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ON GOLDBERG'S CHARACTERISTICS FOR MULTIPLE POWER SERIES

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The article is a review which contains properties of the maximal term, systems of central indices, and the growth characteristics of the sum of multiple power series introduced by A. A. Goldberg.

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Статья является обзором, в котором приводятся свойства максимального члена, систем центральных индексов и характеристик роста суммы кратных степенных рядов, введеных А. А. Гольдбергом.

1. Let

$$f(z) = \sum_{k \in \mathbb{Z}_+^n} a_k z^k = \sum_{k_1,\dots,k_n=0}^{\infty} a_{k_1\dots k_n} z_1^{k_1} \dots z_n^{k_n} \tag{1}$$

be an entire in $\mathbb{C}^n$ function. A. A. Goldberg and I. F. Bitlyan [1] considered for such function besides the maximum modulus $M(r)$ and maximum term $m(r) = \max\{|a_k| r^k : k \in \mathbb{Z}_+^n\}$ the system of central indices $\nu(r) = (\nu_1(r), \dots, \nu_n(r))$, $\nu_j(r) = \max\{k_j \in \mathbb{Z}_+ : |a_k| r^k = m(r)\}$, $j \in \overline{1, n}$ and generalized the well-known Wiman-Valiron theorems.

In articles [2] and [3] A. A. Goldberg defined new growth characteristics for entire functions of several variables, namely the $(G, \rho_1, \dots, \rho_n)$-orders, $(G, \rho_1, \dots, \rho_n)$-types, and also the systems of G-orders and G-types. In these works formulae for the mentioned characteristics are derived by means of Taylor coefficients of functions.

In this paper we describe the main properties of the mentioned characteristics.

2. In [4] a multiple power series

$$A(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k \tag{2}$$

with non-empty domain of convergence $\mathcal{D}_A$ is considered.

We define the maximum term $m_A \colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}_+$ by the equality $m_A(r) = \sup\{|a_k| r^k : k \in \mathbb{Z}^n\}$. Using the terminology and notations of [5] we give the main property of the trace function $x \to \ln m_A(e^x)$, $x \in \mathbb{R}^n$ ($e^x = (e^{x_1}, \dots, e^{x_n})$); of series (2) (this function can take the values $\pm\infty$ ($\ln 0 = -\infty$, $\ln(+\infty) = +\infty$)).

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Theorem 1. *An arbitrary function $C\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is a trace function of some series of form (2) with a non-empty domain of convergence if and only if*

1) *C is a closed proper convex function and $\operatorname{int}(\operatorname{dom}C) \neq \varnothing$ in $\mathbb{R}^n$;*
2) *$\nabla C(x) \in \mathbb{Z}^n$ in any point $x \in \mathbb{R}^n$, where C is differentiable;*
3) *there is a set $K \subset \mathbb{Z}^n$ such that $\operatorname{cl}(\operatorname{conv}K) = \operatorname{cl}(\operatorname{dom}C^*)$.*

In this case the following is true

$$\operatorname{cl}(\operatorname{dom}C^*) = \{x \in \mathbb{R}^n : (C\mathrm{O}^+)(y) = \langle x, y\rangle, \forall y \in \mathbb{R}^n\} =$$
$$= \{x \in \mathbb{R}^n : \sup_{z\in\operatorname{dom}C} [C(y+z) - C(z)] \geqslant \langle x, y\rangle, \forall y \in \mathbb{R}^n\}, \quad \langle x, y\rangle = x_1y_1 + \dots + x_ny_n.$$

Let $\Lambda_A = \{r = (|z_1|, \dots, |z_n|) : z \in \mathcal{D}_A, \quad m_A(r) > 0, \quad r_1 > 0, \dots, r_n > 0\}$. We list some of the main properties of the vector function $r \to \nu(r)$, $r \in \Lambda_A$ (see [4]):

1) $\nu(e^x) = \left(\left(\frac{\partial \ln m_A(e^x)}{\partial x_1}\right)_+, \dots, \left(\frac{\partial \ln m_A(e^x)}{\partial x_n}\right)_+\right)$, $e^x \in \Lambda_A$; and $\nabla \ln m_A(e^x) = \nu(e^x)$ on the set $\ln(\Lambda_A) = \{(\ln r_1, \dots, \ln r_n) : r \in \Lambda_A\}$, except points of some surfaces.
2) Let L be a polygon with ends x' and x'' which lays in the set $\ln(\Lambda_A)$ and in L (except finite number of points) the equation $m_A(r) = |a_{\nu(r)}|r^{\nu(r)}$ holds ($r = e^x$; such a polygon always exists). Then $\int_L \langle \nu(e^x), \mathrm{d}x\rangle = \ln m_A(e^{x''}) - \ln m_A(e^{x'})$.

3. Let G be a closed bounded n-circle region in $\mathbb{C}^n$; $\rho = (\rho_1, \dots, \rho_n) \in \hat{\mathbb{R}}^n$, $\sigma = (\sigma_1, \dots, \sigma_n) \in \hat{\mathbb{R}}^n$, where $\hat{\mathbb{R}}^n = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_n > 0\}$. We put

$$\Gamma(\alpha) = \Gamma(\alpha_1, \dots, \alpha_n) = \max\{|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n} : z \in G\}, \quad \alpha \in \mathbb{R}^n_+;$$
$$M_\rho(r) = \max_{(|z_1|^{\rho_1}/r, \dots, |z_n|^{\rho_n}/r) \in G} |f(z)|, \quad M_{\rho,\sigma}(r) = \max_{(\sigma_1|z_1|^{\rho_1}/r, \dots, \sigma_n|z_n|^{\rho_n}/r) \in G} |f(z)|.$$

A point $\rho \in \hat{\mathbb{R}}^n$ is called the system of G-orders of an entire function f if

$$\limsup_{z\to\infty} \frac{\ln\ln M_\rho(r)}{\ln r} = 1.$$

Let ρ be a system of G-orders of an entire function f. A point $\sigma \in \hat{\mathbb{R}}^n$ is called the system of G-types of f relative to the system ρ if $\limsup_{z\to\infty} \ln M_{\rho,\sigma}(r)/r = 1$. A. A. Goldberg [3] obtained the following formulae for systems of G-orders ρ and G-types σ relative to ρ for an entire function (1):

$$\limsup_{|k|\to\infty} \frac{\langle k, 1/\rho\rangle \ln|k|}{-\ln|a_k|} = 1, \quad 1/\rho = (1/\rho_1, \dots, 1/\rho_n), \quad |k| = k_1 + \dots + k_n, \tag{3}$$

$$\limsup_{|k|\to\infty} \left\{\frac{1}{e}\langle k, 1/\rho\rangle \left[\Gamma\Big(\frac{k}{\rho}\Big)|a_k|\sigma_1^{-k_1/\rho_1} \dots \sigma_n^{-k_n/\rho_n}\right]^{1/\langle k, 1/\rho\rangle}\right\} = 1, \quad \frac{k}{\rho} = (k_1/\rho_1, \dots, k_n/\rho_n). \tag{4}$$

In the case of

$$G = \{z \in \mathbb{C}^n : |z_1| + \dots + |z_n| \leqslant 1\} \tag{5}$$

equations (3) and (4) were derived earlier by M. M. Dzhrbashjan [6]. From (3) it is clear that the system of G-orders does not depend on G.

The shape of the hypersurface of all systems of orders or of all systems of G-types of the entire function f with condition (5) was studied by L. I. Ronkin [7]. Some generalizations of Ronkin's results are made in [8]. We study the shape of the hypersurface of systems of G-types of an entire function f corresponding to a fixed system of orders ρ by the method of directed characteristics [8]–[10]. Let

$$f(z) = \sum_{k\in K} a_k z^k, \quad a_k \neq 0, \quad k \in K \subset \mathbb{Z}_+^n, \tag{6}$$

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \pi(x) = x/\|x\|, \quad \Sigma = \{x \in \mathbb{R}^n : \|x\| = 1\}, \quad \Sigma_+ = \Sigma \cap \mathbb{R}_+^n,$$
$$\Sigma_K = \left\{\alpha \in \Sigma : \alpha = \lim_{m\to\infty} \pi(k^{(m)}), \quad k^{(m)} \in K, \quad \lim_{m\to\infty} |k^{(m)}| = \infty\right\}.$$

As in [9] we can introduce Φ- and (Φ, Ψ)-characteristics respectively.

For the sake of simplicity we rewrite (4) in the form:

$$\max_{\alpha\in\Sigma_K}\left\{\frac{1}{e}\left|\frac{\alpha}{\rho}\right| \limsup_{\pi(k)\to\alpha} \|k\| \left[|a_k|\,\Gamma\left(\frac{k}{\rho}\right)\right]^{1/(\|k\||\alpha/\rho|)} \exp\left[-\frac{1}{|\alpha/\rho|}\sum_{i=1}^n \frac{\alpha_i}{\rho_i}\ln\sigma_i\right]\right\} = 1,$$

or

$$\max_{\alpha\in\Sigma_K}\left\{\left|\pi\left(\frac{\alpha}{\rho}\right)\right| \ln\sigma_f(\alpha) - \left\langle \pi\left(\frac{\alpha}{\rho}\right), \ln\sigma\right\rangle\right\} = 0, \tag{7}$$

where

$$\sigma_f(\alpha) = \frac{1}{e}\left|\frac{\alpha}{\rho}\right| \limsup_{\pi(k)\to\alpha} \|k\| \left[|a_k|\,\Gamma\left(\frac{k}{\rho}\right)\right]^{1/(\|k\||\alpha/\rho|)}$$

is a directed G-type of a function f. Consider the mapping $\beta(\alpha) = \pi(\alpha/\rho)$, $\alpha \in \Sigma$. Obviously, β is a homeomorphism of Σ. Let us denote $\Sigma_K^\beta := \beta(\Sigma_K)$. Then equation (7) takes the form

$$\max_{\varphi\in\Sigma_K^\beta}\left\{|\varphi|\ln\sigma_f(\beta^{-1}(\varphi)) - \langle\varphi, \ln\sigma\rangle\right\} = 0. \tag{8}$$

The equation $\langle\varphi, \ln\sigma\rangle = |\varphi|\ln\sigma_f(\beta^{-1}(\varphi))$ is a normal equation of a hyperplane. For each $\varphi \in \Sigma_K^\beta$ it bounds a half-space in $\mathbb{R}^n$, the boundary of intersections of which is the image of the hypersurface of systems of G-types of f relative to the mapping $\sigma_i \to \ln\sigma_i$, $i \in \overline{1,n}$. On the grounds of the above one can obtain the following description for the hypersurface of systems of G-types.

Theorem 2. *A hypersurface $S \subset \hat{\mathbb{R}}^n$ is the hypersurface of systems of G-types relative to the system of orders $\rho \in \hat{\mathbb{R}}^n$ of an entire function f if and only if the inverse image $\eta^{-1}(S)$ of the hypersurface S with respect to the homeomorphism $\eta : \xi_i \to e^{\xi_i}$, $i \in \overline{1,n}$ coincides with the boundary of the closed convex set Q ($\varnothing \neq Q \neq \mathbb{R}^n$) which is complete in the direction $\left(-\Sigma_K^\beta\right)^\circ$ and satisfies the condition $E^Q \subset -\Sigma_K^\beta$. Here $\circ$ denotes the polar of set and E^Q is the set of those $\alpha \in \Sigma$, for which the supporting half-space $Q_\alpha = \{\xi \in \mathbb{R}^n : \langle\alpha, \xi\rangle \leqslant h(\alpha)\}$ is the extremal half-space of Q.*

4. Now we point out the dependence of the shape of the hypersurface of systems of G-types relative to the system of orders ρ on the local form of hypersurface of the system of orders. Let the image of the boundary in $\hat{\mathbb{R}}^n$ of the convex set H_ρ^f corresponding to mapping

$x_i \to 1/x_i$, $i \in \overline{1,n}$ coincide with the hypersurface of systems of orders of function f [7]. Denote by h_ρ the supporting function of H_ρ^f and by $h_{\rho,\sigma}$ the supporting function of the convex set $H_{\rho,\sigma}^f = Q$, from Theorem 2. We put also $\Omega^f(\rho') = \{\alpha \in \Sigma_+ : \langle \alpha, 1/\rho' \rangle = h_\rho(\alpha)\}$, $\Lambda^f(\rho') = \{\alpha \in \Sigma_+ : h_{\rho',\sigma}(-\alpha/\rho') < \infty\}$, $1/\rho' \in \partial H_\rho^f \cap \hat{\mathbb{R}}^n$. In the directions different from $\alpha \in \Omega^f(\rho')$ the coefficients of the function f converge to zero so quickly that the directed G-type relative to the system of orders ρ' equals to zero. Thus, the following theorem holds.

Theorem 3. *Let $1/\rho' \in \partial H_\rho^f \cap \hat{\mathbb{R}}^n$. The hypersurface S of systems of G-types relative to the system of orders ρ' of an entire function f has the property $\Lambda^f(\rho') \subset \Omega^f(\rho')$. In particular if $1/\rho'$ is a regular boundary point of H_ρ^f and the function f has a finite G-type relative to the system of orders ρ' then $\Lambda^f(\rho') = \Omega^f(\rho') = \{\alpha'\}$, where $\alpha' \in \Sigma_+$. In this case $\eta^{-1}(S)$ is a hyperplane.*

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