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## ON THE GROWTH OF ENTIRE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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The growth and the  $l$ -index boundedness of entire solutions of linear differential equations are investigated.

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Исследуются рост и ограниченность  $l$ -индекса целых решений линейных дифференциальных уравнений.

**1.** Let  $l$  be a positive continuous function on  $[0, +\infty)$ . An entire function  $f$  is said to be of bounded  $l$ -index [1] if there exists  $N \in \mathbb{Z}_+$  such that for all  $n \in \mathbb{Z}_+$  and  $z \in \mathbb{C}$

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \leq \max \left\{ \frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \leq k \leq N \right\}.$$

The least of such  $N$  is called the  $l$ -index of  $f$  and is denoted by  $N(l; f)$ .

Let  $K$  be the class of positive analytic on  $[0, +\infty)$  functions  $l$  such that  $l'(x) = o(l^2(x))$  ( $x \rightarrow +\infty$ ), and  $Q$  be the class of positive continuous on  $[0, +\infty)$  functions  $l$  such that  $l(x + O(1/l(x))) = O(l(x))$  ( $x \rightarrow +\infty$ ). It is known [1] that if  $l \in K$  and  $f$  is of bounded  $l$ -index then

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{L(r)} \leq N(l; f) + 1, \quad L(r) = \int_0^r l(x) dx, \quad (1)$$

where  $M(r, f) = \max\{|f(z)| : |z| = r\}$ .

For an entire function  $g$  with zeros  $a_k$  denote  $G_g(r) = \bigcup_k \{z : |z - a_k| \leq r/l(|a_k|)\}$ ,  $r > 0$ . It is proved [2] that if  $l \in Q$ ,  $g_0, g_1, \dots, g_n, h$  are entire functions of bounded  $l$ -index and for every  $r > 0$  there exists  $M = M(r) > 0$  such that  $|g_j(z)| \leq Ml^j(|z|)|g_0(z)|$  for all  $z \in \mathbb{C} \setminus G_{g_0}(r)$  then an entire solution  $f$  of the equation

$$g_0(z)w^{(n)} + g_1(z)w^{(n-1)} + \dots + g_n(z)w = h(z), \quad (2)$$

is of bounded  $l$ -index. Since there is no estimate from above of  $N(l; f)$  in this statement we cannot use inequality (1) to estimate the growth of  $f$ . But if we impose some additional

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conditions on the coefficients of equation (2) and on function  $l$  we can indicate the estimate of growth of  $f$  without finding a quantity of its  $l$ -index.

2. The following lemma plays an important role in our investigations.

**Lemma 1** [3]. *Let  $l \in Q$ . An entire function  $f$  is of bounded  $l$ -index if and only if there exist numbers  $p \in \mathbb{Z}_+$ ,  $C > 0$  and  $R > 0$  such that*

$$\frac{|f^{(p+1)}(z)|}{l^{p+1}(|z|)} \leq C \max \left\{ \frac{|f^{(k)}(z)|}{l^k(|z|)} : 0 \leq k \leq p \right\} \quad (3)$$

for all  $z \in \mathbb{C}$ ,  $|z| \geq R$ .

We need also the following two lemmas.

**Lemma 2.** *Let  $l \in K$ . If there exist numbers  $p \in \mathbb{Z}_+$  and  $C > 0$  such that for  $z$ ,  $|z| \geq R$ , inequality (3) holds then*

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{L(r)} \leq \max\{1, C\}.$$

*Proof.* Let  $\alpha \in \mathbb{C}$ ,  $|\alpha| = 1$ ,  $x \geq R$  and

$$\Omega(x) = \max \left\{ \frac{|f^{(k)}(\alpha x)|}{l^k(x)} : 0 \leq k \leq p \right\}.$$

The function  $\Omega$  is positive continuous differentiable and

$$\Omega'(x) \leq \max \left\{ \frac{d}{dx} \frac{|f^{(k)}(\alpha x)|}{l^k(x)} : 0 \leq k \leq p \right\}$$

on  $[R, +\infty)$ , except an enumerable set of points. Therefore, using the inequality  $\frac{d}{dx} |\varphi(x)| \leq \left| \frac{d}{dx} \varphi(x) \right|$  which is true for complex-valued function  $\varphi$  of real variable outside an enumerable set of points, in view of (3), we have

$$\begin{aligned} \Omega'(x) &\leq \max \left\{ \frac{|\alpha f^{(k+1)}(\alpha x)|}{l^k(x)} + \frac{|f^{(k)}(\alpha x) k l'(x)|}{l^{k+1}(x)} : 0 \leq k \leq p \right\} \leq \\ &\leq \max \left\{ \frac{|f^{(k+1)}(\alpha x)|}{l^{k+1}(x)} l(x) + \frac{|f^{(k)}(\alpha x)| k |l'(x)|}{l^k(x) l(x)} : 0 \leq k \leq p \right\} \leq \\ &\leq \Omega(x) \left( \max\{1, C\} l(x) + \frac{p |l'(x)|}{l(x)} \right). \end{aligned}$$

Since  $l'(x) = o(l^2(x))$  ( $x \rightarrow +\infty$ ), we obtain  $\Omega'(x) \leq (1 + \varepsilon) \max\{1, C\} l(x) \Omega(x)$  for every  $\varepsilon > 0$  and for all  $x \geq x_0(\varepsilon) \geq R$ , except an enumerable set of points. Therefore, there exists  $r_0 \geq x_0(\varepsilon)$  such that

$$\Omega(r) \leq \Omega(r_0) \exp \left\{ (1 + \varepsilon) \max\{1, C\} \int_{r_0}^r l(x) dx \right\}, \quad r > r_0,$$

whence it follows that  $|f(\alpha r)| \leq \Omega(r_0) \exp\{(1 + \varepsilon) \max\{1, C\} L(r)\}$ , ( $r > r_0$ ), that is, in view of arbitrariness of  $\varepsilon$ , we obtain the required inequality.  $\square$

The proof of following lemma is similar.

**Lemma 3.** *Let  $l \in K$ . If there exist numbers  $p \in \mathbb{Z}_+$  and  $C > 0$  such that for all  $z, |z| \geq R$ ,*

$$\frac{|f^{(p+1)}(z)|}{(p+1)!l^{p+1}(|z|)} \leq C \max \left\{ \frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \leq k \leq p \right\} \tag{4}$$

then

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{L(r)} \leq (p+1) \max\{1, C\}.$$

3. Now we will study the entire solutions of differential equation (2).

**Theorem 1.** *Let  $l \in K \cap Q$ , and for all  $z, |z| \geq R$ , the entire functions  $g_0, g_1, \dots, g_n$  and  $h$  satisfy the conditions: a)  $|g_j(z)| \leq m_j l^j (|z|) |g_0(z)|$ , ( $1 \leq j \leq n$ ), b)  $|g'_j(z)| < M_j l^{j+1} (|z|) |g_0(z)|$ , ( $0 \leq j \leq n$ ) and c)  $|h'(z)| \leq Ml(|z|)|h(z)|$ , where  $m_j, M_j$  and  $M$  are nonnegative constants. If an entire function  $f$  is a solution of (2) then  $f$  is of bounded  $l$ -index and*

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{L(r)} \leq \max\{1, C\}, \quad C = \sum_{j=0}^n M_j + (M+1) \sum_{j=1}^n m_j + M. \tag{5}$$

*Proof.* At first we remark that condition b) with  $j = 0$  implies  $g_0(z) \neq 0$  for  $|z| \geq R$ . Further, since  $f$  satisfies (2),

$$g_0(z)f^{(n+1)}(z) + \sum_{j=0}^n g'_j(z)f^{(n-j)}(z) + \sum_{j=1}^n g_j(z)f^{(n-j+1)}(z) = h'(z),$$

and, in view of c), we have

$$|h'(z)| \leq Ml(|z|)|h(z)| \leq Ml(|z|) \sum_{j=0}^n |g_j(z)||f^{(n-j)}(z)|.$$

Thus, setting  $m_0 = 1$ , we have

$$\begin{aligned} & |f^{(n+1)}(z)| \leq \\ & \leq \frac{1}{|g_0(z)|} \left( Ml(|z|) \sum_{j=0}^n |g_j(z)||f^{(n-j)}(z)| + \sum_{j=0}^n |g'_j(z)||f^{(n-j)}(z)| + \sum_{j=1}^n |g_j(z)||f^{(n-j+1)}(z)| \right) \leq \\ & \leq Ml(|z|) \sum_{j=0}^n m_j l^j (|z|) |f^{(n-j)}(z)| + \sum_{j=0}^n M_j l^{j+1} (|z|) |f^{(n-j)}(z)| + \sum_{j=1}^n m_j l^j (|z|) |f^{(n-j+1)}(z)|, \end{aligned}$$

whence

$$\begin{aligned} & \frac{|f^{(n+1)}(z)|}{l^{n+1}(|z|)} \leq Ml(|z|) \sum_{j=0}^n m_j \frac{|f^{(n-j)}(z)|}{l^{n-j}(|z|)} + \sum_{j=0}^n M_j \frac{|f^{(n-j)}(z)|}{l^{n-j}(|z|)} + \\ & + \sum_{j=1}^n m_j \frac{|f^{(n-j+1)}(z)|}{l^{n-j+1}(|z|)} \leq \left( \sum_{j=0}^n M_j + (M+1) \sum_{j=1}^n m_j + M \right) \max \left\{ \frac{|f^{(k)}(z)|}{l^k(|z|)} : 0 \leq k \leq n \right\}, \end{aligned}$$

i. e.  $f$  is of bounded  $l$ -index by Lemma 1 and estimate (5) holds by Lemma 2. □

**Corollary [4].** *If in differential equation (2) all coefficients  $g_j$  and  $h$  are polynomials with  $\deg g_j \leq \deg g_0 = d$  ( $1 \leq j \leq n$ ) and an entire function  $f$  satisfies (2), then  $f$  is of bounded index ( $l$ -index with  $l(x) \equiv 1$ ) and*

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{r} \leq \max \left\{ 1, \frac{1}{|a_0|} \sum_{j=1}^n |a_j| \right\},$$

where  $a_j$  is the coefficient at  $z^d$  in  $g_j$ .

Indeed, since  $g_j$  and  $h$  are polynomials with  $\deg g_j \leq \deg g_0 = d$  ( $1 \leq j \leq n$ ), therefore  $|g_j(z)| \leq (1 + o(1))|a_j/a_0||g_0(z)|$ , ( $1 \leq j \leq n$ ),  $|g'_j(z)| = o(|g_0(z)|)$ , ( $0 \leq j \leq n$ ) and  $|h'(z)| = o(|h(z)|)$  as  $|z| \rightarrow +\infty$ . Thus, from Theorem 1 we easily obtain the necessary statement.

We remark that estimate (5), in general, is unimprovable. Indeed, consider the differential equation  $\exp\{z^2\}w'' + 2\exp\{z^2\}w = 4z^2$ . The function  $f(z) = \exp\{-z^2\}$  is its solution and of bounded  $l$ -index with  $l(x) = x$  for  $x \geq 1$ . Using Theorem 1 it is easy to show that  $\overline{\lim}_{r \rightarrow +\infty} \ln M(r, f)r^{-2} \leq 2$ . But  $\ln M(r, f) = r^2$ ,  $L(r) = \frac{1+o(1)}{2}r^2$  ( $r \rightarrow +\infty$ ). Thus,  $\overline{\lim}_{r \rightarrow +\infty} \ln M(r, f)r^{-2} = 2$ .

In the case when differential equation is homogeneous, the following more simple statement is true.

**Theorem 2.** *Let  $l \in K \cap Q$ , and entire functions  $g_0, g_1, \dots, g_n$  satisfy the condition  $|g_j(z)| \leq m_j l^j(|z|)|g_0(z)|$ , ( $1 \leq j \leq n$ ) for all  $z$ ,  $|z| \geq R$ . If an entire function  $f$  is a solution of (2) with  $h \equiv 0$  then  $f$  is of bounded  $l$ -index and*

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{L(r)} \leq \max \left\{ 1, \sum_{j=1}^n m_j \right\}. \tag{6}$$

*Proof.* Directly from (2) we have  $|g_0(z)||f^{(n)}(z)| \leq \sum_{j=1}^n |g_j(z)||f^{(n-j)}(z)|$ , whence

$$\frac{|f^{(n)}(z)|}{l^n(|z|)} \leq \sum_{j=1}^n m_j \frac{|f^{(n-j)}(z)|}{l^{n-j}(|z|)} \leq \sum_{j=1}^n m_j \max \left\{ \frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \leq k \leq n-1 \right\},$$

and it remains to use Lemmas 1 and 2. □

In general, one cannot improve estimate (6). The differential equation  $w' + 2zw = 0$  with entire solution  $f(z) = \exp\{-z^2\}$  and  $l(x) = x$  for  $x \geq 1$  demonstrates this.

Finally, using Lemmas 1 and 3 we supplement Theorems 1 and 2 with two statements which give estimates of  $M(r, f)$  that are sometimes better than (5) and (6).

**Theorem 3.** *Let  $l \in K \cap Q$ , and entire functions  $g_0, g_1, \dots, g_n$  and  $h$  satisfy the conditions a), b) and c) of Theorem 1. If an entire function  $f$  is a solution of (2) then it is of bounded  $l$ -index and*

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{L(r)} \leq \max\{n + 1, 2M^*(M + 2)\}, \quad M^* = \max_j \{1, m_j, M_j\}.$$

*Proof.* As in the proof of Theorem 1 we have

$$\begin{aligned} |f^{(n+1)}(z)| &\leq \frac{1}{|g_0(z)|} \left( Ml(|z|) \sum_{j=0}^n |g_j(z)| |f^{(n-j)}(z)| + \right. \\ &\quad \left. + \sum_{j=0}^n |g'_j(z)| |f^{(n-j)}(z)| + \sum_{j=1}^n |g_j(z)| |f^{(n-j+1)}(z)| \right) \leq \\ &\leq M^* \left( (M+1) \sum_{j=0}^n l^{j+1}(|z|) |f^{(n-j)}(z)| + \sum_{j=1}^n l^j(|z|) |f^{(n-j+1)}(z)| \right) \leq \\ &\leq M^*(M+2) \sum_{j=0}^n l^{j+1}(|z|) |f^{(n-j)}(z)|, \end{aligned}$$

whence

$$\begin{aligned} \frac{|f^{(n+1)}(z)|}{(n+1)!l^{n+1}(|z|)} &\leq M^*(M+2) \sum_{j=0}^n \frac{|f^{(n-j)}(z)|}{(n-j)!l^{n-j}(|z|)} \frac{(n-j)!}{(n+1)!} \leq \\ &\leq \frac{2M^*(M+2)}{n+1} \max \left\{ \frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \leq k \leq n \right\}, \end{aligned}$$

because  $\sum_{j=0}^n \frac{(n-j)!}{(n+1)!} \leq \frac{2}{n+1}$ . Therefore, by Lemma 1 the function  $f$  is of bounded  $l$ -index and by Lemma 3 the required estimate holds.  $\square$

**Theorem 4.** Let  $l \in K \cap Q$ , and entire functions  $g_0, g_1, \dots, g_n$  satisfy for all  $z, |z| \geq R$ , the conditions of Theorem 2. If an entire function  $f$  is a solution of (2) with  $h \equiv 0$  then it is of bounded  $l$ -index and

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln M(r, f)}{L(r)} \leq \max\{n, 2M^*\}, \quad M^* = \max_j\{1, m_j\}.$$

The proof of Theorem 4 is analogous to that of Theorems 2 and 3.

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