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MEROMORPHIC ALMOST PERIODIC FUNCTIONS

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We introduce a notion of meromorphic almost periodic function and study properties of this class of functions. In particular, we find a criterion for the product of meromorphic almost periodic functions to be also a meromorphic almost periodic function. We prove that every meromorphic almost periodic function is a quotient of two analytic almost periodic functions.

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Введено понятие мероморфной почти периодической функции, изучены свойства функций этого класса. В частности, найдено необходимое и достаточное условие того, что произведение двух мероморфных почти периодических функций опять является мероморфной почти периодической функцией. Доказано, что любая мероморфная почти периодическая функция является отношением двух аналитических почти периодических функций.

A function $f \in C(\mathbb{R})$ is called *almost periodic* (a.p.) if for every $\varepsilon > 0$ the set of ε-almost periods

$$E_\varepsilon(f) = \{\tau \in \mathbb{R} : |f(x+\tau) - f(x)| < \varepsilon, \quad \forall x \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$. The latter means that for some $L > 0$ and for every $a \in \mathbb{R}$ the segments $[a, a+L]$ have common points with the set $E_\varepsilon(f)$. An analytic function $f(z)$ on a horizontal strip S is said to be *analytic a.p. function* if for every $\varepsilon > 0$ and every substrip $S' \Subset S$ the set of ε, S'-almost periods

$$E_{\varepsilon,S'}(f) = \{\tau \in \mathbb{R} : |f(z+\tau) - f(z)| < \varepsilon, \quad \forall z \in S'\}$$

is relatively dense in $\mathbb{R}$. (For $S = \{z \in \mathbb{C} : a < |\operatorname{Im} z| < b\}$, $-\infty \le a < b \le \infty$, $S' = \{z \in \mathbb{C} : \alpha < |\operatorname{Im} z| < \beta\}$ the enclose $S' \Subset S$ means that $a < \alpha < \beta < b$.)

Theory of analytic a.p. functions was constructed by H. Bohr, K. Bush, B. Jessen; for its detailed presentation, see [1, 2]. The further development of this theory is closely connected with the names of M. G. Krein, B. Ja. Levin, V. P. Potapov, H. Tornehave, L. I. Ronkin ([3]–[10]).

Let us give the following definition:

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A meromorphic function $f(z)$ on a strip S (of finite or infinite width) is called *meromorphic a.p. function* on this strip if for each substrip $S' \Subset S$ and each $\varepsilon > 0$ the set of ε, S'-almost periods

$$E_{\varepsilon,S'}(f) = \{\tau \in \mathbb{R} : \rho(f(z+\tau), f(z)) < \varepsilon, \quad \forall z \in S'\}$$

is relatively dense in $\mathbb{R}$; by ρ we denote the spherical metric on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Notice that analytic a.p. functions are bounded on each substrip $S' \Subset S$ and the spherical metric is equivalent to the Euclidean one on any bounded set; therefore any analytic a.p. function is also a meromorphic a.p. function. It is easy to see that any uniformly continuous function on $\overline{\mathbb{C}}$ takes meromorphic a.p. functions to meromorphic a.p. functions. Hence each linear-fractional transformation takes a meromorphic a.p. function (in particular, an analytic a.p. function) to a meromorphic a.p. function.

We show that the class of meromorphic a.p. functions is not closed under the operations of addition and multiplication. Nevertheless the class of meromorphic a.p. functions inherits many properties of that of analytic a.p. functions: the uniform limit of meromorphic a.p. functions is a meromorphic a.p. function; zeros and poles of meromorphic a.p. functions form almost periodic sets; Bochner's criterion of almost periodicity is valid for meromorphic a.p. functions. Also, we show that every meromorphic a.p. function is a quotient of analytic a.p. functions; this result is based on the theorems about a.p. discrete sets from [11, 12]. Finally, using methods of [13, 14], we give a criterion for a pair of a.p. sets to be the zero set and the pole set of some meromorphic a.p. function.

At first we prove the following simple statement.

Theorem 1. *A meromorphic a.p. function $f(z)$ on a strip S is uniformly continuous with respect to the metric ρ on any substrip $S' \Subset S$.*

Proof. Take $\varepsilon > 0$. Let L be a positive number such that each segment $[a, a+L]$, $a \in \mathbb{R}$, contains an ε, S'-almost period of f. A meromorphic function is uniformly continuous with respect to the spherical metric on every compact set, therefore for some $\delta > 0$ and any $z, z' \in S' \cap \{z : |\mathrm{Re}\, z| < L+1\}$ such that $|z - z'| < \delta$ we have $\rho(f(z), f(z')) < \varepsilon$. If z, z' are arbitrary points in S' and $|z - z'| < \delta$, then there exists $\tau \in E_{\varepsilon,S'}(f)$ such that $|\mathrm{Re}\,(z+\tau)| < L+1$, $|\mathrm{Re}\,(z'+\tau)| < L+1$. We get

$$\rho(f(z), f(z')) \le \rho(f(z), f(z+\tau)) + \rho(f(z+\tau), f(z'+\tau)) + \rho(f(z'+\tau), f(z')) < 3\varepsilon.$$

□

Corollary 1. *If f is a meromorphic a.p. function and N, P its zero set and pole set respectively, then we have*

$$\inf\{|z - z'| : z \in N \cap S',\, z' \in P \cap S'\} \ge \delta(S', f) > 0 \quad \forall S' \Subset S. \tag{1}$$

Proof. It follows from Theorem 1 that there are no sequences z_n, $z'_n \in S'$ such that $f(z_n) = 0$, $f(z'_n) = \infty$ and $|z_n - z'_n| \to 0$ as $n \to \infty$. □

We shall say that some property is valid *inside* S if it is valid on any substrip $S' \Subset S$. In particular, we shall say that sets N and P are *separated* inside S if they satisfy (1).

Now we can give examples of meromorphic a.p. functions such that their sum or product is not a meromorphic a.p. function.

Take $f_1(z) = \sin\sqrt{2}\pi z$, $f_2(z) = 1/\sin\pi z$. By Kronecker's theorem (see for example [2]), for any $\delta > 0$ there exist $t \in \mathbb{R}$ and $m, n \in \mathbb{Z}$ such that $|t - n| < \delta$, $|t/\sqrt{2} - m| < \delta$. the points $z' = n/\sqrt{2}$ and $z'' = m$ belong to the set of zeros and the set of poles, respectively, for the product $f_1(z)f_2(z) = \sin\sqrt{2}\pi z/\sin\pi z$. Furthermore, we have $|n/\sqrt{2} - m| \le |n/\sqrt{2} - t/\sqrt{2}| + |t/\sqrt{2} - m| \le (1 + 1/\sqrt{2})\delta$. Since the choice of δ is arbitrary, zeros and poles of $f_1(z)f_2(z)$ are not separated and $f_1(z)f_2(z)$ is not a meromorphic a.p. function. By the same argument, the sum $1/\sin\pi z + 1/\sin(2\sqrt{2} - 1)\pi z$ is not a meromorphic a.p. function, too.

However the class of meromorphic a.p. functions is closed with respect to the uniform convergence:

Theorem 2. *If a sequence of meromorphic a.p. functions $f_n(z)$ on S converges uniformly inside S, then the limit function $f(z)$ is a meromorphic a.p. function on S.*

Proof. It can easily be checked that the uniform limit of meromorphic functions with respect to the spherical metric is also a meromorphic function. Now let n be large enough and $\tau \in E_{\varepsilon,S'}(f_n)$, then we have

$$\rho(f(z+\tau), f(z)) \le \rho(f(z+\tau), f_n(z+\tau)) + \rho(f_n(z+\tau), f_n(z)) + \rho(f_n(z), f(z)) < 3\varepsilon.$$

Hence $f(z)$ is a meromorphic a.p. function on S. □

The following result (Bochner's criterion) is very useful for the sequel.

Theorem 3. *The following conditions are equivalent:*

(i) $f(z)$ is a meromorphic a.p. function on S;

(ii) for any sequence of real numbers t_n there exists a subsequence t'_n such that the sequence of functions $f(z + t'_n)$ converges uniformly inside S.

Proof. The proof of this theorem is as same as that of Bochner's criterion for a.p. functions on the axis (see, for example, [2]). □

Note that it follows from theorem 2 that the sequence of functions $f(z + t'_n)$ converges to a meromorphic a.p. function inside S.

Let us prove the following statement.

Theorem 4. *Let $f(z)$ be a meromorphic a.p. function on S and let P_r be the union of the disks of radius r with the centers at the poles of $f(z)$, then $f(z)$ is bounded on the set $S' \setminus P_r$ for any substrip $S' \Subset S$.*

Proof. Assume the contrary. Then there exists a sequence of points $z_n = x_n + iy_n \in S' \setminus P_r$ such that $f(z_n) \to \infty$. Taking into account Theorem 3, we may assume that the sequence of functions $f(z+x_n)$ converges uniformly with respect to the spherical metric to a meromorphic a.p. function $g(z)$ inside S and the points iy_n converge to $iy_0 \in S$. By uniform continuity of $f(z)$ we get that the functions $f(z + z_n - iy_0)$ also converge uniformly to $g(z)$. In particular, the point iy_0 is a pole of $g(z)$. Suppose B_0 is a disk of radius $r' < r$ with the center iy_0 such that $\overline{B_0} \subset S$ and $g(z)$ has no zeros on $\overline{B_0}$ and no poles on ∂B_0; then $|g(z)| \ge \alpha > 0$ for $z \in \overline{B_0}$. Hence the uniform convergence of the functions $f(z + z_n - iy_0)$ to $g(z)$ with respect to the spherical metric implies the convergence of the functions $1/f(z + z_n - iy_0)$ to $1/g(z)$ with respect to the Euclidean metric. Hurwitz' theorem yields that the functions $f(z + z_n - iy_0)$ have poles w_n in B_0 for n large enough. So the points $w_n + z_n - iy_0$ are poles of $f(z)$ and $|w_n + z_n - iy_0 - z_n| < r$. This contradicts to the choice of z_n. □

Let N_r be the union of the disks of radius r with the centers at the zeros of f. Applying Theorem 4 to the function $1/f$, we get the inequality

$$\inf\{|f(z)| : z \in S' \setminus N_r\} > 0.$$

Corollary 2. *If a meromorphic a.p. function has no poles on S, then it is an analytic a.p. function on S.*

Proof. By the previous theorem, the function $|f(z)|$ is bounded on each substrip $S' \Subset S$. At the same time the spherical metric is equivalent to the Euclidean one on any bounded set. □

Now we can give a simple condition for the product of meromorphic a.p. functions to be a meromorphic a.p. function.

Theorem 5. *Let $f_1(z)$, $f_2(z)$ be meromorphic a.p. functions. A necessary and sufficient conditions for the product $f_1(z)f_2(z)$ to be a meromorphic a.p. function is that zeros and poles of this product be separated inside S.*

Proof. The necessity follows from Corollary 1. Let us prove the sufficiency. Taking into account Theorem 3, we shall show that the uniform convergence of $f_1(z + t_n)$ to $g_1(z)$ and $f_2(z + t_n)$ to $g_2(z)$ (with respect to the spherical metric) inside S implies the uniform convergence of the functions $f_1(z + t_n)f_2(z + t_n)$ to the function $g_1(z)g_2(z)$.

First we shall show that the distance between zeros and poles of the function $g_1(z)g_2(z)$ in each substrip $S' \Subset S$ equals the distance between zeros and poles of the function $f_1(z)f_2(z)$.

Suppose $g_1(z')g_2(z') = 0$, $g_1(z'')g_2(z'') = \infty$, z', $z'' \in S' \Subset S$. Let $C(z')$, $C(z'')$ be the circles of radius δ with the centers at the points z', z'' respectively such that no zeros and poles of the functions $g_1(z)$, $g_2(z)$ lie either on these circles or inside these circles, except for the centers. It can be assumed also that $C(z') \subset S'$, $C(z'') \subset S'$. Using Hurwitz' theorem for the functions $f_1(z+t_n)$ and $1/f_1(z+t_n)$ we obtain that for n large enough, each function $f_1(z + t_n)$ has just the same number of zeros and poles (with the multiplicity) inside the circles $C(z')$, $C(z'')$ as the function $g_1(z)$ has at the points z', z'' respectively. the same assertion is true for the functions $f_2(z + t_n)$ and $g_2(z)$. Hence the number of zeros of the product $f_1(z + t_n)f_2(z + t_n)$ inside the circle $C(z')$ is greater than the number of its poles. Conversely, the number of poles of this product inside the circle $C(z'')$ is greater than the number of its zeros. It follows that the function $f_1(z + t_n)f_2(z + t_n)$ has at least one zero inside the circle $C(z')$ and at least one pole inside the circle $C(z'')$. Therefore the distance between zeros and poles of the product $f_1(z)f_2(z)$ is not greater than $|z' - z''| + 2\delta$ with an arbitrary small δ.

On the other hand, the uniform convergence of $g_1(z - t_n)$ to $f_1(z)$ and $g_2(z - t_n)$ to $f_2(z)$ inside S implies that the distance between zeros and poles of the product $g_1(z)g_2(z)$ is not greater than $|w' - w''| + 2\delta$ for an arbitrary zero $w' \in S'$ and a pole $w'' \in S'$ of the product $f_1(z)f_2(z)$.

Furthermore, suppose our theorem is not true. Then there exist $\gamma > 0$ and sequence of points $z_n = x_n + iy_n \in S' \Subset S$ such that

$$\rho(f_1(z_n + t_n)f_2(z_n + t_n), g_1(z_n)g_2(z_n)) \geq \gamma. \tag{2}$$

Note that $g_1(z)$, $g_2(z)$ are meromorphic a.p. functions. Hence we can assume without loss of generality that the functions $g_j(z + x_n)$ converge uniformly inside S to meromorphic a.p. functions $h_j(z)$, $j=1,2$, respectively and the points iy_n converge to a point $iy_0 \in S$. It is clear that the functions $f_j(z+t_n+x_n)$ converge uniformly as $n\to\infty$ to the same functions $h_j(z)$, $1,2$. Since the functions $f_j(z)$ are uniformly continuous, we see that $f_j(z+z_n+t_n-iy_0)$ also converge uniformly inside S to the functions $h_j(z)$, $j=1,2$.

If the both functions $h_j(z)$ are finite at the point iy_0, then the sequences of functions $f_j(z+z_n+t_n-iy_0)$ and $g_j(z+z_n-iy_0)$, $j=1,2$, are bounded at this point. Therefore the sequences of numbers $f_1(z_n+t_n)f_2(z_n+t_n)$ and $g_1(z_n)g_2(z_n)$ have the common limit $h_1(iy_0)h_2(iy_0)$. this contradicts inequality (2).

If the both functions $h_j(z)$ are non vanishing at the point iy_0, then the sequences of functions $1/f_j(z+z_n+t_n-iy_0)$ and $1/g_j(z+z_n-iy_0)$, $j=1,2$, are bounded at this point. therefore the sequences of numbers $1/[f_1(z_n+t_n)f_2(z_n+t_n)]$ and $1/[g_1(z_n)g_2(z_n)]$ have the common limit $1/[h_1(iy_0)h_2(iy_0)]$. This also contradicts to inequality (2).

Now suppose that the function $h_1(z)$ has a zero of multiplicity k at the point iy_0 and the function $h_2(z)$ has a pole of multiplicity $p\le k$ at the same point. Let C be a circle $\{z:\ |z-iy_0|=\delta\}$ such that $C\subset S$ and the functions h_1, h_2 have neither zeros nor poles on the set $\{z:\ 0<|z-iy_0|\le\delta\}$. Hurwitz' theorem yields that each function $f_1(z+z_n+t_n-iy_0)$, $g_1(z+z_n-iy_0)$ has exactly k zeros inside C for $n>n_1$ and so has $g_1(z+z_n-iy_0)$. For the same reason, the functions $f_2(z+z_n+t_n-iy_0)$, $g_2(z+z_n-iy_0)$ have exactly p poles inside C for $n>n_2$. Since zeros and poles of the functions $f_1(z)f_2(z)$ and $g_1(z)g_2(z)$ are separated inside S, we see that only zeros of the products $f_1(z+x_n+t_n)f_2(z+x_n+t_n)$, $g_1(z+x_n)g_2(z+x_n)$ can be in the disk $\{z:\ |z-iy_0|<\delta\}$ for δ small enough.

Further, the moduli of the functions $f_1(z+x_n+t_n)$, $f_2(z+x_n+t_n)$, $g_1(z+x_n)$, $g_2(z+x_n)$ are bounded from above and bounded away from zero uniformly on the circle C for n large enough. Hence for all $z\in C$ and $n>n_3$,

$$|f_1(z+z_n+t_n-iy_0)f_2(z+z_n+t_n-iy_0)-g_1(z+z_n-iy_0)g_2(z+z_n-iy_0)|>\gamma.$$

By the Maximum Principle, we obtain that the same assertion is valid for $z=iy_0$. This also contradicts to inequality (2).

The same argument for the functions $1/[f_1(z_n+t_n)f_2(z_n+t_n)]$ and $1/[g_1(z_n)g_2(z_n)]$ leads to the contradiction with (2) in the case $k<p$. □

Corollary 3. *The quotient of two analytic a.p. functions on S is a meromorphic a.p. function on S iff zeros and poles of this quotient are separated inside S.*

For studying zeros and poles of meromorphic a.p. functions we use the concept of divisor.

A divisor d in the domain $G\subset\mathbb{C}$ is a set of pairs $\{(a_n,k_n)\}$ such that the support of the divisor $\operatorname{supp} d_n=\{a_n\}$ is a discrete set in G without limit points in G and *multiplicities* k_n are numbers from $\mathbb{Z}\setminus\{0\}$. *The divisor d_f of a meromorphic function f* on G is the set $\{(a_n,k_n)\}$ such that a_n are zeros or poles of $f(z)$ and $|k_n|$ are multiplicities of zeros and poles, with $k_n>0$ for zeros and $k_n<0$ for poles. A divisor $\{(a_n,k_n)\}$ can be identified with the charge concentrated on the set of points a_n with masses k_n; the charge for the divisor of a meromorphic function f is equal to $(1/2\pi)\triangle\log|f|$, where the Laplace operator is considered in the sense of distributions. The sum of divisors $d=\{a_n,k_n\}$, $d'=\{a'_n,k'_n\}$ is a divisor with the support in $\{a'_n\}\cup\{a_n\}$ and the corresponding multiplicities; in particular, the multiplicity at the point $a_m=a'_n\in\operatorname{supp} d\cap\operatorname{supp} d'$ is equal to $k_m+k'_n$. It is clear that

$d_{fg} = d_f + d_g$. Further, let $d = \{(a_n, k_n)\}$ be a divisor; define $|d| = \{(a_n, |k_n|) : (a_n, k_n) \in d\}$, $d^+ = \{(a_n, |k_n|) : (a_n, k_n) \in d,\ k_n > 0\}$, $d^- = \{(a_n, |k_n|) : (a_n, k_n) \in d,\ k_n < 0\}$. Note that $|d| = d^+ + d^-,\ d + d^- = d^+$.

A divisor d in a strip S is called *almost periodic* if for any smooth (infinite differentiable) function $\chi(z)$ with the support in S the sum $\sum k_n \chi(a_n + t)$, i.e. the convolution of the charge d with the function χ, is an a.p. function of $t \in \mathbb{R}$ (see [10]).

A divisor $d = \{(a_n, k_n)\}$ is *positive* if $k_n > 0$ for all n; this divisor can be identified with the sequence $\{b_m\}$ of points $\{a_n\}$ where each a_n appears k_n times. The sequence b_m in S is said to be *almost periodic* if for any $S' \Subset S$ and $\varepsilon > 0$ the set

$$E_{\varepsilon,S'} = \{t \in \mathbb{R} :\ \text{there exists a bijection } \alpha\colon \mathbb{N} \to \mathbb{N} \\ \text{such that } b_m \in S' \ \&\ b_{\alpha(m)} \in S' \Rightarrow |b_m + t - b_{\alpha(m)}| < \varepsilon\}$$

is relatively dense in $\mathbb{R}$ (see [5] and, in a special case, [3]). This definition is equivalent to the definition of almost periodicity for the corresponding positive divisor (see [11, 12]).

It was proved in [5] that the divisor d_f of every analytic a.p. function f is almost periodic. We extend this result to meromorphic f.p. functions:

Theorem 6. *Suppose f is a meromorphic a.p. function on a strip S; then its divisor d_f, divisor of zeros d_f^+, divisor of poles d_f^-, and divisor $|d|$ are almost periodic.*

Proof. We need the following simple generalization of Lemma 3.1 from [10]:

Lemma 1. *Let g, f_n, $n \in \mathbb{N}$, be meromorphic a.p. functions on a domain $G \subset \mathbb{C}$. If $\rho(f_n(z), g(z)) \to 0$ uniformly on compact subset of G, then the functions $\log|f_n|$ considered as the distributions on G converge to $\log|g|$, and the charges d_{f_n} considered as the distributions on G converge to the charge d_g.*

Proof. It suffices to check the convergence of the functions $\log|f_n|$ to $\log|g|$ on a neighborhood of each point $z' \in G$. If $g(z') \neq \infty$, then $g(z)$ is bounded on some neighborhood U of z'. Hence the functions $f_n(z)$ converge uniformly on U to $g(z)$ with respect to the Euclidean metric. Using Lemma 3.1 from [10] we get the convergence of the distributions $\log|f_n|$ to $\log|g|$ on U. If $g(z') = \infty$, then the functions $1/f_n(z)$ converge uniformly on some neighborhood of z' to $1/g(z)$ with respect to the Euclidean metric and we can use Lemma 3.1 from [10] again. Note that $d_{f_n} = (1/2\pi)\triangle \log|f_n|$ and $d_g = (1/2\pi)\triangle \log|g|$; since the differentiation keeps the convergence in the sense of distributions, we obtain the last assertion of the lemma. □

We continue the proof of the theorem. Let us show that the convolution

$$(d_f * \chi)(t) = \sum_n k_n \chi(a_n + t)$$

of the charge $d_f = \{a_n,\, k_n\}$ with an infinite differentiated function $\chi(z)$ with the support in S is an a.p. function of $t \in \mathbb{R}$.

Let $\{s_n\}$ be a sequence of real numbers. Taking into account Theorem 3, we may assume that the functions $f(s_n + z)$ converge, with respect to the spherical metric uniformly inside S, to a meromorphic a.p. function $g(z)$. Let us check that the functions $(d_f * \chi)(s_n + t)$ converge uniformly on $t \in \mathbb{R}$ to the function $(d_g * \chi)(t)$.

Assume the contrary. Then for any $\delta > 0$ there exists a sequence $t_{n'} \in \mathbb{R}$ such that

$$|(d_f * \chi)(s_{n'} + t_{n'}) - (d_g * \chi)(t_{n'})| \geq \delta. \tag{3}$$

As before, it can be assumed that the functions $f(s_{n'} + t_{n'} + z)$ converge, with respect to the spherical metric uniformly inside S, to a meromorphic a.p. function $h(z)$ and so do the functions $g(t_{n'} + z)$. By the lemma, it follows that the divisors of the functions $f(s_{n'} + t_{n'} + z)$ converge in sense of distributions to the divisor d_h and so do the divisors of the functions $g(t_{n'} + z)$. therefore the functions of t $(d_f * \chi)(s_{n'} + t_{n'} + t)$, $(d_g * \chi)(t_{n'} + t)$ converge to the same function $(d_h * \chi)(t)$. This contradicts (3).

Now it follows from Bochner's criterion for a.p. functions on the axis (see [2]) that $(d_f * \chi)(t)$ is an a.p. function. Since this statement is true for all smooth and supported in S functions $\chi(z)$, we see that d_f is an a.p. divisor.

Further, let $\phi(z)$ be a nonnegative smooth function with the support in a disk $B_r \subset S' \Subset S$ of radius r. Since zeros and poles of f are separated inside S, we see that for $r < r(S')$ and for every $t \in \mathbb{R}$ the support of the function $\phi(z + t)$ does not contains simultaneously zeros and poles of $f(z)$. Hence we have $(d_f^+ * \phi)(t) = \max\{(d_f * \phi)(t), 0\}$. Consequently the function $(d_f^+ * \phi)(t)$ is an a.p. function of t. Since any smooth function with support in $S' \Subset S$ is a linear combination of smooth functions with supports in disks of radius $r < r(S')$, we see that d_f^+ is an a.p. divisor. Evidently, $d_f^- = d_f + d_f^+$ and $|d_f| = d_f + 2d_f^+$ are also a.p. divisors. □

Corollary 4. *Numbers of zeros and poles of meromorphic a.p. function inside the rectangle*

$$\Pi_1(S', t) = \{z \in S' : |\mathrm{Re}\, z - t| < 1\}, \quad t \in \mathbb{R},\ S' \Subset S,$$

are uniformly bounded from above by a constant depending on S' only.

Proof. We shall prove that the numbers $|d|(\Pi_1(S', t))$ are bounded uniformly with respect to $t \in \mathbb{R}$. Let $\phi(z)$ be a nonnegative such that $\chi(z) = 1$ on the set $\Pi_1(S', 0))$. Since the convolution $(\chi * |d|)(t)$ is an a.p. function on $\mathbb{R}$, we see that this convolution is bounded. therefore the numbers $|d|(\Pi_1(S', t)) \leq (\chi * |d|)(t)$ are bounded uniformly with respect to $t \in \mathbb{R}$. □

The following theorem with Corollary 3 gives the complete description of meromorphic a.p. functions.

Theorem 7. *Any meromorphic a.p. function $f(z)$ on a strip S is a quotient of two analytic a.p. functions on S.*

Proof. By Theorem 6, poles of f form the a.p. divisor d_f^-. It follows from [11, 12] that there exists an a.p. divisor d' in S such that $d^- + d'$ is the divisor of some a.p. analytic function $h(z)$ on S. Then the function $g(z) = h(z)f(z)$ is a meromorphic function without poles. By Theorem 5 and Corollary 2, we obtain that $g(z)$ is an analytic a.p. function. So we have $f = g/h$. □

Corollary 5. *Let $f_1(z)$, $f_2(z)$ be meromorphic a.p. functions. A necessary and sufficient conditions for the sum $f_1(z) + f_2(z)$ to be a meromorphic a.p. function is that zeros and poles of this sum be separated inside S.*

Proof. It follows from Theorem 7 that the sum $f_1(z) + f_2(z)$ is a quotient of two analytic a.p. functions; so the assertion follows from Corollary 3. □

Consider the problem of realizability of an a.p. divisor as the divisor of a meromorphic a.p. function. this problem was solved in [13, 14] for positive a.p. divisors and analytic a.p. functions by the methods of cohomology theory. Namely, it was proved that to each positive a.p. divisor d in a strip S a class of Čech cohomology $c(d)$ of the group $H^2(K_{\mathbb{R}}, \mathbb{Z})$ is assigned, $K_{\mathbb{R}}$ being the universal Bohr compactification of $\mathbb{R}$; $c(d) = 0$ iff d is the divisor of an a.p. function on S, and $c(d_1 + d_2) = c(d_1) + c(d_2)$. Moreover, the element $c(d)$ remains the same for the restriction of d to any $S' \subset S$ and for the image of d under every homeomorphism of S onto $\tilde{S}$ of the form

$$\Gamma(x + iy) = x + i\gamma(y). \tag{4}$$

We do not give the definition of the Bohr compactification here (one can find it in [15]). We need only that the group $H^2(K_{\mathbb{R}}, \mathbb{Z})$ can be identified with the factor group $\mathbb{R} \wedge_{\mathbb{Z}} \mathbb{R} = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\{a \otimes a : a \in \mathbb{R}\}$ (see [16]). For example, the element $c(d) = \lambda \wedge \mu \in \mathbb{R} \wedge_{\mathbb{Z}} \mathbb{R}$ corresponds to the a.p. divisor $d^{\lambda\mu}$ with the support $\{(\lambda+i\mu)^{-1}n_1+i(\lambda+i\mu)^{-1}n_2 : n_1, n_2 \in \mathbb{Z}\}$ with the multiplicities $k_{n_1,n_2} = 1$. If λ/μ is rational, then $c(d) = 0$; otherwise, $c(d^{\lambda\mu}) \neq 0$ and the divisor $d^{\lambda\mu}$ is the divisor of no analytic a.p. function; this coincides with the corresponding result of [6].

Theorem 8. *A divisor d in a strip S is the divisor of a meromorphic a.p. function on S if and only if the following conditions are fulfilled:*

i) the supports of the divisors d^+ and d^- are separated inside S,

ii) d^+, d^- are a.p. divisors,

iii) $c(d^+) = c(d^-)$.

Proof. If $d = d_f$ for some meromorphic a.p. function on S, then i) follows from Corollary 1, ii) follows from Theorem 6. Now by Theorem 7 we have $f = g/h$, where g, h are analytic a.p. function on S, therefore $c(d_g) = c(d_h) = 0$. Since $d_g = d_f^+ + d'$, $d_h = d_f^- + d'$ for any a.p. divisor d' in S, we obtain iii).

On the other hand, suppose i), ii), iii) are fulfilled, and $c(d^+) = \lambda \wedge \mu$. We have

$$c(d^+ + d^{\mu\lambda}) = c(d^+) + \mu \wedge \lambda = 0,$$

hence there exists an analytic a.p. function $g(z)$ on S such that $d_g = d_f^+ + d^{\mu\lambda}$. By iii) we have

$$c(d^- + d^{\mu\lambda}) = c(d^+) + \mu \wedge \lambda = 0,$$

therefore $d^- + d^{\mu\lambda} = d_h$ for some analytic a.p. function $h(z)$ on S. It follows from i) that $f = g/h$ is a meromorphic a.p. function on S. Finally, $d_f = d_g - d_h = d$. □

Corollary 6. *Suppose d is a divisor in S such that conditions i) and ii) of Theorem 8 are fulfilled. If the restriction of d to $S' \subset S$ is the divisor of some meromorphic a.p. function on S', then d is the divisor of a meromorphic a.p. function on S; if Γ is a homeomorphism of S onto $\tilde{S}$ of the form (4) and d is the divisor of some meromorphic a.p. function on S, then Γd is the divisor of some meromorphic a.p. function on $\tilde{S}$.*

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