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ON ORDER OF GROWTH OF ANALYTIC SOLUTIONS FOR ALGEBRAIC DIFFERENTIAL EQUATIONS HAVING LOGARITHMIC SINGULARITY

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Let w be an analytic solution with logarithmic singularity in ∞ of the differential equation $P(z, \ln z, w, w') = 0$, where P is a polynomial of all variables. Then the order of the growth of w is finite and the estimate for the order is described. Asymptotic properties of such solutions are studied.

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Пусть w аналитическое решение дифференциального уравнения $P(z, \ln z, w, w') = 0$ с логарифмической особой точкой в ∞; P — многочлен всех переменных. Доказано, что w имеем конечный порядок роста и дается оценка порядка. Изучаются асимптотические свойства таких решений.

We use notations of the theory of meromorphic functions [1]. The entire, rational, algebraic solutions of the differential equation (d. e.) are analytic functions having in ∞ respectively the pole, algebraic branch point, or an essential singularity. By Painleve's theorem, the equation of the first order, algebraic over unknown function and its derivative, cannot have in integrals movable transcendental and essentially singular points [2, p. 54]. But in these integrals there can be fixed transcendental and essentially singular points. For example, the integral of the equation $2zww' = 1$ has the form $w(z) = \sqrt{\ln(z/c)}$, $c = \text{const}$; the function $w(z) = \exp(\ln^2 z)$ is a solution of d. e. $zw' = 2w \ln z$. It is known that any entire solution of d. e. $P(z, w, w') = 0$, where P is a polynomial on all variables, has a finite order of growth [3, p. 217]. A. A. Gol'dberg established [4] that the order of growth of meromorphic solutions of this equation is also finite. It will be shown that if specified d. e. has a solution with a logarithmic singularity, the order of growth of such solution is finite. From here, taking this into account, finiteness of any analytic solution of the d. e. $P(z, w, w') = 0$ follows, with an isolated singular point in ∞.

Suppose we have a d. e.

$$\sum_{j=0}^{m} (w')^j \sum_{n=0}^{k_j} w^n (c_{jn} + o(1)) z^{\alpha_{jn}} (\ln z)^{h_{jn}} = 0, \quad z \to \infty,\ c_{jn} \in \mathbb{C}\,; \alpha_{jn}, h_{jn} \in \mathbb{R}; \qquad (1)$$

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where all coefficients are analytic in $G = \{z : r_0 \le |z| < \infty\}$ functions. Consider also an equation of more general form

$$(w')^m Q_m + \cdots + (w')^j Q_j + \cdots + Q_0 = 0, \tag{2}$$

$$Q_j = \sum_{n=0}^{k_j} a_{jn}(z) b_{jn}(w), \quad j = 0, 1, \ldots, m; \tag{3}$$

(the functions $b_{jn}(w)$ are meromorphic in the whole plane outside some set of isolated points; a_{jn} are analytic functions,

$$a_{mn} = (c_{mn} + o(1)) z^{\alpha_{mn}} (\ln z)^{h_{mn}}, \quad n = 0, 1, \ldots, k_m;\ z \in G,\ z \to \infty, \tag{4}$$

$$|a_{jn}(z)| = O(|z|^{\alpha_{jn}}), \quad z \to \infty,\ j = 0, 1, \ldots, m-1;\ n = 0, 1, \ldots, k_j; \tag{5}$$

$c_{mn} \ne 0$, $c_{mn} \in \mathbb{C}$; $\alpha_{mn}, \alpha_{jn}, h_{mn} \ge 0$. We denote $\nu = \max \alpha_{mn}$, $n = 0, 1, \ldots, k_m$; $T = \{n : n \in \mathbb{N}, \alpha_{mn} = \nu\}$, $\mu = \max h_{mn}$, $n \in T$; $J = \{n : n \in \mathbb{N}, \alpha_{mn} \wedge h_{mn} = \mu\}$,

$$p = \max(\alpha_{jn} - \nu)/(m - j), \quad j = 0, 1, \ldots, k_j. \tag{6}$$

Let $\sigma(w) = \sum_{n \in J} c_{mn} b_{mn}(w) \not\equiv 0$. Then $(z \to \infty)$

$$Q_m = z^\nu (\ln z)^\mu \sigma(w) + \sum_{n=0}^{k_m} \tilde{a}_{mn}(z)(z) b_{mn}(w), \quad |\tilde{a}_{mn}(z)| = o(|z^\nu \ln^\mu z|), \tag{7}$$

Let us specify how do we understand the operations on multifunctions. We set $g = \{z : |z - r_0| < \varepsilon\}$, $r_0, \varepsilon > 0$, where ε sufficiently small. Choose any regular elements [5, v. 2, p. 481] $\exp(\alpha_{jn} \ln_0 z)$, $(\ln_0 z)^{h_{jm}}$, $z \in g$ respectively, of functions $z^{\alpha_{jn}} = \exp(\alpha_{jn})$, $(\ln z)^{h_{jn}}$, $j = 0, \ldots, m$; $n = 0, \ldots, k_j$. From properties of these functions it follows that the taken regular elements can be analytically continued along every continuous curve in the area $G = \{z : r_0 \le |z| < \infty\}$. We shall assume that there exists a regular element $w_0(z)$, $z \in g$, such that at the substitution $w_0(z)$, $\exp(\alpha_{jn} \ln_0 z)$, $(\ln_0 z)^{h_{jn}}$, $z \in g$ in (2) instead of w, $z^{\alpha_{jn}}$, $(\ln z)^{h_{jn}}$ respectively gives the identity. Suppose that the element $w_0(z)$ can be analytically continued along any continuous curve $z = \lambda(t)$, $t_0 \le t \le t_1$, $\lambda(t_0) = z_0$, $\lambda(t_1) = z_1$ belonging to G, and the result of this continuation always is either a regular element $w_1(z)$, $z \in \{z : |z - z_1| < \varepsilon_1\}$, $\varepsilon_1 > 0$ or an element having a nonramified pole at a point (an element of the form $(\sum_{n=-s}^{\infty} a(z - z_1)^n$, $s \in \mathbb{N})$. Suppose that for any $z_1 \in G$ there exists an infinite set of the distinct regular elements with center z_1 which are immediate analytic continuations of the element $w_0(z)$, $z \in g$. The set of all such elements will be denoted by $w_(z)$, $z \in G$. We say that $w(z)$, $z \in G$ is a meromorphic function with a logarithmic singularity in ∞. Further, we assume that the poles $w(z)$, $z \in G$ considered on its Riemann surface, are isolated, and ∞ is the unique possible point of condensation of these poles. In particular, if at all analytic continuations of the element $w_0(z)$, $z \in g$ in the domain G the result of a continuation is a regular element, then $w(z)$, $z \in G$ has in ∞ an isolated logarithmic singularity.

Choose some α, β, $-\infty < \alpha < \beta < +\infty$. Let, for example, $\alpha > 0$. Consider the curve $z = r_0 e^{it} = \mu(t)$, $0 \le t \le \alpha$, $\mu(\alpha) = r_0 e^{i\alpha}$. Analytically continue the elements $w_0(z)$, $\exp(\alpha_{jn} \ln_0 z)$, $(\ln_0 z)^{h_{jn}}$, $z \in g$, $j = 0, 1, \ldots, m$; $n = 0, 1, \ldots, k_j$ along the curve $\mu(t)$, $o \le t \le \alpha$. As the result of continuation, we receive the elements

$$w_\alpha(z), \quad \exp(\alpha_{jn} \ln_\alpha z), \quad (\ln_\alpha z)^{h_{jn}} \tag{8}$$

with centers at the point $z_1 = r_0 \exp(i\alpha)$. Then we shall continue elements (8) analytically along every possible curves $z = r(t)\exp(i\theta(t))$, $t_1 \le t \le t_2$, where $r(t)$, $\theta(t)$, $t_1 \le t \le t_2$ are continuous functions such that $r_0 \le r(t) < \infty$, $\alpha \le \theta(t) \le \beta$, $t_1 \le t \le t_2$. It is possible that $\beta - \alpha \ge 2\pi$. To each element from (8) there corresponds the set of all elements obtained by such continuations. These sets are denoted respectively by

$$\begin{gathered} w(z), \quad z \in g_{\alpha\beta} = \{z : z = re^{i\theta}, r_0 \le r < \infty, \alpha \le \theta \le \beta\}; \\ z^{\alpha_{jn}},\ z \in g_{\alpha\beta}; \quad \ln^{h_{jn}} z,\ z \in g_{\alpha\beta}; j = 0, \dots, m;\ n = 0, \dots, k_j, \end{gathered} \tag{9}$$

$g_{\lambda\beta}$ is the angular domain on the appropriate Riemann surface. If $\beta - \alpha < 2\pi$, by the monodromy theorem [5, v. 2, p. 488] functions (9) are univalent analytic functions in $g_{\alpha\beta} \subset \mathbb{C}$. If $\beta - \alpha \ge 2\pi$, the domain $g_{\alpha\beta}$ is a simply connected domain on a Riemann surface. In this domain the monodromy theorem can be also applied. Therefore functions (9) are univalent meromorphic functions on a piece of a Riemann surface $g_{\alpha\beta}$.

Assume, that asymptotic relations (1), (4)–(6) are fulfilled uniformly on θ, $\alpha \le \theta \le \beta$ in any fixed angular domain. For example, relation (4) is understood as: $\forall \alpha, \beta$, $-\infty < \alpha < \beta < +\infty$, $\forall \varepsilon > 0$, $\exists \rho = \rho(\alpha, \beta)$: $a_{mn}(z) = (c_{mn} + \nu_{mn}(z)) z^{\alpha_{mn}} \ln^{h_{mn}} z$, $|\nu_{mn}(z)| < \varepsilon$, $z \in g_{\alpha\beta} = \{z = re^{i\theta} : \rho \le r < +\infty, \alpha \le \theta \le \beta\}$, $\nu_{mn}(z)$ is some analytic function.

In order to define the order of the growth of a multifunction $w(z)$, $z \in G$ we shall consider Nevanlinna's characteristics of function meromorphic in angular domain. Let ζ, $\zeta \in g_{\alpha\beta}$ be a meromorphic function, . We set $-\infty < \alpha < \beta < +\infty$. We set

$$\ln^+ x = \max(\ln x, 0), \quad x \ge 0; \quad k = \pi/(\beta - \alpha) > 0. \tag{10}$$

Let $b_n = |b_n| \exp(i\theta_n)$ be the poles of the function $f(\zeta)$, $\zeta \in g_{\alpha\beta}$. We shall consider the characteristics of the function f [1, p. 40]:

$$\begin{aligned} A_{\alpha,\beta}(r, f) &= \frac{k}{\pi} \int_{r_0}^{r} \Big(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}}\Big)(\ln^+ |f(te^{i\alpha}) + \ln^+ |f(te^{i\beta})|)\, dt, \\ B_{\alpha,\beta}(r, f) &= \frac{2k}{\pi r^k} \int_{\alpha}^{\beta} \ln^+ |f(re^{i\theta})| \sin k(\theta - \alpha)\, d\theta, \\ C_{\alpha,\beta}(r, f) &= 2k \int_{r_0}^{r} c_{\alpha,\beta}(t, f) \Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)\, dt, \\ c_{\alpha,\beta}(t, f) &= c_{\alpha,\beta}(t, \infty) = \sum_{\substack{r_0 < |b_n| < t, \\ \alpha \le \theta_n \le \beta}} \sin k(\theta_n - \alpha) \end{aligned} \tag{11}$$

is the counting function of poles; each pole is counted according to its multiplicity. Let

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f), \quad r_0 \le r < \infty \tag{12}$$

Define the order of growth of a multifunction $w(z)$, $z \in G$. Choose arbitrary α, β, $-\infty < \alpha < \beta < +\infty$. Consider the angular domain $g_{\lambda\beta}$ and the corresponding univalent branch (see (9). First we assume that $w(z)$, $z \in G$ has no poles. Let

$$M_{\alpha\beta}(r, w) = \max |w(z)|, z \in \{z = te^{i\theta} : r_0 \le t \le r, \alpha \le \theta \le \beta\}. \tag{13}$$

For $x \in \mathbb{R}$ we denote $\hat{x} = \max(x, 1)$. Let

$$\rho^0_{\alpha\beta} = \varlimsup_{r \to +\infty} \frac{\ln \ln M_{\alpha\beta}(r, w)}{\ln r}. \tag{14}$$

The number

$$\rho^0 = \sup_{-\infty<\alpha<\beta<+\infty} \rho^0_{\alpha\beta} \tag{15}$$

is called the order of the growth of function $w(z)$, $z \in G$. If $w(z)$, $z \in G$ is a meromorphic function with a logarithmic singularity in ∞, we consider the Nevanlinna characteristic $S_{\alpha,\beta}(r, w)$ of the branch $w(z)$, $z \in g_{\alpha\beta}$ and put

$$\rho_{\alpha\beta} = \overline{\lim_{r\to+\infty}} \frac{\ln \hat{S}_{\alpha,\beta}(r, w)}{\ln r}. \tag{16}$$

Denote the growth of the function $w(z)$, $z \in G$ by

$$\rho = \sup_{-\infty<\alpha<\beta<+\infty} \rho_{\alpha\beta}. \tag{17}$$

It is possible to show that if $w(z)$, $z \in G$ has no poles, the orders ρ and ρ^0 both are finite or infinite.

Theorem 1. *Let a meromorphic function $w(z)$, $z \in G$ with logarithmic singularity in ∞ be a solution of d. e. (2). Then the order of growth of the solution $\rho \le \max(0, 2p + 2)$. If this solution has an isolated logarithmic singularity in ∞, then for every α, β, $-\infty < \alpha < \beta < +\infty$, there exists $c = c(\alpha, \beta) > 0$ such that*

$$\ln^+ |w(re^{i\theta})| < \begin{cases} cr^{2p+1}, & p > \frac{1}{2}, \\ cr\ln r, & p = \frac{1}{2}, \\ cr, & p < \frac{1}{2}, \end{cases} \quad r > r_1 > r_0, \alpha \le \theta \le \beta, \tag{18}$$

By E we denote some set of circles with a finite sum of radii belongings completely or partially to the angular domain $g_{\alpha\beta}$ (or $g_{\alpha_s\beta_s}$) (see (9)) of Riemann surface of function $w(z)$, $z \in G$.

Theorem 2. *Let a meromorphic function $w(z)$, $z \in G$ with a logarithmic singularity in ∞ be a solution of d. e. (1). Then two cases are possible:*

1) $(\forall\varepsilon > 0)$ $(\forall\alpha, \beta, -\infty < \alpha < \beta < +\infty)$ $(\exists = d(\alpha, \beta, \varepsilon))$ *such that for the branch $w(z)$, $z \in g_{\alpha\beta}$, $|z| > d$, $Z \notin E$ is fulfilled* $\ln w(z) = \ln^{\tau+1} z((\tau + 1)^{-1}\phi + \nu(z))$, $|\nu(z)| < \varepsilon$, $\tau > 0$, $\operatorname{Re}\phi \ne 0$;
2) *the function w has the properties:*
   a) *there are nonintersecting segments $(\alpha_s\beta_s)$, $s \in J$, (J is some (probably, empty) set of indexes) such, that for the branches $w(z)$, $z \in g_{\alpha_s\beta_s} = \{z = r^{i\theta} : r_0 \le r < +\infty, \alpha_s \le \theta \le \beta_s\}$ we have:* $(\forall\eta > 0)(\forall\varepsilon > 0)(\exists R = R(s, \eta, \varepsilon))$: $\ln w(z) = (\phi_s\rho_s^{-1} + \nu(z))z^{\rho_s} \ln^{\tau_s} z$, $|\nu(z)| < \varepsilon$, $z \in \{z = r\exp(i\theta) : r \ge R, \alpha_s + \eta \le \theta \le \beta_s - \eta\} \setminus E$, $\ln |w(re^{i\alpha_s})|$, $\ln |w(re^{i\beta_s})| = o(r^\rho \ln^\tau r)$, $r \in [R, +\infty) \setminus \Delta$, $\operatorname{mes}\Delta < +\infty$;
   b) *on the complement of angular domains $g_{\alpha_s\beta_s}$, $s \in J$ (considered with respect to the Riemann surface of the function $w(z)$, $z \in G$)*

$$\ln |w(re^{i\theta})| < \varepsilon \ln^{\tau+1} r, \quad \varepsilon > 0, r > r(\theta), r \notin \Delta.$$

The numbers are determined by the form of equation (1).

Consider a meromorphic function $f(\zeta)$, $\zeta \in \bar{D} = \{\zeta = te^{i\theta} : r_0 \le t \le r, \alpha \le \theta \le \beta\}$. Choose $a \in \mathbb{C}$. Let $\{a_m\}$ be the set of zeros, $\{b_n\}$ the set of poles of the function $f(\zeta)-a$, $\zeta \in \bar{D}$. We apply counting function (11) to $1/(f(\zeta)-a)$, $\zeta \in \bar{D}$. Let $c_{\alpha,\beta}(t,a) = c_{\alpha,\beta}(t, 1/(f-a))$ be the counting function of a-points.

Lemma 1. *If $f(\zeta), \zeta \in \bar{D}$ is a meromorphic function, then we have*

$$2k\int_{r_0}^{r} c_{\alpha,\beta}(t,a)\Big(\frac{1}{t^{k+1}}+\frac{t^{k-1}}{r^{2k}}\Big)\,dt = 2k\int_{r_0}^{r} c_{\alpha,\beta}(t,\infty)\Big(\frac{1}{t^{k+1}}+\frac{t^{k-1}}{r^{2k}}\Big)\,dt+ \\ +\frac{2k}{\pi r^k}\int_{\alpha}^{\beta}\ln|f(re^{i\theta})-a|\sin k(\theta-\alpha)\,d\theta+\frac{k}{\pi}\int_{r_0}^{r}\Big(\frac{1}{t^{k+1}}-\frac{t^{k-1}}{r^{2k}}\Big)\times \\ \times\ln|(f(te^{i\alpha})-a)(f(te^{i\beta})-a)|\,dt - Q_1(r_0,r,\alpha,\beta)-P_1(r_0,r,\alpha,\beta), \tag{19}$$

where

$$Q_1(r_0,r,\alpha,\beta) = \frac{k}{\pi}\int_{\lambda}^{\beta}\Big(\frac{1}{r_0^{k+1}}+\frac{r_0^{k-1}}{r^{2k}}\Big)\sin k(\theta-\alpha)\ln|f(r_0e^{i\theta})-a|\,d\theta+ \\ +\frac{1}{\pi}\int_{\alpha}^{\beta}\Big(\frac{1}{r_0^{k}}+\frac{r_0^{k}}{r^{2k}}\Big)\sin k(\theta-\alpha)\,\frac{\partial\ln|f(te^{i\theta})-a|}{\partial t}\bigg|_{t=r_0}\,dt, \tag{20}$$

$$P_1(r_0,r,\alpha,\beta) = \Big(\frac{1}{r_0^{k}}-\frac{r_0^{k}}{r^{2k}}\Big)\bigg(\sum_{\substack{|a_m|=r_0,\\ \alpha<\varphi_m<\beta}}\sin k(\varphi_m-\alpha)-\sum_{\substack{|b_n|=r_0,\\ \alpha<\psi_n<\beta}}\sin k(\psi_n-\alpha)\bigg), \tag{21}$$

where $a_m = r_0e^{i\varphi_m}$ are zeros of the function $f(\zeta)-a$, $b_n = r_0e^{i\psi_n}$ its poles which lay on the arc $\{\zeta = r\exp(i\theta) : r = r_0, \alpha \le \theta \le \beta\}$.

Remark. In [1, p. 20, (3.2)] the formula similar to (19) is proved. In it the summand $P_1(r_0,r,\alpha,\beta)$, (see (21)), which accounts the number of a-points of the function f on the arc $\{\zeta = r\exp(i\theta) : r = r_0, \alpha \le \theta \le \beta\}$ is dropped. For a fixed one can choose r_0 so that $f(re^{i\theta}) \ne a, \infty$; $\alpha \le \theta \le \beta$. Then formula (3.2) with defined in [1, p. 20] residual term $Q^{(1)}(\dots)$ is true. However, below we take by both parts of (19) the integrated average, when a varies on the ring $\{a : \gamma \le |a| \le \Gamma\}$. Therefore it is impossible simply to reject a summand $P_1(\dots)$. Taking this into account, we reduce a proof of formula (19).

Let $f(\zeta)$, $\zeta \in \{\zeta = \rho e^{i\theta} : r_0 \le \rho \le +\infty, \alpha \le \theta \le \beta\}$ be a meromorphic function. Choose some γ, Γ, $0 <\ge< \Gamma < +\infty$. Denote

$$P = \{\zeta = \rho e^{i\theta} : r_0 \le \rho \le t, \alpha \le \theta \le \beta, \gamma \le |f(z)| \le \Gamma\}, \quad F = \{a : \gamma \le |a| \le \Gamma\}, \\ |F| = \pi(\Gamma^2-\gamma^2), \quad \Omega(t,f) = |F|^{-1}\iint_P |f'(\rho e^{i\theta})|^2\sin k(\theta-\alpha)\rho\,d\rho\,d\theta. \tag{22}$$

Lemma 2. *Let $f(\zeta)$, $\zeta \in \{\zeta = \rho e^{i\theta} : r_0 \le \rho < +\infty, \alpha \le \theta \le \beta\}$ be a meromorphic function, $f(r_0e^{i\theta}) \ne \infty$, $\alpha \le \theta \le \beta$. Then*

$$S_{\alpha,\beta}(r,f)+O(1) = 2k\int_{r_0}^{r}\Omega(t,f)\Big(\frac{1}{t^{k+1}}+\frac{t^{k-1}}{r^{2k}}\Big)\,dt \overset{\text{def}}{=} S^*_{\alpha,\beta}(r,f). \tag{23}$$

Lemma 3. *If the function $f(\zeta) \not\equiv 0$ is holomorphic in the domain $U = \{\zeta = re^{i\varphi} : t_0 \le r \le \lambda, 0 \le \varphi \le \pi\}$, $f(t_0e^{i\varphi}) \ne 0$, $0 \le \varphi \le \pi$, then*

$$\ln^+|f(re^{i\varphi})| \le \frac{\lambda^2 - r^2}{\lambda^2 \sin^3\varphi} rA_{0,\pi}(\lambda, f) + \frac{r\lambda^2}{\lambda^2 - r^2} B_{0,\pi}(\lambda, f) + c,$$
$$0 < \varphi < \pi,\ r_* < r < \lambda,\ r_* = \max(t_0 + 1; 3t_0),\ c = c(t_0) = const. \quad (24)$$

Let $w(z)$, $z \in G$ be a meromorphic function with a logarithmic singularity in ∞ having a finite order ρ. We choose some α, β, $-\infty < \alpha < \beta < +\infty$, and consider a univalent branch $w(z)$, $z \in g_{\alpha\beta}$, of this function on a piece of the Riemann surface $g_{\alpha\beta}$ (see (9)).

Lemma 4. *If $w(z)$, $z \in G$ is a meromorphic function with logarithmic singularity in ∞ having finite order ρ, then for any branch $w(z)$, $z \in g_{\alpha\beta}$ (see (9)) and $\forall\varepsilon > 0$ $\exists d = d(\alpha, \beta, \varepsilon)$:*

$$|w'(z)/w(z)| < |z|^{2\rho+1+\varepsilon}, \quad z \in g_{\alpha\beta} \setminus E, |z| \ge d,$$

where E is the set of disks on a part of a Riemann surface $g_{\alpha\beta}$, with centers in zeros and poles of a branch $w(z)$, $z \in g_{\alpha\beta}$, and sum of its radii is finite.

Lemma 5. *Let $f(z)$, $z = x + iy \in D$ be a holomorphic in a domain D function,* $\operatorname{Re} f(z) = u(z) = u(x, y)$, $\operatorname{Im} f(z) = \nu(z) = \nu(x, y)$; *$l$ a vector on the plane XOY, φ the angle between l and the axes OX. Let $\partial/\partial l$ be the operator of differentiation in direction l. Then*

$$\frac{\partial u(z)}{\partial l} = \operatorname{Re}[e^{i\varphi} f'(z)], \quad \frac{\partial \nu(z)}{\partial l} = \operatorname{Im}[e^{i\varphi} f'(z)] \quad (25)$$

Proof of Theorem 1. We use a method applied in [4]. Choose in (22) γ and Γ so that in the domain $F = \{w : \gamma \le |w| \le \Gamma\}$ there are no zeros and singular points (including poles) of the functions $\sigma(w)$, b_{jn}, $j = 0, \dots, m$; $n = 0, \dots, k_j$. The set F is closed, therefore there are numbers $m_1 > 0$ and $M < \infty$ such that $|\sigma(w)| \ge m_1$, $b_{jn}(w)| < M$, $w \in F$, $j = 0, \dots, m$; $n = 0, \dots, k_j$. If $|z| \ge r_1 \ge r_0$, r_1 is sufficiently large, then taking into account (7), $M \sum_{n=0}^{k_m} |\tilde{a}_{mn}(z) z^{-\nu} \ln^{-\mu}(z)| \le m_1/2$. Therefore

$$|Q_m z^{-\nu} \ln^{-\mu}(z)| \ge |\sigma(w)| - M \sum_{n=0}^{k_m} |\tilde{a}_{mn}(z) z^{-\nu} \ln^{-\mu}(z)| \le m_1/2 \quad (26)$$

Equation (2) can be rewritten as

$$(w')^m = -\sum_{j=0}^{m-1} (w')^j \frac{Q_j}{Q_m};$$

then

$$|w'|^2 \le \sum_{j=0}^{m-1} |w'|^{2j/m} \frac{|Q_j|}{|Q_m|}^{2/m}.$$

Therefore, taking into consideration (22) and the Hölder inequality, we obtain

$$\Omega(t, w) = \frac{1}{|F|} \iint_P |w'(\rho e^{i\theta})|^2 \sin k(\theta - \alpha) \rho \, d\rho \, d\theta \le$$

$$\le \frac{m}{|F|}\iint_P \sum_{j=0}^{m-1} |w'|^{\frac{2j}{m}} \Big|\frac{Q_j}{Q_m}\Big|^{\frac{2}{m}} \sin k(\theta-\alpha)\rho\, d\rho\, d\theta \le$$

$$\le \frac{m}{|F|}\sum_{j=0}^{m-1}\Big\{\Big(\iint_P \big(|w'|^{2j}\sin k(\theta-\alpha)\rho\, d\rho\, d\theta\Big)^{\frac{j}{m}} \Big(\iint_P \Big|\frac{Q_j}{Q_m}\Big|^{\frac{2}{m-j}} \sin k(\theta-\alpha)\rho\, d\rho\, d\theta\Big)^{\frac{m-j}{m}}\Big\} \le$$

$$\le \frac{m^2}{|F|^{\frac{m-s}{m}}}\Omega(t,w)^{\frac{s}{m}}\Big(\iint_P \Big|\frac{Q_s}{Q_m}\Big|^{\frac{2}{m-s}}\rho\, d\rho\, d\theta\Big)^{\frac{m-s}{m}},$$

where $s = s(t)$ is the number of the greatest summand in the last sum, $0 \le s \le m$. Then $\Omega(t,w) \le m^{2m}|F|^{-1}\iint_P |Q_s/Q_m|^{2/(m-s)}\rho\, d\rho\, d\theta$. If $|z| \ge r_1$ and $\gamma \le |w| \le \Gamma$, then $|b_{jn}(w)| < M$ and inequality (26) is true. Taking this into account and also (5), (6),

$$\Omega(t,w) \le \text{const}\int_{r_1}^{t}\int_{\alpha}^{\beta}\Big(\sum_{n=0}^{k_s}|a_{sn}(z)z^{-\nu}\ln^{-\mu} z|\Big)^{\frac{2}{m-s}}\rho\, d\rho\, d\theta \le$$

$$\le \text{const}\int_{r_1}^{t}\rho^{2p+1}\, d\rho \le \begin{cases} C_1 r^{2p+2}, & p > -1,\\ C_1 \ln t, & p \le -1,\end{cases}$$

where $C_1 = \text{const} > 0$. From this and from (23) we obtain ($r > r_0$, $k = \pi/(\beta-\alpha) > 0$)

$$S_{\alpha,\beta}(r,w) \le 4k\int_{r_0}^{r}\Omega(t,w)t^{-k-1}\, dt + O(1) \le \begin{cases} O(r^{2p+2-k}, & p > -1+\frac{k}{2},\\ O(\ln r), & p = -1+\frac{k}{2},\\ O(1), & p < -1+\frac{k}{2}.\end{cases} \tag{27}$$

From this and (16), (17) the first statement of the theorem follows.

Assume, that the solution $w(z)$, $z \in G$ of d. e. (2) has in ∞ an isolated logarithmic singularity. Choose α, $-\infty < \alpha < +\infty$, and consider a univalent branch $w(z)$, $z \in g_{\alpha,\alpha+\pi}$ (see (9)). The function $z = \zeta e^{i\alpha}$ bijectively maps the domain $D_1 = \{\zeta = re^{i\varphi} : r_0 \le r < +\infty, 0 \le \varphi \le \pi\}$ onto the domain $g_{\alpha,\alpha+\pi}$, and the domain $\{\zeta = re^{i\varphi} : r_0 \le r < +\infty, \delta \le \varphi \le \pi-\delta\}$ maps onto the domain $\{z = re^{i\theta} : r_0 \le r < +\infty, \alpha+\delta \le \theta \le \alpha+\pi\delta\}$, ($\delta > 0$, $re^{i\theta} = z = \zeta e^{i\alpha} = re^{i(\varphi+\alpha)}$). The conditions of Lemma 3 are fulfilled for the holomorphic function $f(\zeta) = w(\zeta e^{i\alpha})$, $\zeta \in D_1$. Indeed, the function $w(z)$, $z \in G$ has at most countable set of zeros, therefore there is t_0 such that $W(t_0e^{i\theta}) \ne 0$, $-\infty < \theta < +\infty$, hence $f(te^{i\varphi}) \ne 0$, $0 \le \varphi \le \pi$. If $\lambda = 2r$, $\delta \le \varphi \le \pi-\delta$, then $\sin\varphi \ge \delta/2$ and from (24) we obtain

$$\ln^+|w(re^{i\theta})| = \ln^+|w(\zeta e^{i\alpha})| = \ln^+|f(re^{i\varphi})| \le \frac{6r}{\delta^3}A_{0,\pi}(2r,f)+2rB_{0,\pi}(2r,f)+C_0,\ C_0 = \text{const}\,.$$

Thus as [1, p. 41], taking into account (12), (27), it follows from the previous inequality that ($\beta = \alpha+\pi$, $k = \pi/(\beta-\alpha) = 1$)

$$\ln^+|w(re^{i\theta})| \le \frac{6r}{\delta^3}A_{\alpha,\alpha+\pi}(2r,w) + 2rB_{\alpha,\alpha+\pi}(2r,w) + C_0 \le$$

$$\le C_1 r S_{\alpha,\alpha+\pi}(2r,w) + C_0 \le \begin{cases} C_2 r^{2p+2}, & p > \frac{1}{2},\\ C_2 r\ln r, & p = \frac{1}{2}, \alpha+\delta \le \theta \le \alpha+\pi-\delta,\\ C_2 r, & p, \frac{1}{2},\end{cases} \tag{28}$$

$C_1, C_2 = \text{const}$, $C_2 = C_2(\alpha, \delta)$, $r > r_1 > r_0$, $-\infty < \alpha < +\infty$, α is arbitrary. For any segment $[\alpha, \beta]$ it is possible to find numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_t$ such that $[\alpha, \beta] \subset \bigcup_{j=1}^{t} [\alpha_j + \delta, \alpha_j + \pi - \delta]$; since (28) holds on each segment $\alpha_j + \delta \le \theta \le \alpha_j + \pi - \delta$, inequality (18) is proved. □

Proof of Lemma 1. Let $\{c_\nu\}_{\nu=1}^{q}$ be the set-theoretic sum of the sets $\{a_m\}$, $\{b_n\}$; $c_\nu = |c_\nu| e^{i\theta_\nu}$. We consider the function $u(\zeta) = \ln |f(\zeta) - a|$. In a neighbourhood of the point c_ν the function $u(\zeta)$ has the form

$$u(\zeta) = d_\nu \ln |\zeta - c_\nu| + u_\nu(\zeta), \tag{29}$$

where $d_\nu \in \mathbb{R}$, $u_\nu(\zeta)$ is a twice continuously differentiable function in a neighbourhood of c_ν; if c_ν is a zero (pole) of order k_ν of the function $f(\zeta) - a$, then in a neighbourhood of c_ν we have (29) with $d_\nu = k_\nu$ ($d_\nu = -k_\nu$). Let Γ be the boundary of $D = \{\zeta : \alpha < \arg \zeta < \beta, r_0 < |\zeta| < r\}$. We exclude from the domain D disks of sufficiently small radius ε with centers at the points $c_1, \dots, c_q$. We receive the domain, which we denote by D_ε, and the part of the curve Γ outside the disks is denoted by Γ_ε. By $C(\varepsilon, c_\nu)$ we denote the part of the circle $\{\zeta : |\zeta - c_\nu| = \varepsilon\}$ which belongs to D. Apply to the domain D_ε the second Green formula

$$\iint_{D_\varepsilon} (u \Delta \nu - \nu \Delta u) \, d\sigma = -\Big(\int_{\Gamma_\varepsilon} + \sum_{\nu=1}^{q} \int_{C(\varepsilon, c_\nu)} \Big) \Big(u \frac{\partial \nu}{\partial n} - \nu \frac{\partial u}{\partial n} \Big) \, ds, \tag{30}$$

where

$$\nu = -\operatorname{Im} \Big(\frac{1}{\zeta^k e^{-i\alpha k}} + \frac{\zeta^k e^{-i\alpha k}}{r^{2k}} \Big) = \Big(\frac{1}{t^k} - \frac{t^k}{r^{2k}} \Big) \sin k(\theta - \alpha), \tag{31}$$

$$u = u(\zeta) = \ln |f(\zeta) - a|, \quad \zeta = t e^{i\theta}, \ t \in \mathbb{R}, \ k = \pi / (\beta - \alpha), \tag{32}$$

$d\sigma$ is an area element, ds an element of length of an arc, $\frac{\partial}{\partial n}$ the operator of differentiation on interior normal to Γ_ε, Δ the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Taking into account that $\Delta u = \Delta \nu = 0$ formula (30) is of the form

$$0 = \Big(\int_{\Gamma_\varepsilon} + \sum_{\nu+1}^{q} \int_{C(\varepsilon, c_\nu)} \Big) \Big(u \frac{\partial \nu}{\partial n} - \nu \frac{\partial u}{\partial n} \Big) \, ds. \tag{33}$$

The functions $\nu(\zeta)$, $u(\zeta)$ have the properties:

a) on the arc $\{r e^{i\theta} : \alpha \le \theta \le \beta\}$

$$\nu = 0, \ \frac{\partial \nu(\zeta)}{\partial n} \Big|_{|\zeta| = r} = - \frac{\partial \nu(t e^{i\theta})}{\partial t} \Big|_{t = r} = \frac{2k}{r^{2k+1}} \sin k(\theta - \alpha); \tag{34}$$

b) on the rays $\{\zeta : \arg \zeta = \alpha\}$ and $\{\zeta : \arg \zeta = \beta\}$ respectively,

$$\nu = 0, \frac{\partial \nu(\zeta)}{\partial n} = \frac{\partial \nu(t e^{i\theta})}{\partial \theta} \frac{1}{t} \Big|_{\zeta = t e^{i\alpha}} = \Big(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}} \Big) k, \tag{35}$$

$$\nu = 0, \frac{\partial \nu(\zeta)}{\partial n} = - \frac{\partial \nu(t e^{i\theta})}{\partial \theta} \frac{1}{t} \Big|_{\zeta = t e^{i\beta}} = \Big(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}} \Big) k, \tag{36}$$

c) on the arc $\{\zeta : |\zeta| = r_0, \alpha \le \arg\zeta \le \beta\}$

$$\frac{\partial\nu(\zeta)}{\partial n}\bigg|_{|\zeta|=r_0} = \frac{\partial\nu(te^{i\theta})}{\partial t}\bigg|_{t=r_0} = -k\Big(\frac{1}{r_0^{k+1}} + \frac{r_0^{k-1}}{r^{2k}}\Big)\sin k(\theta-\alpha). \tag{37}$$

Find the limits at $\varepsilon \to 0$ of integrals in (33). By the mean value theorem

$$\int_{C(\varepsilon,c_\nu)}\Big(u\frac{\partial\nu}{\partial n} - \nu\frac{\partial u}{\partial n}\Big)\,ds = (\text{length}\,C(\varepsilon,c_\nu))\Big(u\frac{\partial\nu}{\partial n} - \nu\frac{\partial u}{\partial n}\Big)_{\zeta^*}, \tag{38}$$

where ζ^* is some point on $C(\varepsilon, c_\nu)$. If $c_\nu \in D$, then length $C(\varepsilon,c_\nu) = 2\pi\varepsilon$ and on $C(\varepsilon,c_\nu)$ asymptotic as $\varepsilon \to 0$ relations hold:

$$u = d_\nu \ln\varepsilon + O(1), \quad \frac{\partial u}{\partial n} = \frac{\partial u}{\partial\varepsilon} = \frac{d_\nu}{\varepsilon} + O(1),$$
$$\nu = \Big(\frac{1}{|c_\nu|^k} - \frac{|c_\nu|^k}{r^{2k}}\Big)\sin k(\theta_\nu - \alpha) + o(1), \quad \frac{\partial\nu}{\partial n} = O(1). \tag{39}$$

Hence, if $c_\nu \in D$, then

$$\lim_{\varepsilon\to 0}\int_{C(\varepsilon,c_\nu)}\Big(u\frac{\partial\nu}{\partial n} - \nu\frac{\partial u}{\partial n}\Big)\,ds = -2\pi d_\nu\Big(\frac{1}{|c_\nu|^k} - \frac{|c_\nu|^k}{r^{2k}}\Big)\sin k(\theta_\nu - \alpha). \tag{40}$$

Let $c_\nu \in \Gamma_1 = \{\zeta : |\zeta| = r, \alpha \le \arg\zeta \le \beta\} \cup \{\zeta : \zeta = te^{i\alpha}, r_0 \le t \le r\} \cup \{\zeta : \zeta = te^{i\beta}, r_0 \le t \le r\}$. Length $C(\varepsilon,c_\nu) = O(\varepsilon)$, $\varepsilon \to 0$; as $\nu(\zeta) = 0$, if $\zeta \in \Gamma_1$, then on $C(\varepsilon,c_\nu)$ the following estimates are true:

$$u = d_\nu \ln\varepsilon + O(1), \quad \frac{\partial u}{\partial n} = \frac{d_\nu}{\varepsilon} + O(1), \quad \nu = o(1), \quad \frac{\partial\nu}{\partial n} = O(1).$$

Therefore, taking into account (38),

$$\lim_{\varepsilon\to 0}\int\limits_{C(\varepsilon,c_\nu),c_\nu\in\Gamma_1}\Big(u\frac{\partial\nu}{\partial n} - \nu\frac{\partial u}{\partial n}\Big)\,ds = 0. \tag{41}$$

If $c_\nu \in \{\zeta : |\zeta| = r_0, \alpha < \arg\zeta < \beta\}$, so length $C(\varepsilon,c_\nu) \sim \pi\varepsilon$, $\varepsilon \to 0$ and on $C(\varepsilon,c_\nu)$ formulas (39) are true. Therefore, taking into account (38), we have

$$\lim_{\varepsilon\to 0}\int\limits_{C(\varepsilon,c_\nu),|c_\nu|=r_0}\Big(u\frac{\partial\nu}{\partial n} - \nu\frac{\partial u}{\partial n}\Big)\,ds = -\pi d_\nu\Big(\frac{1}{r_0^k} - \frac{r_0^k}{r^2k}\Big)\sin k(\theta-\alpha). \tag{42}$$

Passing in (33) to the limit as $\varepsilon \to 0$ and taking into account (40)–(42), we receive ($\nu(\zeta) = 0$, $\zeta \in \Gamma_1$)

$$\int_{\Gamma_1} u(\zeta)\frac{\partial\nu(\zeta)}{\partial n}\,ds + \int\limits_{\substack{|\zeta|=r_0,\\ \alpha\le\arg\zeta\le\beta}}\Big(u\frac{\partial\nu}{\partial n} - \nu\frac{\partial u}{\partial n}\Big)\,ds =$$
$$= 2\pi\sum_{a_m\in D}\Big(\frac{1}{r_m^k} - \frac{r_m^k}{r^{2k}}\Big)\sin k(\varphi_m - \alpha) - 2\pi\sum_{b_n\in D}\Big(\frac{1}{\rho_n^k} - \frac{\rho_n^k}{r^{2k}}\Big)\sin k(\psi_n - \alpha) +$$
$$+\pi\Big(\frac{1}{r_0^k} - \frac{r_0^k}{r^{2k}}\Big)\Big(\sum_{|a_m|=r_0}\sin k(\varphi_m - \alpha) - \sum_{|b_n|=r_0}\sin k(\psi_n - \alpha)\Big), \tag{43}$$

$r_m e^{i\varphi_m}$ are zeros, $\rho_n e^{i\psi_n}$ poles of the function $f(\zeta) - a$ laying in $D = \{\zeta : r_0 < |\zeta| < r, \alpha < \arg\zeta < \beta\}$; $r_0 e^{i\varphi_m}$ are zeros, $r_0 e^{i\psi_n}$ poles laying on the arc $\{\zeta : |\zeta| = r_0, \alpha \le \arg\zeta \le \beta\}$.

Summands which correspond to multiple zeros or poles, are counted the appropriate number of times in the right part of (43). From (34)-(36), it follows

$$\int_{\Gamma_1} u(\zeta)\frac{\partial\nu(\zeta)}{\partial n}\,ds = \frac{2k}{r^k}\int_\alpha^\beta \ln|(f(re^{i\theta}) - a|\sin k(\theta-\alpha)\,d\theta + \\ + k\int_{r_0}^r \Big(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}}\Big)\ln|(f(te^{i\alpha}) - a)(f(te^{i\beta}) - a)|\,dt. \quad (44)$$

Taking into account (37), we obtain

$$\int\limits_{\substack{|\zeta|=r_0,\\ \alpha\le\arg\zeta\le\beta}} \Big(u\frac{\partial\nu}{\partial n} - \nu\frac{\partial u}{\partial n}\Big)\,ds = \int\limits_\alpha^\beta \Big[-k\Big(\frac{1}{r_0^{k+1}} + \frac{r_0^{k-1}}{r^{2k}}\Big)\sin k(\theta-\alpha)\ln|f(r_0e^{i\theta}) - a| - \\ -\Big(\frac{1}{r_0^k} - \frac{r_0^k}{r^{2k}}\Big)\sin k(\theta-\alpha)\,\frac{\partial\ln|f(te^{i\theta}) - a|}{\partial t}\Big|_{t=r_0}\Big]r_0\,d\theta \quad (45)$$

From properties of the Stieltjes integral [6, v. 3, p. 97, 103], and the definition of characteristics $c_{\alpha,\beta}(t,\infty)$, $c_{\alpha,\beta}(t,a) = c_{\alpha,\beta}(t, 1/(f-a))$ (11), it follows that

$$\sum_{a_m\in D}\Big(\frac{1}{r_m^k} - \frac{r_m^k}{r^{2k}}\sin k(\varphi_m - \alpha) = \int_{r_0}^r \Big(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\Big)dc_{\alpha,\beta}(t,a) = \Big(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\Big)c_{\alpha,\beta}(t,a)\Big|_{r_0}^r + \\ +k\int_{r_0}^r c_{\alpha,\beta}(t,a)\Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)\,dt = k\int_{r_0}^r c_{\alpha,\beta}(t,a)\Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)\,dt, \quad (46)$$

$$\sum_{b_n\in D}\Big(\frac{1}{\rho_n^k} - \frac{\rho_n^k}{r^{2k}}\Big)\sin k(\psi_n - \alpha) = k\int_{r_0}^r c_{\alpha,\beta}(t,\infty)\Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)\,dt. \quad (47)$$

Substituting (44)–(47) in (43), we receive (19). □

Proof of Lemma 2. Let F be the domain defined in (22). Further we shall integrate both parts of (19) on the domain F. By $d\sigma(a)$ we shall denote an area element in a-plane. Taking into account the formula [1, p. 34] $(2\pi)^{-1}\int_0^{2\pi}\ln w - e^{i\varphi}|\,d\varphi = \ln^+|w|$, we have

$$|F|^{-1}\iint_F \ln|f(te^{i\theta}) - a|\,d\sigma(a) = |F|^{-1}\int_\gamma^\Gamma |a|\,d|a|\int_0^{2\pi}\ln|f(te^{i\theta}) - 1|\,d\arg a = \\ = \frac{2}{\Gamma^2 - \gamma^2}\int_\gamma^\Gamma \ln[\max(|f(re^{i\theta})|, |a|)]|a|\,d|a| = \ln^+|f(te^{i\theta})| + Q(f), \quad (48)$$

$|Q(f)| < M$, M is a constant independent of f. Swapping the order of integration and taking into account the previous relation, we receive as $r \to +\infty$

$$\frac{1}{|F|}\iint_F d\sigma(a)\int_\alpha^\beta \ln|f(re^{i\theta}) - a|\sin k(\theta-\alpha)\,d\theta = \int_\alpha^\beta \ln^+|f(re^{i\theta})|\sin k(\theta-\alpha)\,d\theta + O(1), \quad (49)$$

where $O(1)$ is independent of f. The swapping of the order of integration is possible; this can be shown as in [1, c. 35].

For the meromorphic function $f(z)$, $z \in \{z : r_0 \le |z| \le t, \alpha \le \arg z \le \beta\}$ the formula [7, p. 262, Theorem 3.2.3]

$$\iint_P |f'(z)|^2 \sin k(\arg z - \alpha)\, d\sigma(z) = \iint_F c(t,a)\, d\sigma(a), \tag{50}$$

is true. Here P is defined in (22), $d\sigma(z)$ an area element in z-plane, $c(t,a)$ the counting function of a-points. Taking into account (50), (22), we obtain

$$|F|^{-1} \iint_F \Big\{ \int_{r_0}^{r} c(t,a)\Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)\, dt \Big\} d\sigma(a) = |F|^{-1} \int_{r_0}^{r} \Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big) \times$$
$$\times \Big\{ \iint_F c(t,a)\, d\sigma(a) \Big\}\, dt = \int_{r_0}^{r} \Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big) \Omega(t,f)\, dt \overset{\text{def}}{=} S^*_{\alpha,\beta}(r,f)(2k)^{-1}, \tag{51}$$

where $\Omega(t,f)$ is defined in (22). We estimate $\iint_F |f(r_0e^{i\theta}) - a|^{-1}\, d\sigma(a)$. Let $K = \{a : |f(r_0e^{i\theta}) - a| \le (\Gamma - \gamma)2^{-1}\}$. Then

$$\iint_K |f(r_0e^{i\theta}) - a|^{-1}\, d\sigma(a) = \int_0^{2\pi} d\varphi \int_0^{(\Gamma-\gamma)/2} \rho \frac{d\rho}{\rho} = \pi(\Gamma - \gamma);$$

if $a \in F \setminus K$, then $|f(r_0e^{i\theta})| - a| > (\Gamma - \gamma)/2$, and

$$\iint_{F\setminus K} |f(r_0e^{i\theta}) - a|^{-1}\, d\sigma(a) \le 2(\Gamma - \gamma)^{-1} \iint_F d\sigma(a) = 2\pi(\Gamma + \gamma).$$

Therefore

$$\iint_F |f(r_0e^{i\theta}) - a|^{-1}\, d\sigma(a) \le \pi(3\Gamma + \gamma). \tag{52}$$

As $|\partial \ln |f(te^{i\theta}) - a|/\partial t| \le |f'(te^{i\theta})/(f(te^{i\theta}) - a)|$, [1, p. 88, (1.3.9)], taking into account (52), $(r \to +\infty)$

$$\Big| \iint_F d\sigma(a) \int_\alpha^\beta \frac{\partial \ln |f(te^{i\theta}) - a|}{\partial t}\Big|_{t=r_0} \Big(1 - \frac{r_0^{2k}}{r^{2k}}\Big) \sin k(\theta - \alpha)\, d\theta \Big| \le$$
$$\le \iint_F d\sigma(a) \int_\alpha^\beta |f'(r_0e^{i\theta})/(f(r_0e^{i\theta}) - a)|\, d\theta \le$$
$$\le \max_{\alpha\le\theta\le\beta} |f'(r_0e^{i\theta})| \int_\alpha^\beta d\theta \iint_F |f(r_0e^{i\theta}) - a|^{-1} d\sigma(a) = O(1). \tag{53}$$

Let $n(a)$ be the number of solutions of the equation $f(\zeta) = a$, $\zeta \in \{\zeta : r_0 \le |\zeta| \le r_0 + 1, \alpha \le \arg \zeta \le \beta\}$. By the Cauchy theorem on residues, [8, p. 198, (5.3)] $\exists l = \max n(a)$, $a \in \mathbb{C}$. Therefore, in (21) $\sum\limits_{|a_m|=r_0, \alpha<\varphi_m<\beta} \sin k(\varphi_m - \alpha) < l$, $l = l(\alpha, \beta, f)$. It follows from the last inequality that

$$\Big| \iint_F \Big(\frac{1}{r_0^k} - \frac{r_0^k}{r^{2k}}\Big) \Big(\sum_{|a_m|=r_0} \sin k(\varphi_m - \alpha) - \sum_{|b_n|=r_0} \sin k(\psi_n - \alpha)\Big)\, d\sigma(a) \Big| < K, \tag{54}$$

where $K = l\pi(\Gamma^2 - \gamma^2)r_0^{-k}$. Taking into account (48), we obtain

$$\frac{1}{|F|}\iint_F d\sigma(a)\iint_{r_0}^r\Big(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}}\Big)\ln|(f(te^{i\alpha}) - a)(f(te^{i\beta}) - a)|\,dt =$$
$$= \int_{r_0}^r\Big(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}}\Big)(\ln^+|f(te^{i\alpha})| + \ln^+|f(te^{i\beta})|)\,dt + O(1). \quad (55)$$

Let us take now integrated mean on F in both parts of (19); taking into account (49)–(55), and also (12), we obtain

$$S^*_{\alpha,\beta}(r,f) = 2k\int_{r_0}^r\Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)\Omega(t,f)\,dt =$$
$$2k\int_{r_0}^r c_{\alpha,\beta}(t,\infty)\Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)\,dt + \frac{2k}{\pi r^k}\int_\alpha^\beta \ln^+|f(re^{i\theta})|\sin k(\theta-\alpha)\,d\theta +$$
$$+\frac{k}{\pi}\int_{r_0}^r\Big(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}}\Big)(\ln^+|f(te^{i\alpha})| + \ln^+|f(te^{i\beta})|)\,dt + O(1) =$$
$$= C_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + A_{\alpha,\beta}(r,f) + O(1) = S_{\alpha,\beta}(r,f) + O(1). \quad (56)$$

Lemma 2 is proved. □

From (56), (22) it follows that the function $S^*_{\alpha,\beta}(r,f)$ is nondecreasing. Therefore, up to a bounded value, function $S_{\alpha,\beta}(r,f)$, $r_0 \le r < +\infty$ is nondecreasing.

Proof of Lemma 3. The following formula is known (similar to proved in [1, p. 16]): if a function $f(\zeta) \not\equiv 0$ is holomorphic in the domain $\{\zeta = re^{i\varphi} : t_0 \le r \le \lambda, 0 \le \varphi \le \pi\}$, $f(t_0e^{i\varphi}) \ne 0$, $0 \le \varphi \le \pi$, then

$$\ln|f(\zeta)| = I_1 + I_2 - S + I_3, \quad (57)$$

where

$$I_1 = \frac{1}{\pi}\int\limits_{[-\lambda,-t_0]\cup[t_0,\lambda]} \ln|f(t)|\Big\{\frac{|\zeta|\sin\varphi}{|t-\zeta|^2} - \frac{\lambda^2|\zeta|\sin\varphi}{|\lambda^2-\zeta t|^2}\Big\}\,dt,$$
$$I_2 = \frac{1}{2\pi}\int_0^\pi \ln|f(\eta)|\,\mathrm{Re}\Big\{\frac{\eta+\zeta}{\eta-\zeta} - \frac{\bar\eta+\zeta}{\bar\eta-\zeta}\Big\}_{\eta=\lambda e^{i\theta}}d\theta,$$
$$S = \sum_{t_0<|a_m|<\alpha}\mathrm{Re}\,F(\zeta,a_m), \quad F(\zeta,\eta) = \ln\Big(\frac{\lambda^2-\zeta\bar\eta}{\zeta-\eta}\,\frac{\zeta-\bar\eta}{\lambda^2-\zeta\eta}\Big),$$
$$I_3 = \frac{t_0}{2\pi}\int_0^p i\Big\{\ln|f(t_0e^{i\theta})|\left.\frac{\partial\,\mathrm{Re}\,F(\zeta,re^{i\theta})}{\partial r}\right|_{r=t_0} - \mathrm{Re}\,F(\zeta,t_0e^{i\theta})\left.\frac{\partial\ln|f(re^{i\theta})|}{\partial r}\right|_{r=t_0}\Big\}\,d\theta,$$

where a_m are zeros of the function $f(\zeta)$, $\zeta \in \{\zeta = re^{i\varphi} : t_0 < r < \alpha, 0 < \varphi < \pi\}$; the summands corresponding to multiple zeros, are repeated in the sum S the corresponding number of times. It is known [1, p. 11, (1.5); p. 12, (1.11); p. 393] that for $\zeta \in \{\zeta = re^{i\varphi} : t_0 < r < \alpha, 0 < \varphi < \pi\}$

$$\frac{|\zeta|\sin\varphi}{|t-\zeta|^2} - \frac{\lambda^2|\zeta|\sin\varphi}{|\lambda^2-\zeta t|^2} > 0, \quad (58)$$

$$\operatorname{Re}\Big\{\frac{\eta+\zeta}{\eta-\zeta}-\frac{\bar\eta+\zeta}{\bar\eta-\zeta}\Big\}_{\eta=\lambda e^{i\theta}}=\frac{\lambda^2-|\zeta|^2}{|\eta-\zeta|^2}-\frac{\lambda^2-|\zeta|^2}{|\eta-\bar\zeta|^2}\Big|_{\eta=\lambda e^{i\theta}}>0. \tag{59}$$

Let us estimate the integral I_1. We have $\zeta=|\zeta|\exp(i\varphi)$, $0<\varphi<\pi$, $t\in\mathbb{R}$, $t_0<|t|<\lambda$. Then $|t-\zeta|\geq|t|\sin\varphi$, $|\lambda^2-\zeta t|\geq\lambda^2\sin\varphi$, and, taking into account (58), we obtain

$$0<\frac{1}{|t-\zeta|^2}-\frac{\lambda^2}{|\lambda^2-\zeta t|^2}=\frac{(\lambda^2-t^2)(\lambda^2-|\zeta|^2)}{|t-\zeta|^2|\lambda^2-\zeta t|^2}\leq\Big(\frac{1}{t^2}-\frac{1}{\lambda^2}\Big)\frac{\lambda^2-|\zeta|^2}{\lambda^2\sin^4\varphi}.$$

From this and from the definition of I_1 it follows

$$|I_1|\leq\frac{1}{\pi}\int_{[-\lambda,-t_0]\cup[t_0,\lambda]}\ln^+|f(t)|\Big\{\frac{|\zeta|\sin\varphi}{|t-\zeta|^2}-\frac{\lambda^2|\zeta|\sin\varphi}{|\lambda^2-\zeta t|^2}\Big\}\,dt\leq$$
$$\frac{(\lambda^2-|\zeta|^2)|\zeta|}{\pi\lambda^2\sin^3\varphi}\int_{[-\lambda,-t_0]\cup[t_0,\lambda]}\Big(\frac{1}{t^2}-\frac{1}{\lambda^2}\Big)\ln^+|f(t)|\,dt=\frac{\lambda^2-|\zeta|^2}{\lambda^2\sin^3\varphi}|\zeta|A_{0,\pi}(\lambda,f). \tag{60}$$

Let us estimate the integral I_2. As $\eta=\lambda\exp(i\theta)$, $0\leq\theta\leq\pi$, we have

$$\Big|\frac{\eta+\zeta}{\eta-\zeta}-\frac{\bar\eta+\zeta}{\bar\eta-\zeta}\Big|=\Big|\frac{-i4\zeta\lambda\sin\theta}{(\eta-\zeta)(\bar\eta-\zeta)}\Big|\leq\frac{4|\zeta|\lambda\sin\theta}{(\lambda-|\zeta|)^2}.$$

Therefore, taking into account (59) and the definition of I_2, we obtain

$$|I_2|\leq\frac{1}{2\pi}\int_0^\pi\ln^+|f(\eta)|\operatorname{Re}\Big\{\frac{\eta+\zeta}{\eta-\zeta}-\frac{\bar\eta+\zeta}{\bar\eta-\zeta}\Big\}_{\eta=\lambda e^{i\theta}}\,d\theta\leq$$
$$\leq\frac{2|\zeta|\lambda}{\pi(\lambda-|\zeta|)^2}\int_0^\pi\ln^+|f(\lambda e^{i\theta})|\sin\theta\,d\theta=\frac{|\zeta|\lambda^2}{(\lambda-|\zeta|)^2}B_{0,\pi}(\lambda,f). \tag{61}$$

Let us estimate I_3. From the definition of $F(\zeta,\eta)$ it follows that

$$\frac{\partial F(\zeta,re^{i\theta})}{\partial r}\Big|_{r=t_0}=-\frac{\zeta e^{-i\theta}}{\lambda^2-\zeta t_0e^{-i\theta}}+\frac{e^{i\theta}}{\zeta-t_0e^{i\theta}}-\frac{e^{-i\theta}}{\zeta-t_0e^{-i\theta}}+\frac{\zeta e^{i\theta}}{\lambda^2-\zeta t_0e^{i\theta}}.$$

Let

$$r_*<|\zeta|<\lambda,\quad r_*=\max(3t_0,t_0+1). \tag{62}$$

Then $|\lambda^2-\zeta t_0e^{-i\theta}|>\lambda^2-\lambda t_0$; $|\zeta/(\lambda^2-\zeta t_0e^{-i\theta})|<\lambda/(\lambda^2-\lambda t_0)<1$; $|\zeta-t_0e^{i\theta}|>1$; $|\zeta-t_0e^{-i\theta}|>1$; $|\lambda^2-\zeta t_0e^{i\theta}|>\lambda^2-\lambda t_0$, $|\zeta/(\lambda^2-\zeta t_0e^{i\theta})|<1$. Therefore $\frac{\partial F(\zeta,re^{i\theta})}{\partial r}\Big|_{r=t_0}<4$. If $|\eta|=t_0$, from (62) and the definition of $F(\zeta,\eta)$ it follows

$$\Big|\frac{\lambda^2-\zeta\bar\eta}{\lambda^2-\zeta\eta}\Big|<\frac{\lambda+t_0}{\lambda-t_0}<2,\quad\Big|\frac{\zeta-\bar\eta}{\zeta-\eta}\Big|<\frac{|\zeta|+t_0}{|\zeta|-t_0}<2,\quad|F(\zeta,t_0e^{i\theta})|<4.$$

Thus,

$$I_3<4t_0\max\Big\{\max_{0\leq\theta\leq\pi}|\ln|f(t_0e^{i\theta})||,\max\Big|\frac{|\partial\ln|f(re^{i\theta})|}{\partial r}\Big|_{r=t_0}\Big|\Big\}=C. \tag{63}$$

By formula (1.5) from [1, p. 11], the summands in S (57) represent values of the Green function for the semi-disk $\{\zeta:|\zeta|<\lambda,\operatorname{Im}\zeta>0\}$ and, hence, are nonnegative. Dropping S (57) and taking into account (60)–(63), we obtain (24). □

Proof of Lemma 4. Choose α, $-\infty < \alpha < +\infty$, and consider a branch $w(z)$, $z \in g_{\alpha,\alpha+\pi} = \{z = re^{i\theta} : r_0 \le r < +\infty, \alpha \le \theta \le \alpha+\pi\}$ of the function $w(z)$, $z \in G$. The function $z = \zeta e^{i\alpha}$ bijectively maps the domain $D_1 = \{\zeta = re^{i\varphi} : r_0 \le r < +\infty, 0 \le \varphi \le \pi\}$ on $g_{\alpha,\alpha+\pi}$. Let $f(\zeta) \stackrel{\text{def}}{=} w(\zeta e^{i\alpha})$, $\zeta \in D_1$. The Nevanlinna characteristics $S_{0,\pi}(r, f) = S_{\alpha,\alpha+\pi}(r, w)$, [1, p. 41]. Therefore, if ρ_0 is the order of the function $f(\zeta) = w(\zeta e^{i\alpha})$, $\zeta \in D_1$, so (see (16), (17) $\rho_0 \le \rho = \sup\limits_{-\infty<\alpha<\beta<+\infty} \rho_{\alpha\beta}$, $\rho_{\alpha\beta}$ is the order of growth of the characteristics $S_{\alpha,\beta}(r, w)$, ρ is the order of function $w(z)$, $z \in G$. Since $\rho < +\infty$, we have $\rho_0 < +\infty$. Let $\{c_k\}$ be the set of zeros and poles of the branch $w(z)$, $z \in g_{\alpha,\alpha+\pi}$, $c_k = |c_k|e^{i\theta_k}$. Under the bijective map $z = \zeta e^{i\alpha}$ of the domain D_1 on $g_{\alpha,\alpha+\pi}$, any zero (pole) $c_k \in \{c_k\}$ of branch $w(z)$, $z \in g_{\alpha,\alpha+\pi}$ corresponds to a zero (pole) $\zeta_k = c_k e^{-i\alpha} = |c_k|e^{i(\theta_k-\alpha)}$ of function $f(\zeta)$, $\zeta \in D_1$. Consider the circle of the radius $\delta_k = |\zeta_k|^{-\rho-1-\varepsilon/3} \sin\varphi_k$, $\varepsilon > 0$, $\varphi_k = \theta_k - \alpha$, and the center at the point $\zeta_k = |\zeta_k|e^{i\varphi_k}$. By E_0 denote the set of the points of the domain D_1 laying inside of all these circles. A circle of the radius δ_k with center at the point ζ_k at the map $z = \zeta e^{i\alpha}$ of the domain D_1 on $g_{\alpha,\alpha+\pi}$ passes in to a circle of the radius δ_k with the center at the point c_k; the set of circles E_0 passes in some set of circles E_* of the domain $g_{\alpha,\alpha+\pi}$.

In [9] is proved that if $f(\zeta)$, $\zeta \in D_1$ is a meromorphic function of the order $\rho_0 < +\infty$, then $\exists d = d(\varepsilon) \ge r_0$:

$$\Big|\frac{f'(\zeta)}{f(\zeta)}\Big| < \frac{|\zeta|^{2\rho_0+2+2\varepsilon/3}}{\sin^2\varphi}, \quad \zeta = re^{i\varphi} \in D_1 \setminus E_0, |\zeta| > d; \tag{64}$$
$$\sum_{r_0<|\zeta_k|<+\infty} |\zeta_k|^{-\rho_0-1-\frac{\varepsilon}{3}} \sin\varphi_k < c = \text{const}\,.$$

As $f(\zeta) = w(\zeta e^{i\alpha})$, $\zeta \in D_1$, so $|w'(z)/w(z)| = |f'(\zeta)/f(\zeta)|$, $re^{i\theta} = z = \zeta e^{i\alpha} = re^{i(\varphi+\alpha)}$, $\varphi = \theta - \alpha$, $\varphi_k = \theta_k - \alpha$, and from (64) it follows

$$|w'(z)/w(z)| < \frac{|z|^{2\rho_0+2+2\varepsilon/3}}{\sin^2(\theta-\alpha)}, \quad z = re^{i\theta} \in g_{\alpha,\alpha+\pi} \setminus E_*, |z| > d;$$
$$\sum_{\substack{r_0<|c_k|<+\infty,\\ \alpha\le\theta_k\le\alpha+\pi}} |c_k|^{-\rho_0-1-\frac{\varepsilon}{3}} \sin(\theta_k-\alpha) < c = \text{const}\,, \quad \rho_0 \le \rho \tag{65}$$

where E_* is the set of circles with centers at the points $c_k \in \{c_k\}$, $c_k \in g_{\alpha,\alpha+\pi}$ and with radii $\delta_k = |c_k|^{-\rho_0-1-\frac{\varepsilon}{3}} \sin(\theta_k - \alpha)$ whose sum is finite.

Consider an arbitrary branch $w(z)$, $z \in g_{\alpha\beta}$, $-\infty < \alpha < \beta < +\infty$, of the function $w(z)$, $z \in G$. For some $m, n \in \mathbb{Z}$ we have $\pi(m+1)/2 < \alpha < \beta < \pi(m+n-1)/2$. Let $\alpha_j = \pi j/2$, $j = m, m+1, \dots, m+n$. Consider the angular domain $g_{\alpha_j,\alpha_j+\pi} = \{z = re^{i\theta} : r_0 \le r < +\infty, \alpha_j \le \theta \le \alpha_j + \pi\}$ and a univalent branch connected to it, $w(z)$, $z \in g_{\alpha_j,\alpha_j+\pi}$. For this branch relations (65) are fulfilled, in which $\alpha = \alpha_j$, ρ_0 takes some value $\rho_0 = \rho_j$, probably, different for various $j = m, \dots, m+n$; $\rho_j \le \rho$. Instead of the set of circles E_* it is necessary to write the set E_j, defined by the following way: by $\{c_{kj}\}$ denote the set of zeros and poles of a branch $w(z)$, $z \in g_{\alpha,\alpha+\pi}$ and from each point $c_{kj} = |c_{kj}| \exp(i\theta_{kj}) \in \{c_{kj}\}$ as the a center, we draw the circle of the radius $\delta_{kj} = |c_{kj}|^{-\rho_j-1-\frac{\varepsilon}{3}} \sin(\theta_{kj} - \alpha)$. Then E_j is the set of points the part $g_{\alpha,\alpha+\pi}$ for the Riemann surface of the function $w(z)$, $z \in G$, belonging to all these circles. Besides that instead of c_k, θ_k in (65) in the considered case it is necessary to write c_{kj}, θ_{kj}. Choose some δ, $0 < \delta < \pi/4$. If $\alpha_j + \delta \le \theta \le \alpha_j + \pi - \delta$, $\alpha_j + \delta \le \theta_{kj} \le \alpha_j + \pi - \delta$, so $\sin(\theta-\alpha_j)$, $\sin(\theta_{kj}-\alpha_j) \geq \sin\delta > 0$ and from here and from (65) it follows

$$\Big|\frac{w'(z)}{w(z)}\Big| < \frac{|z|^{2\rho_j+2+\frac{2\varepsilon}{3}}}{\sin^2(\theta-\alpha_j)} < \frac{|z|^{2\rho_j+2+\frac{2\varepsilon}{3}}}{\sin^2\delta} < |z|^{2\rho_j+2+\varepsilon},$$
$$|z| \geq d_j, z \in g(j) = \{z = re^{i\theta} : \alpha_j+\delta \leq \theta \leq \alpha_j+\pi-\delta, r_0 \leq r\}, z \notin E_j,$$
$$\sin\delta \sum_{\substack{\alpha_j+\delta\leq\theta_{kj}\leq\alpha_j+\pi-\delta,\\ r_0\leq|c_{kj}|<+\infty}} |z|^{-\rho_j-1-\frac{\varepsilon}{3}} \leq \sum_{\substack{\alpha_j\leq\theta_{kj}\leq\alpha_j+\pi,\\ r_0\leq|c_{kj}|<+\infty}} |z|^{-\rho_j-1-\frac{\varepsilon}{3}} \sin(\theta_{jk}-\alpha_j) < c_j,$$
$$c_j = \text{const}\,, j = m, m+1, \ldots, m+n. \tag{66}$$

From each point $c_{kj} \in g(j)$ we draw a circle of the radius $\sigma_{kj} = |c_{kj}|^{-\rho_j-1-\frac{\varepsilon}{3}} > \delta_{kj}$. By E_j we denote a point set of a part $g(j)$ of the Riemann surface of the function $w(z)$, $z \in G$ belonging to all these circles. Taking into account (66), we obtain

$$\sum_{j=m}^{m+n} \sum_{\substack{\alpha_j+\delta\leq\theta_{kj}\leq\alpha_j+\pi-\delta,\\ r_0\leq|c_{kj}|<+\infty}} \sigma_{kj} < c = \text{const}\,, \quad \sigma_{kj} = |c_{kj}|^{-\rho_j-1-\frac{\varepsilon}{3}}. \tag{67}$$

From the definition of the numbers a_j, $j = m, m+1, \ldots, m+n$ it follows that a segment $[\alpha,\beta] \subset \bigcup_{j=m}^{m+n}[\alpha_j, \alpha_j+\pi]$, and part of the Riemann surface $g_{\alpha\beta} \subset \bigcup_{j=m}^{m+n} g(j)$ (the last sum we consider as the join of pieces of a Riemann surface $w(z)$, $z \in G$). Let E be the sum of the sets E_j, $j = m, \ldots, m+n$, considered on the mentioned Riemann surface. By (67) the sums of radii of the circles forming the set E is finite. As $g_{\alpha\beta} \subset \bigcup_{j=m}^{m+n} g(j)$, $E = \bigcup_{j=m}^{m+n} E(j)$, $\rho_j \leq \rho$, then (66), (67) imply the statement of Lemma 4. □

Proof of Theorem 2 is similar to that of asymptotic relations, reduced in [11], for the solution of d. e. (1) meromorphic in a half-plane and is based on application of Theorem 1 and Lemma 4.

Examples. The function $w(z) = \exp(\ln^2 z)$ is a solution of the d. e. $zw' = 2w\ln z$, for this solution the first statement of Theorem 2 is true. The Weierstrass function $\wp(z)$, $z \in \mathbb{C}$ is a solution of the d. e. $(w')^2 = 4w^3 + g_2 w + g_3$, $g_i = \text{const}$, [5, v. 2, p. 359], and the function $\wp(z) + \ln z$, $z \in \{z : |z| \geq r_0\}$ is the solution of the equation $(w' - z^{-1})^2 = 4(w - \ln z)^3 + g_2(w - \ln z) + g_3$. For the Weierstrass function $|\wp(z)| < |z|^\nu$, $z \in \mathbb{C} \setminus E$, $\nu = \text{const} > 0$, is true, where E is some set of circles with a finite sum of radii. Then for the function $\wp(z) + \ln z$ statement b) of Theorem 2 is true.

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