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STABILITY PHENOMENON AND PROBLEMS FOR COMPLEX DIFFERENTIAL EQUATIONS WITH RELATIONS TO SHARED VALUES

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Three results and a collection of problems for complex algebraic differential equations and their systems related with so-called stability phenomenon are formulated. The latter, in particular, means that meromorphic (entire) solutions of algebraic differential equation $P(z, f, f', f, \dots, f^{(k)}) = 0$, where $P(\cdot)$ is a polynomial in all variables, retain properties which they have on a "small" subset of $\mathbb{C}$.

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Сформулированы три результата и серия задач для комплексных алгебраичесих дифференциальных уравнений и их систем, связанных с так называемым явлением устойчивости. Последнее, в частности, означает, что мероморфные (целые) решения алгебраического дифференциального уравнения $P(z, f, f', f \dots, f^{(k)}) = 0$, где $P(\cdot)$ — многочлен по всем переменным, сохраняют свойства, которыми они обладают на "малом"подмножестве $\mathbb{C}$.

Meromorphic solutions of algebraic differential equations of the form

$$P(z, f, f', f \dots, f^{(k)}) = 0, \tag{1}$$

where $P(z, u_0, u_1, \dots, u_k)$ is a polynomial in all variables, have been intensively investigated in the complex plane for about half a century. The basic idea of this article is to propose a collection of open problems into this field, aiming to offer a new point of view for research. More precisely, we prove a variant of the classical result due to A. A. Gol'dberg, see [6], Theorem 1, resulting in the following stability phenomenon: To obtain a property for meromorphic solutions f of (1), it is often enough to require that f satisfies (1) just in a "small" subset of $\mathbb{C}$. In fact, we believe that a number of known results concerning meromorphic solutions of complex differential equations may be revised by this kind of stability. Moreover, such renewed versions obviously reveal similarity with problems in shared values and uniqueness. We also shortly describe the method of estimating derivatives in complex differential equations, as this idea may be useful in such stability studies.

In what follows, let f be a meromorphic function in $\mathbb{C}$, $\rho(f)$ be its order and let $a_1, \dots, a_q$ stand for a finite set of disjoint complex values.

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Theorem 1. *Let f satisfy the equation*

$$P(z, f, f') = 0 \tag{2}$$

on all points $z \in \mathbb{C}$ such that $f(z) = a_j$, $j = 1, \ldots, 5$. Then $\rho(f)$ is finite.

Proof. See for a sketch at the end of this paper. □

Of course, Theorem 1 improves the classical theorem of A. A. Gol'dberg in [6] that all meromorphic solutions of (2) are of finite order. Therefore, Theorem 1 presents an example of a stability phenomenon described above. This gives rise to a general setting of problems. To this end, given an algebraic differential equation (1) and a property $Q[f]$ for a meromorphic function f, consider the following situations:

(A) $Q[f]$ holds for all meromorphic solutions f of (1);

(B) $Q[f]$ holds for all meromorphic functions f satisfying (1) in a "small" subset $Z \subset \mathbb{C}$ of the complex plane.

We now proceed to describe several classes of problems.

Problems of class 1: If (A) is true, find (the best possible) $Z \subset \mathbb{C}$ such that (B) remains true.

Obviously, this class admits numerous variations depending on f or equation (1). As an example, we may consider the situation in Theorem 1 assuming that f satisfies (2) on all points $z \in \mathbb{C}$ such that $f(z) = a_j$, $j = 1, \ldots, q$, where $q < 5$. One may ask about those additional, minimal conditions which ensure that $\rho(f)$ is finite. Another group of such problems can be formulated by starting from the entire version of Theorem 1:

Theorem 2. *Let an entire function f satisfy equation (2) on all points $z \in \mathbb{C}$ such that $f(z) = a_j$, $j = 1, 2, 3$. Then $\rho(f)$ is finite.*

Of course, similar problems arise by considering algebraic differential equations (1) instead of (2). As an example, one may try to argue similarly as to above, starting from

Theorem 3. *Let f satisfy the equation $P_1(z, f, f') = P_2(z, f, f', f'', \ldots, f^{(k)})$ on all points $z \in \mathbb{C}$ such that $f(z) = a_j$, $j = 1, \ldots, 5$. Assume that for some constant $K < \infty$,*

$$|P_2(z, f, f', f'', \ldots, f^{(k)})| \leq |z|^K$$

on the same set. Then $\rho(f)$ is finite.

Problems of class 2: Assuming f satisfies (1) on given $Z \subset \mathbb{C}$, does it follow that f is a solution of (1), and so (B) implies (A).

Again, there are a number of such problems, depending on classes of (1) and on classes of the "small" sets $Z \subset \mathbb{C}$. Obviously, several results, e.g. in [8], could be tested from this point of view. Actually, problems of class 2 present some kind of stability problems: To find a required set Z means to find conditions that result in the validity of (1) for all $z \in \mathbb{C}$. Therefore, if (B) implies (A) in a problem situation of class 2, we say that the pair (P, Z) is *stable*. For instance, for a given equation (1), we may ask for geometric conditions on Z to imply that (P, Z) is stable. Also, we may ask for classes of equations (1) such that (P, Z) is stable as Z is the real axis or, more generally, a given family of lines in $\mathbb{C}$. Further, obvious stability problems appear by taking as Z certain collections of $a_1, \ldots, a_q$-points of f, f' or f

and f' simultaneously. In particular, considering as Z the totality of $a_1, \dots, a_q$-points of f, it is pertinent to ask, for what number q we may conclude that (P, Z) is stable.

Problems of class 3: We now pass to problems related to meromorphic solutions of a pair of complex differential equations

$$\begin{cases} P_1(z, f, g, f', g') = 0 \\ P_2(z, f, g, f', g') = 0, \end{cases} \tag{3.1}$$

studied in very few papers only, see e.g. [9]. It is natural to ask for a counterpart to the classical Gol'dberg theorem [6], i.e. for situations such that f and g both would be of finite order. Unfortunately, the situation is more complicated as one can see by looking at the pair

$$\begin{cases} f' - fg = 0 \\ f' - fg' = 0 \end{cases}$$

solved by $f = \exp(\exp z)$, $g = \exp z$. Therefore, it is non-trivial to find such pairs of differential equations (3.1) that f, g would be of finite order. More generally, we may consider

$$\begin{cases} P_1(z, f, g, f^{(n)}, g^{(m)}) = 0 \\ P_2(z, f, g, f^{(n)}, g^{(m)}) = 0 \end{cases} \tag{3.2}$$

with $n, m \geq 2$. It is natural to ask, whether f and g should have the same order of growth. In fact, we conjecture that this conclusion holds even if (3.2) is replaced by just one equation

$$P(z, f, g, f^{(n)}, g^{(m)}) = 0, \quad n, m \geq 2.$$

Problems of class 4: To study stability for pairs of differential equations (3.1), (3.2) in the sense described above.

Problems of class 5: We now proceed to consider problems related to shared values. To this end, let $P(x, y)$ be a polynomial in two variables with no separable factors and let f, g be meromorphic functions that satisfy an algebraic equation

$$P(f, g) = 0 \tag{4}$$

on all points $z \in \mathbb{C}$ such that $f(z) = a_j, j = 1, \dots, q$. We now ask for q such that f, g satisfy (4) for all $z \in \mathbb{C}$. Of course, this generalizes problems in shared values as one can see by taking $P(f, g) := f - g$, see Mues [11] and Yang [15] for surveys of shared value problems.

Problems of class 6: The previous type of problems may be generalized by introducing derivatives into (4), as well as by looking pairs of equations. As an example, suppose that f, g are two meromorphic functions satisfying

$$P_1(f, g, f', g') = 0 \tag{5.1}$$

on a set Z_1 such that $f(z) \in \{a_1, \dots, a_q\}$, $g(z) \in \{a_1, \dots, a_q\}$

$$P_2(f, g, f', g') = 0 \tag{5.2}$$

on another set Z_2 such that $f(z) \in \{a_1, \dots, a_q\}, g(z) \in \{a_1, \dots, a_q\}$. The problem is now to describe classes of P_1, P_2, Z_1, Z_2 such that the above conditions imply that f, g are solutions of (5.1) and (5.2) for all $z \in \mathbb{C}$.

Problems of class 7: What can be said about the mutual behaviour of two distinct meromorphic functions f_1, f_2 satisfying (B)? To describe this, recall the following problem related to uniqueness, due to Yang, see [14], p. 169: Is it true that for two polynomials $P(z), Q(z)$ of the same degree and the same preimage for $\{0, 1\}$, either $P(z) = Q(z)$ or $P(z) = 1 - Q(z)$? This problem has attracted attention in complex analysis and algebraic geometry, see e.g. Dobbertin and Schmieder [5] and Moh [10], with some partial results obtained. Only quite recently the problem was completely solved by Pakovitch [13]. Furthermore, Pakovitch, Ostrovskii and Zaidenberg [12] generalized and resolved the problem for two nonconstant polynomials P and Q that share a compact set K, with cardinality of $K \geq 3$. Of course, extensions to two meromorphic functions (sharing a finite set of values or small functions) would be interesting. We refer the reader to the survey article by Hua and Yang [7] for more of the studies. An interesting open question in this regard is: What is the smallest cardinality of a set K such that whenever two nonconstans meromorphic functions f and g share the set K, then $f = g$?

Proof sketches for theorems

We now indicate the method of estimating derivatives, which seems to be useful in considering the above problems, see [1] for a more complete presentation. Moreover, we indicate below how the above theorems may be proved.

Let w be meromorphic in the complex plane. Then $|w'(z(a_\nu, w))| \geq \frac{r}{(A(r,w))^{1/2+\varepsilon}}$ for majority of $a_1, \ldots, a_q$-points $z(a_\nu, w)$ of w, $\nu = 1, \ldots, q$, in $|z| < r, r \notin E$, where E is an exceptional set of finite logarithmic measure and $A(r, w)$ stands for the Ahlfors-Shimizu characteristic.

This may now be applied to prove the classical Gol'dberg theorem, see [4] for a detailed proof. By inspection of that proof, and taking into account a result in [1], it appears that the conclusion holds as soon as equation (2) is satisfied on all points $z \in \mathbb{C}$ such that $f(z) = a_j$, $j = 1, \ldots, 5$. This proves Theorem 1. Similar arguments may be used to prove Theorem 2 and Theorem 3.

Similar estimates for higher derivatives may be found in [2], see also [3]. We feel that these results considering the problem classes described above. For applications to complex differential equations, see [4].

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