$$1 \geq 2\xi_1 \geq \frac{3}{2}\xi_2 + 3\eta_2 \geq \dots \geq \frac{n+1}{n}\xi_n + \frac{n+1}{n-1}\eta_n \geq \dots > 0, \tag{12}$$

then (11) holds.

First, we prove the close-to-convexity of function (9). We put $\xi_1 = 1/2$, $\eta_1 = 1$ and

$$\xi_n = \frac{|\beta_0|}{n}, \quad \eta_n = \frac{|\gamma_0|}{n(n-1)}, \quad (n \geq 2). \tag{13}$$

From (10) we have $a_{n+1} = \xi_n a_n + \eta_n a_{n-1}\,(n \geq 1)$. Since $1 \geq 2\xi_1$, $\frac{3}{2}\xi_2 + 3\eta_2 = = \frac{3}{4}(|\beta_0| + 2|\gamma_0|) \leq 1 = 2\xi_1$ and the sequence

$$\frac{n+1}{n}\xi_n + \frac{n+1}{n-1}\eta_n = \frac{n+1}{n^2}|\beta_0| + \frac{n+1}{n(n-1)^2}|\gamma_0| \quad n \geq 2,$$

is decreasing, all inequalities (12) hold. By Lemmas 1 and 2 function (9) is close-to-convex.

Now, let $k \geq 1$. Since

$$a^{(k)}(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n, \quad a_n^{(k)} = \frac{(n+k)!}{n!}a_{n+k} \tag{14}$$

and $a_{n+1} = \xi_n a_n + \eta_n a_{n-1}\,(n \geq 1)$, where ξ_n and η_n are defined by (13), i.e. $a_{n+k} = \frac{n!}{(n+k)!}a_n^{(k)}$ and $a_{n+1+k} = \xi_{n+k}a_{n+k} + \eta_{n+k}a_{n-1+k}\,(n \geq 0)$, we have

$$\begin{aligned} a_{n+1}^{(k)} &= \frac{(n+1+k)!}{(n+1)!}\left(\frac{n!}{(n+k)!}\xi_{n+k}a_n^{(k)} + \frac{(n-1)!}{(n-1+k)!}\eta_{n+k}a_{n-1}^{(k)}\right) = \\ &= \frac{n+1+k}{n+1}\xi_{n+k}a_n^{(k)} + \frac{(n+1+k)(n+k)}{n(n+1)}\eta_{n+k}a_{n-1}^{(k)}, \quad n \geq 1. \end{aligned} \tag{15}$$

But $a_n^{(k)} > 0\,(k \geq 1, n \geq 0)$. Therefore, the function $a^{(k)}$ is close-to-convex if and only if the function

$$\frac{a^{(k)}(z) - a_0^{(k)}}{a_1^{(k)}} = z + \sum_{n=2}^{\infty} a_{n,k} z^n,$$

is close-to-convex, where $a_{0,k} = 0,\ a_{1,k} = 1,\ a_{n,k} = a_n^{(k)}/a_1^{(k)}\ (n \geq 2)$.

From (13) and (15) we obtain $a_{n+1,k} = \xi_{n,k}a_{n,k} + \eta_{n,k}a_{n-1,k}$, where

$$\xi_{n,k} = \frac{n+1+k}{n+1}\xi_{n+k} = \frac{(n+1+k)|\beta_0|}{(n+1)(n+k)},$$

$$\eta_{n,k} = \frac{(n+1+k)(n+k)}{n(n+1)}\eta_{n+k} = \frac{(n+1+k)|\gamma_0|}{n(n+1)(n+k-1)}.$$

Thus, by Lemma 2 we need to prove the following inequalities

$$1 \geq 2\xi_{1,k} \geq \frac{3}{2}\xi_{2,k} + 3\eta_{2,k} \geq \cdots \geq \frac{n+1}{n}\xi_{n,k} + \frac{n+1}{n-1}\eta_{n,k} \geq \cdots > 0. \tag{16}$$

Since $|\beta_0| \leq \frac{2}{3}$, we have $|\beta_0| \leq \frac{k+1}{k+2}$ for all $k \geq 1$ and $2\xi_{1,k} = \frac{(k+2)|\beta_0|}{k+1} \leq 1, \quad k \geq 1$. Further, since $|\gamma_0| \leq 5|\beta_0|/6$,

$$\frac{3}{2}\xi_{2,k} + 3\eta_{2,k} = \frac{(k+3)|\beta_0|}{2} + \frac{(k+3)|\gamma_0|}{2(k+1)} \leq \frac{(k+3)|\beta_0|}{2(k+2)}\left(\frac{1}{k+2} + \frac{5}{6(k+1)}\right) \leq$$
$$\leq \frac{k+2}{k+1}|\beta_0| = 2\xi_{1,k}, \quad k \geq 1.$$

Finally, the sequence

$$\frac{n+1}{n}\xi_{n,k} + \frac{n+1}{n-1}\eta_{n,k} = \frac{n+k+1}{n}\left(\frac{|\beta_0|}{n+k} + \frac{|\gamma_0|}{(n-1)(n+k-1)}\right)$$

is decreasing for $n \geq 2$. Therefore, inequalities (16) hold and thus $a^{(k)}$ is close-to-convex.

Now we prove (8). We put $\alpha_n = \frac{a_{n-1}}{(n-1)a_n}$. Since

$$a_n = \frac{|\beta_0|}{n-1}a_{n-1} + \frac{|\gamma_0|}{(n-1)(n-2)}a_{n-2}, \quad n \geq 3,$$

we have

$$\frac{1}{\alpha_n} = |\beta_0| + |\gamma_0|\alpha_{n-1}, \quad n \geq 3. \tag{17}$$

We denote $\alpha_* = \varliminf_{n\to\infty} \alpha_n$, $\alpha^* = \varlimsup_{n\to\infty} \alpha_n$. Then from (17) we obtain $\frac{1}{\alpha_*} = |\beta_0| + |\gamma_0|\alpha^*$ and $\frac{1}{\alpha^*} = |\beta_0| + |\gamma_0|\alpha_*$. Hence

$$\frac{1}{\alpha_*} = |\beta_0| + \frac{|\gamma_0|}{|\beta_0| + |\gamma_0|\alpha_*}, \quad \frac{1}{\alpha^*} = |\beta_0| + \frac{|\gamma_0|}{|\beta_0| + |\gamma_0|\alpha^*},$$

i. e. α_* and α^* are solutions of the equation $|\gamma_0|x^2 + |\beta_0|x - 1 = 0$. The equation have the solutions of various signs and $\alpha_* \geq 0$, $\alpha^* \geq 0$. Therefore,

$$\alpha_* = \alpha^* = \frac{\sqrt{|\beta_0|^2 + 4|\gamma_0|} - |\beta_0|}{2|\gamma_0|} = \frac{2}{\sqrt{|\beta_0|^2 + 4|\gamma_0|} + |\beta_0|}.$$

We put $\sigma = \frac{1}{2}(\sqrt{|\beta_0|^2 + 4|\gamma_0|} + |\beta_0|)$. Then $\frac{(n-1)a_n}{a_{n-1}} \to \sigma$ $(n \to \infty)$, whence $a_n = \frac{1+\delta_n}{n}\sigma a_{n-1}$, where $\delta_n \to 0$ $(n \to \infty)$. Hence

$$a_n = \frac{\sigma^{n-1}}{n!}\prod_{j=2}^{n}(1+\delta_j) = \frac{\sigma^{n-1}}{n!}(1+\varepsilon_n)^n,$$

where

$$\varepsilon_n = \exp\left\{\frac{1}{n}\sum_{j=2}^{n}\ln(1+\delta_j)\right\} - 1 \to 0, \quad n \to \infty.$$

Since the coefficients of (8) are positive,

$$M_a(r) = a(r) = r + \frac{r^2}{2} + \sigma\sum_{n=3}^{\infty}\frac{\sigma^{n-1}}{n!}(1+\varepsilon_n)^n,$$

whence $\ln M_a(r) = (1+o(1))\sigma r$ $(r \to +\infty)$. The proof of Theorem 1 is complete.

REFERENCES

1. Shah S. M. *Univalence of a function f and its successive derivatives when f satisfies a differential equation, II,* J. Math. anal. and appl. **142** (1989), 422–430.

2. Шеремета З. М., *О свойствах целых решений одного дифференциального уравнения,* Дифф. урав. **36** (2000), №6, 921–929.

3. Goodman A. W. *Univalent function, Vol. II,* Mariner Publishing Co., 1983, 158 pp.

Institute of Applied Problems of Mechanics and Mathematics,
3b Naukova str., Lviv, Ukraine

Received 16.12.1999