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N. V. Zorii

ON EXISTENCE OF A CONDENSER MEASURE

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A criterion for the existence of a condenser measure in a locally compact space is obtained.

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Получен критерий существования меры конденсатора в локально компактном пространстве.

1. Introduction

Let $\mathbf{X}$ be a locally compact Hausdorff space. By a *kernel* on $\mathbf{X}$ a lower semicontinuous function $\kappa\colon \mathbf{X}\times\mathbf{X}\to[\,0,+\infty\,]$ is meant. We also assume that κ is symmetric, i. e. $\kappa(x,y)=\kappa(y,x)$, $\forall\, x,\ y\in\mathbf{X}$.

Referring to [1, 2] for an exposition of the theory of potentials in a locally compact space, we restrict ourselves to listing the following basic concepts.

Let $\mathfrak{M}$ be the class of all real-valued Radon measures μ on $\mathbf{X}$. For a (Radon) measure μ, we use the canonical decomposition $\mu=\mu^1-\mu^0$, $\mu^0,\ \mu^1\geqslant 0$, and write $|\mu|:=\mu^0+\mu^1$. The *potential* of $\mu\in\mathfrak{M}$ at a point $x\in\mathbf{X}$ is defined by

$$\kappa(x,\mu):=\int\kappa(x,y)\,d\mu(y)=\kappa(x,\mu^1)-\kappa(x,\mu^0)$$

provided $\kappa(x,\mu^1)$ and $\kappa(x,\mu^0)$ are not both infinite. (In particular, the potential of a nonnegative measure is defined everywhere and represents a nonnegative lower semicontinuous function on $\mathbf{X}$.)

The *mutual energy* of two measures μ and ν is defined by

$$\kappa(\mu,\nu):=\int\kappa(x,y)\,d(\mu\otimes\nu)(x,y)=\kappa(\mu^0,\nu^0)+\kappa(\mu^1,\nu^1)-\kappa(\mu^0,\nu^1)-\kappa(\mu^1,\nu^0)$$

provided $\kappa(\mu^0,\nu^0)+\kappa(\mu^1,\nu^1)$ or $\kappa(\mu^0,\nu^1)+\kappa(\mu^1,\nu^0)$ is finite; thus in particular if $\mu\geqslant 0$ and $\nu\geqslant 0$. Since κ is assumed to be symmetric, Fubini's theorem implies $\kappa(\mu,\nu)=\int\kappa(x,\nu)\,d\mu(x)=\int\kappa(x,\mu)\,d\nu(x)$ whenever $\kappa(\mu,\nu)$ is defined. For $\nu=\mu$ we obtain the *energy* $\kappa(\mu,\mu)$ of μ.

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For a set $B \subset \mathbf{X}$, let $\mathfrak{M}^+(B)$ be the class of all nonnegative measures μ supported by B. Write $w(B) := \inf\ \kappa(\mu,\mu)$, the infimum being taken over the class $\mathfrak{M}^+(B,1) := \bigl\{\mu \in \mathfrak{M}^+(B) :\ \mu(\mathbf{X}) = 1\bigr\}$.

The *interior capacity* of B is defined by $\operatorname{cap} B := [\,w(B)\,]^{-1}$; it is countably subadditive on sets which are measurable with respect to every measure on $\mathbf{X}$ [1].

The sets $N \subset \mathbf{X}$ such that $\operatorname{cap} N = 0$ play an important role as negligible sets. A proposition involving a variable point $x \in Q$ (where Q denotes a given subset of $\mathbf{X}$) is said to subsist *nearly everywhere* (n.e.) in Q if $\operatorname{cap} N = 0$, N being the set of all points of Q for which the proposition fails to hold.

The following property [1] is useful: $\operatorname{cap} N = 0$ if and only if $\mu_*(N) = 0$ for every $\mu \in \mathfrak{M}^+(\mathbf{X})$ of finite energy (where $\mu_*(\,\cdot\,)$ denotes the interior μ–measure of a set).

Lemma 1. [1] *For every compact set $K \subset \mathbf{X}$ with $w(K) < +\infty$, there exists a measure $\lambda \in \mathfrak{M}^+(K,1)$ such that $\kappa(\lambda,\lambda) = w(K)$.*

2. A condenser measure

Let A_1 and A_0 be a couple of nonintersecting nonempty closed sets in $\mathbf{X}$. The ordered pair $\mathcal{A} := (\mathcal{A}_\infty, \mathcal{A}_\prime)$ is called a *condenser*; $\mathcal{A}$ is said to be *compact* if the set $A := A_0 \cup A_1$ is compact. Given a condenser $\mathcal{A}$, let $\mathfrak{M}(\mathcal{A})$ be the class of all measures μ such that $\mu^i \in \mathfrak{M}^+(A_i)$, $i = 0,\,1$.

Definition 1. [3] *A measure $\sigma = \sigma_{\mathcal{A}} \in \mathfrak{M}(\mathcal{A})$ is said to be a measure of the condenser $\mathcal{A}$ if the following two conditions are fulfilled:*

$$\kappa(x,\sigma) = i \quad \text{n. e. in} \ \ A_i, \quad i = 0, 1\,, \tag{1}$$

$$0 \leqslant \kappa(x,\sigma) \leqslant 1 \quad \forall\, x \in \mathbf{X}\,. \tag{2}$$

The principal aim of this paper is to solve the following problem.

Problem 1. *What conditions on a kernel κ and a condenser $\mathcal{A}$ are sufficient for the existence of a measure $\sigma_{\mathcal{A}}$?*

In the case where the condensers under consideration are compact, such a problem was solved by M. Kishi [3]. In the present paper, due to a certain new approach worked out here, Problem 1 is solved for not necessarily compact condensers.

3. Preliminaries

3.1. The vague topology. The *vague topology* on the linear space $\mathfrak{M}$ is defined by the seminorms $\mu \mapsto |\int f\,d\mu|$, f being an arbitrary real-valued continuous function with the compact support $S(f)$ [4, p. 75]. The space $\mathfrak{M}$ is then a Hausdorff space.

Lemma 2. *For a condenser $\mathcal{A}$, let a net (= a directed family [5, p. 20]) $(\mu_s)_{s\in S} \subset \mathfrak{M}(\mathcal{A})$ converge vaguely to μ. Then $(\mu_s^i)_{s\in S}$, $i = 0,\,1$, converges vaguely to μ^i. In addition, $\mu \in \mathfrak{M}(\mathcal{A})$, thus the class $\mathfrak{M}(\mathcal{A})$ is vaguely closed.*

Proof. The first statement of the lemma follows by the same arguments as in the case of the Euclidean space $\mathbb{R}^n$ (see [6, Proof of Lemma]), with application of the Urysohn-Tietze theorem on continuous extension of functions in a normal topological space [5, p. 30]. The second one is obvious, because $\mathfrak{M}^+(F)$, F being a closed set in $\mathbf{X}$, is closed in the vague topology [4, p. 86]. □

Remark. The first statement of Lemma 2 does not remain valid in general if the condition that $(\mu_s)_{s\in S}$ is contained in $\mathfrak{M}(\mathcal{A})$ is deleted from the hypotheses.

3.2. The strong topology. A kernel κ is said to be *definite* (=positive definite) [1] if the energy $\kappa(\mu,\mu)$ is nonnegative whenever defined.

Throughout sections 3.2 and 3.3, the kernel κ is assumed to be definite. Then $\mathcal{E} := \{\, \mu \in \mathfrak{M} :\ \kappa(\mu,\mu) < +\infty \,\}$ is a pre-Hilbert space (over the field of real numbers) with the scalar product $\kappa(\mu,\nu)$ and the seminorm $\|\mu\| = \sqrt{\kappa(\mu,\mu)}$ [1]. The topology on $\mathcal{E}$ defined by the seminorm $\|\mu\|$ is said to be the *strong topology.*

For any collection of measures $\mathfrak{N}$, the set $\mathfrak{N}^\circ := \mathfrak{N} \cap \mathcal{E}$ is considered to be a semimetric space with the semimetric inherited from $\mathcal{E}$; the topology of this space is called likewise the strong topology on $\mathfrak{N}^\circ$. For the semimetric space $\mathfrak{M}^+(F) \cap \mathcal{E}$, F being a closed set in $\mathbf{X}$, the notation $\mathcal{E}^+(F)$ is used as well.

Simple examples show that the pre-Hilbert space $\mathcal{E}$ is in general incomplete (in the strong topology). As was proved by H. Cartan [7], this is so even in the case of the Newtonian kernel $|x-y|^{2-n}$ in $\mathbb{R}^n$, $n \geqslant 3$.

For a condenser $\mathcal{A}$ and $c \geqslant 0$, write $\mathfrak{M}(\mathcal{A}, \leqslant c) := \{\, \nu \in \mathfrak{M}(\mathcal{A}) :\ |\nu|(\mathbf{X}) \leqslant c \,\}$.

As was proved by the author [8, 9], in the case of the Riesz kernels $|x-y|^{\alpha-n}$, $0<\alpha<n$, in $\mathbb{R}^n$, $n \geqslant 3$, the space $\mathfrak{M}^\circ(\mathcal{A}, \leqslant c)$ is strongly complete for any c, and strong convergence in $\mathfrak{M}^\circ(\mathcal{A}, \leqslant c)$ implies vague convergence to the same limit.

This leads us to the following notion of $\mathcal{A}$*–perfectness* (compare with [1, p. 166]).

3.3. $\mathcal{A}$-perfect kernel. Consider a definite kernel κ, and denote by $\mathbb{B}(\mathcal{A})$ the class of all strong Cauchy nets $(\mu_s)_{s\in S} \subset \mathfrak{M}^\circ(\mathcal{A})$ such that

$$\sup_{s\in S} |\mu_s|(\mathbf{X}) < +\infty \,. \tag{3}$$

Definition 2. *A (definite) kernel is called $\mathcal{A}$-perfect if for every $(\mu_s)_{s\in S} \in \mathbb{B}(\mathcal{A})$, the following two conditions are fulfilled:*

($\mathcal{A}P_1$) *$(\mu_s)_{s\in S}$ converges strongly in $\mathfrak{M}^\circ(\mathcal{A})$;*

($\mathcal{A}P_2$) *if $\mu \in \mathfrak{M}^\circ(\mathcal{A})$ is a strong limit of $(\mu_s)_{s\in S}$, then $\mu_s \to \mu$ vaguely.*

Example. The Riesz kernels $|x-y|^{\alpha-n}$, $0<\alpha<n$, are $\mathcal{A}$-perfect for any $\mathcal{A}$ in $\mathbb{R}^n$, $n \geqslant 3$ [8, 9]. If $G \subset \mathbb{R}^n$, $n \geqslant 3$, is an open set, then the Green kernel $\kappa = g_G$ is $\mathcal{A}$-perfect for any $\mathcal{A} = (A_1, A_0)$ in G provided κ is bounded on $A_1 \times A_0$ [10].

Lemma 3. *A kernel is $\mathcal{A}$-perfect if and only if it possesses the following two properties:*

($\mathcal{A}C$) *if μ is a vague cluster point for $(\mu_s)_{s\in S} \in \mathbb{B}(\mathcal{A})$, then $\mu \in \mathcal{E}$ and $(\mu_s)_{s\in S}$ converges strongly to μ;*

($\mathcal{A}SD$) *if $(\mu_s)_{s\in S} \in \mathbb{B}(\mathcal{A})$ converges strongly to μ', $\mu'' \in \mathfrak{M}^\circ(\mathcal{A})$, then $\mu' = \mu''$.*

Proof. Fix $(\mu_s)_{s\in S} \in \mathbb{B}(\mathcal{A})$. By virtue of (3), the set $\{\mu_s :\ s \in S\}$ is bounded in the vague topology, and hence vaguely relatively compact [4, p. 76]. Consequently, there exists its vague cluster point (say μ), and $\mu \in \mathfrak{M}(\mathcal{A})$ by Lemma 2.

Suppose first that κ satisfies ($\mathcal{A}C$) and ($\mathcal{A}SD$). It follows from ($\mathcal{A}C$) that $\mu \in \mathcal{E}$ and $\mu_s \to \mu$ strongly, which gives ($\mathcal{A}P_1$). In view of ($\mathcal{A}SD$), ($\mathcal{A}P_2$) will be proved if we show that $\mu_s \to \mu$ vaguely. Let μ' be a vague cluster point for $(\mu_s)_{s\in S}$. We infer from ($\mathcal{A}C$) and Lemma 2 that $\mu' \in \mathfrak{M}^\circ(\mathcal{A})$ and $\mu_s \to \mu'$ strongly, and hence that $\mu' = \mu$, by ($\mathcal{A}SD$).

Thus the vague adherence of $(\mu_s)_{s\in S}$ is nonvoid and reduces to the single measure μ; and consequently $\mu_s \to \mu$ vaguely (see [5, p. 34]).

Let now κ be $\mathcal{A}$–perfect, and let $(\mu_t)_{t\in T}$ be a subnet of $(\mu_s)_{s\in S}$ which converges vaguely to μ. Clearly, $(\mu_t)_{t\in T} \in \mathbb{B}(\mathcal{A})$, therefore it converges strongly and vaguely to some measure (say μ_0) in $\mathfrak{M}^\circ(\mathcal{A})$. Since the vague topology is separated, we conclude that $\mu_0 = \mu$, hence $\mu_t \to \mu$ strongly, and finally $\mu_s \to \mu$ strongly (see [5, p. 50]). This proves $(\mathcal{A}C)$. Since $(\mathcal{A}P_2)$ yields $(\mathcal{A}SD)$, the proof is complete. □

Lemma 4. *If a kernel κ possesses the property $(\mathcal{A}SD)$, then for every bounded measure ν supported by A, $\|\nu\| = 0$ implies $\nu = 0$.*

Proof. One can write $\nu = \nu_1^1 - \nu_1^0 + \nu_0^1 - \nu_0^0$, where $\nu_i^j \in \mathfrak{M}^+(A_i)\ \ \forall\, i,\, j = 0,\, 1$. Therefore $\|\nu\| = 0$ implies that the stationary sequence $(\nu_1^1 - \nu_0^0) \in \mathbb{B}(\mathcal{A})$ converges strongly to $\nu_1^0 - \nu_0^1$. Since this one also converges to $\nu_1^1 - \nu_0^0,\ \nu = 0$ by $(\mathcal{A}SD)$. □

Corollary 1. *If a kernel κ possesses the property $(\mathcal{A}SD)$, then $\operatorname{cap} K < +\infty$ for all compact sets $K \subset A$ (and therefore $\kappa(x, x) > 0$ for all $x \in A$).*

Proof. This follows immediately from Lemmas 1 and 4. □

Lemma 5. *Suppose that κ is $\mathcal{A}$-perfect and that K is a compact set contained in A_1 (respectively, A_0). Then the space $\mathcal{E}^+(K)$ is complete.*

Proof. Taking into account Lemmas 3 and 4, we obtain, by reason of homogeneity,

$$\operatorname{cap} K = \sup_{\nu}\ \big(\,\nu(\mathbf{X})\|\nu\|\,\big)^2 \qquad \big(\,\nu \in \mathfrak{M}^+(K)\,,\ \nu \neq 0\,\big)\ . \tag{4}$$

Let $(\mu_s)_{s\in S} \subset \mathcal{E}^+(K)$ be a strong Cauchy net. Since we can certainly assume it to be strongly bounded, (4) and Corollary 1 imply that $\mu_s(\mathbf{X})$, $s \in S$, is bounded. Thus $(\mu_s)_{s\in S}$ belongs to $\mathbb{B}(\mathcal{A})$ and therefore converges strongly to every its vague cluster point, by Lemma 3. As the vague adherence of $(\mu_s)_{s\in S}$ is nonvoid and the class $\mathfrak{M}^+(K)$ is vaguely closed, the lemma follows. □

Lemma 6. *Suppose κ possesses the property $(\mathcal{A}C)$, and let $F \subset A_i$, $i = 0, 1$, be a closed set with $\operatorname{cap} F < +\infty$. Then there exists a so-called interior capacity distribution on F, i.e. a measure $\theta = \theta_F$ with the support $S(\theta) \subset F$ such that*

$$\|\theta\|^2 = \theta(\mathbf{X}) = \operatorname{cap} F\,, \tag{5}$$

$$\kappa(x, \theta) \geqslant 1 \quad \textit{n. e. in}\ \ F\,, \tag{6}$$

$$\kappa(x, \theta) \leqslant 1 \quad \forall\, x \in S(\theta)\,.$$

Proof. The lemma follows by the same arguments as in [1, Proof of Th. 4.1]. □

Corollary 2. *Under the hypotheses of Lemma 6, let $F' \subset F$ be a closed set. Then*

$$\|\theta_F - \theta_{F'}\|^2 \leqslant \|\theta_F\|^2 - \|\theta_{F'}\|^2,$$

$\theta_{F'}$ being an interior capacity distribution on F'.

Proof. This follows from Lemma 6 by virtue of [1, Lemmas 3.2.2 and 4.1.1]. □

3.4. A continuity property of potentials. We will need the following assertion.

Lemma 7. *Suppose κ is continuous and bounded on $A_0 \times A_1$. For any bounded measure $\nu \in \mathfrak{M}(\mathcal{A})$, then $\kappa(x, \nu^i)$, $i = 0, 1$, is continuous on $A \setminus A_i$, and therefore $\kappa(x, \nu)$ is lower (respectively, upper) semicontinuous on A_1 (respectively, A_0).*

Proof. Under the above conditions, there exists a finite number b such that the function $\kappa' := -\kappa + b$ is nonnegative and continuous on $A_0 \times A_1$. Therefore for any compact set $K \subset A_i$, $i = 0, 1$, the potential $\kappa'(x, \nu^i_K)$, ν^i_K being the trace of ν^i on K, is continuous on $A \setminus A_i$ (see [4, p. 106]). Since $\kappa'(x, \nu^i) = \sup\ \kappa'(x, \nu^i_K)$, where K ranges over the class of all compact subsets of A_i [1, Lemma 1.2.2], we conclude that $\kappa'(x, \nu^i)$ is lower semicontinuous on $A \setminus A_i$, and hence so is $\kappa(x, -\nu^i)$. Since $\kappa(x, -\nu^i)$ is upper semicontinuous on $\mathbf{X}$, it is continuous on $A \setminus A_i$. □

3.5. The complete maximum principle. A kernel κ is said to satisfy the *complete maximum principle* [11] if for every number $b \geqslant 0$ and for any positive measures ν, μ such that ν has compact support and finite energy, the inequality $\kappa(x, \nu) \leqslant \kappa(x, \mu) + b \quad \forall\, x \in S(\nu)$ implies $\kappa(x, \nu) \leqslant \kappa(x, \mu) + b \quad \forall\, x \in \mathbf{X}$.

Remark. One can see that for the kernel in question $\kappa \geqslant 0$, the condition of compactness of $S(\nu)$ in the above definition may be omitted.

4. A criterion for the existence of a condenser measure

From now on the kernel in question is assumed to be continuous and bounded on $A_0 \times A_1$, to satisfy the complete maximum principle, and also to be $\mathcal{A}$-perfect.

Furthermore, we can certainly assume that $\operatorname{cap} A_1 > 0$, since otherwise the measure $\sigma_{\mathcal{A}} = 0$ meets relations (1) and (2) in Definition 1.

Let $\mathcal{E}_b$ denote the subclass of $\mathcal{E}$ consisting of all bounded measures.

Theorem 1. *The following statements are equivalent:*

(i$_1$) $\operatorname{cap} A_1 < +\infty$;

(ii$_1$) in $\mathcal{E}_b$ there exists a measure $\sigma_{\mathcal{A}}$ of the condenser $\mathcal{A}$.

In the class $\mathcal{E}_b$ a measure $\sigma_{\mathcal{A}}$ is unique (if exists).

Proof of uniqueness. Let σ, $\omega \in \mathcal{E}_b$ be two measures of $\mathcal{A}$. For any $i = 0, 1$, writing $N_i := \{x \in A_i : \ \kappa(x, \sigma) \neq \ i\}$, we have $\operatorname{cap} N_i = 0$, and hence $\omega^i_*(N_i) = 0$. Observing that the set N_i is ω^i–integrable (because it is ω^i–measurable and ω^i is bounded), we conclude (see [5, p. 275]) that $\kappa(x, \sigma) = i \ \ \omega^i$–almost everywhere ($\omega^i$–a.e.), and hence that $\kappa(\sigma, \omega) = \int \kappa(x, \sigma)\, d\omega(x) = \omega^1(\mathbf{X})$.

Similarly, $\kappa(\omega, \sigma) = \sigma^1(\mathbf{X})$ (and therefore $\omega^1(\mathbf{X}) = \sigma^1(\mathbf{X})$), $\|\sigma\|^2 = \sigma^1(\mathbf{X})$, $\|\omega\|^2 = \omega^1(\mathbf{X})$, which gives $\|\sigma - \omega\| = 0$. Applying Lemma 4, we get $\sigma = \omega$. □

To prove the statement of existence, we consider the following extremal problem.

5. An auxiliary extremal problem

5.1. Statement of a problem. Set $\mathfrak{M}_1(\mathcal{A}) := \bigl\{\nu \in \mathfrak{M}(\mathcal{A}) : \ \nu^1(\mathbf{X}) = 1\bigr\}$ and

$$w_1(\mathcal{A}) := \inf_{\nu \in \mathfrak{M}_1(\mathcal{A})} \kappa(\nu, \nu) .$$

Observe that $w_1(\mathcal{A}) < +\infty$, which is clear from the assumption $\operatorname{cap} A_1 > 0$.

Problem 2. *Does there exist a measure $\gamma \in \mathfrak{M}_1(\mathcal{A})$ such that $\|\gamma\|^2 = w_1(\mathcal{A})$?*

On account of Lemmas 3 and 4, the fact that $\mathfrak{M}_1(\mathcal{A})$ is convex implies in the usual way that in the class of bounded measures such a γ is unique (if exists).

Lemma 8. $w_1(\mathcal{A}) = \inf\ w_1(\mathcal{K})$, *where $\mathcal{K}$ ranges over the class of all compact condensers $\mathcal{K} = (K_1, K_0)$ such that $K_i \subset A_i \ \ \forall\, i = 0,\, 1$.*

Proof. The proof is similar to [8, Proof of Lemma 2] (see also [1, p. 152]) and may be omitted. □

5.2. Projections of measures. Consider the following representation of $w_1(\mathcal{A})$:

$$w_1(\mathcal{A}) = \inf_{\mu \in \mathfrak{M}^+(A_1,\, 1)} p(\mu) , \quad \text{where} \quad p(\mu) := \inf_{\omega \in \mathcal{E}^+(A_0)} \|\mu - \omega\|^2 .$$

Fix $\mu \in \mathfrak{M}^+(A_1, 1)$. A measure $P\mu \in \mathcal{E}^+(A_0)$ such that $\|\mu - P\mu\|^2 = p(\mu)$ is called a *projection* of μ onto $\mathcal{E}^+(A_0)$ [5, p. 139–140]. A projection $P\mu$ exists if the space $\mathcal{E}^+(A_0)$ is complete (therefore, by Lemma 5, this does so if A_0 is compact).

If ω_1 and ω_2 are two projections of μ onto $\mathcal{E}^+(A_0)$, then $\|\omega_1 - \omega_2\| = 0$; consequently, by Lemmas 3 and 4, in the class $\mathcal{E}_b$ a projection $P\mu$ is unique (if exists).

We have thus proved that it holds

$$w_1(\mathcal{A}) = \inf_{\mu \in \mathfrak{M}^+(A_1,\, 1)} \|\mu - P\mu\|^2 \qquad \text{provided } \ A_0 \ \text{ is compact} . \tag{7}$$

Lemma 9. *If $P\mu \in \mathcal{E}_b$, then the following relations hold:*

$$\kappa(x, P\mu) \leqslant \kappa(x, \mu) \quad \forall\, x \in \mathbf{X} , \tag{8}$$

$$\kappa(x, P\mu) = \kappa(x, \mu) \quad \textit{n. e. in} \ \ A_0 . \tag{9}$$

Proof. First, taking into account [5, Prop. 1.12.4] and applying arguments similar to those in [12, Proof of Th. 4.16], one can see that

$$\kappa(x, P\mu) \geqslant \kappa(x, \mu) \quad \text{n. e. in} \ \ A_0 , \tag{10}$$

$$\kappa(x, P\mu) = \kappa(x, \mu) \quad P\mu\text{–a. e.} \tag{11}$$

Fix a point $x \in S(P\mu)$, and consider the set $\mathcal{B}(x)$ of all open neighborhoods of x in A_0. Since for any $U \in \mathcal{B}(x)$ we have $P\mu(U) > 0$, it follows from (11) that there exists $x_U \in U$ with $\kappa(x_U, P\mu) = \kappa(x_U, \mu)$. But the net $(x_U)_{U \in \mathcal{B}(x)}$ converges to x, when $\mathcal{B}(x)$ is considered to be a decreasing directed set. Letting $x_U \to x$, we obtain

$$\kappa(x, P\mu) \leqslant \kappa(x, \mu) \quad \forall\, x \in S(P\mu) ,$$

by virtue of Lemma 7. Application of the definition given in Sec. 3.5 (see also the remark below it) yields (8). Finally, (9) follows by combining (8) and (10). □

Lemma 10. *If A_0 is compact, then $P\mu(\mathbf{X}) \leqslant 1$.*

Proof. Applying the complete maximum principle, we infer from Corollary 1 and Lemma 6 that there exists $\theta \in \mathcal{E}_b$ with $S(\theta) \subset A_0$, $\kappa(x,\theta) = 1$ n. e. in A_0, and $\kappa(x,\theta) \leqslant 1 \ \ \forall x \in \mathbf{X}$. As in the proof given in § 4, from this and (9) we obtain

$$P\mu(\mathbf{X}) = \int \kappa(x,\theta)\, d(P\mu)(x) = \int \kappa(x,\mu)\, d\theta(x) = \int \kappa(x,\theta)\, d\mu(x) \leqslant \mu(\mathbf{X}) = 1\,,$$

which is the desired conclusion. □

Corollary 3. $w_1(\mathcal{A}) = \inf\ \kappa(\nu,\nu)$*, the infimum being taken over the class* $\mathfrak{M}_1'(\mathcal{A}) := \big\{\nu \in \mathfrak{M}_1(\mathcal{A}) :\ \nu^0(\mathbf{X}) \leqslant 1\big\}$.

Proof. For a compact $\mathcal{A}$, the desired formula is derived from (7) by Lemma 10. In the case of a noncompact condenser, this one is proved by applying Lemma 8. □

5.3. Existence of a minimal measure. Let us give a solution to Problem 2.

Theorem 2. *The following statements are equivalent:*

(i$_2$) $\operatorname{cap} A_1 < +\infty$*;*

(ii$_2$) $w_1(\mathcal{A}) > 0$*;*

(iii$_2$) there exists $\gamma \in \mathfrak{M}_1'(\mathcal{A})$ *such that* $\|\gamma\|^2 = w_1(\mathcal{A})$.

Proof. We begin by proving that (i$_2$) implies (iii$_2$). In accordance with Corollary 3, there exists a sequence $(\nu_n)_{n\in\mathbb{N}} \subset \mathfrak{M}_1'(\mathcal{A})$ with $\lim_{n\in\mathbb{N}}\ \|\nu_n\|^2 = w_1(\mathcal{A})$. In view of the boundedness of κ on $A_0 \times A_1$, we infer from the definition of $\mathfrak{M}_1'(\mathcal{A})$ that $\kappa(\nu_n^0,\nu_n^1) \leqslant c < +\infty \ \ \forall n \in \mathbb{N}$. Since $w_1(\mathcal{A}) < +\infty$, this gives

$$\sup_{n\in\mathbb{N}}\ \|\nu_n^i\| < +\infty \quad \forall\, i = 0,\, 1\,. \tag{12}$$

The fact that $\mathfrak{M}_1'(\mathcal{A})$ is convex implies in the usual way that $(\nu_n)_{n\in\mathbb{N}}$ is a strong Cauchy sequence. Since the set $\{\nu_n :\ n \in \mathbb{N}\}$ is vaguely bounded (and hence vaguely relatively compact), there exists its vague cluster point (say ν). Let $(\nu_s)_{s\in S}$ be a subnet of $(\nu_n)_{n\in\mathbb{N}}$ which converges vaguely to ν. Since $(\nu_s)_{s\in S} \in \mathbb{B}(\mathcal{A})$, Lemmas 2 and 3 show that $\nu \in \mathfrak{M}^\circ(\mathcal{A})$, $\nu_s \to \nu$ strongly, and also that $\nu^i(\mathbf{X}) \leqslant 1 \ \ \forall\, i = 0,\, 1$.

Let (i$_2$) hold. If we prove that $\nu^1(\mathbf{X}) = 1$, (iii$_2$) follows. To this end, consider the increasing directed set $\{K\}$ of all compact subsets K of $\mathbf{X}$. As $(\nu_s^1)_{s\in S}$ converges to ν^1 vaguely (see Lemma 2), for every K we have $\nu^1(K) \geqslant \sup\ \nu_s^1(K)$, $s \in S$, and hence

$$1 \geqslant \nu^1(\mathbf{X}) = \lim_{K\in\{K\}}\ \nu^1(K) \geqslant \limsup_{(s,\,K)\in S\times\{K\}}\ [1 - \nu_s^1(\mathbf{X}\setminus K)]\,,$$

$S \times \{K\}$ being the directed product of S and $\{K\}$. It remains to show that

$$\liminf_{(s,\,K)\in S\times\{K\}}\ \nu_s^1(\mathbf{X}\setminus K) = 0\,. \tag{13}$$

For every $K \in \{K\}$, let K^* denote the closure of $A_1 \setminus K$. According to Lemma 6, (i$_2$) implies that there exists an interior capacity distribution θ_{K^*} on K^*. In view of monotonicity of the interior capacity, from (5) and Corollary 2 we infer that the net $(\theta_{K^*})_{K\in\{K\}}$ is of the class $\mathbb{B}(\mathcal{A})$. To show that it converges to zero vaguely, we consider a continuous function f with compact support. If K_0 is a compact neighborhood of $S(f)$, then for every $K\in\{K\}$ such that $K\supset K_0$ we have $f=0$ θ_{K^*}–a.e., which leads immediately to the desired conclusion. Applying Lemma 3, we deduce from what has just been proved that

$$\lim_{K\in\{K\}} \|\theta_{K^*}\|^2 = 0\,. \tag{14}$$

Let φ_{K^*} denote the characteristic function associated with K^*. Since $\kappa\geqslant 0$, we infer from (6) that

$$\nu_s^1(\mathbf{X}\setminus K)\leqslant \nu_s^1(K^*)=\int \varphi_{K^*}\,d\nu_s^1\leqslant \int \kappa(x,\theta_{K^*})\varphi_{K^*}(x)\,d\nu_s^1(x)\leqslant \kappa(\theta_{K^*},\nu_s^1)\,,$$

and hence that $\nu_s^1(\mathbf{X}\setminus K)\leqslant \|\theta_{K^*}\|\cdot\|\nu_s^1\|$ for all $(s,K)\in S\times\{K\}$, by the Cauchy-Schwarz inequality. Taking into account (12) and (14), we deduce relation (13) (and hence (iii$_2$)) from the latter one by passing to the limit along $S\times\{K\}$.

Observing that the statement (ii$_2$)$\Rightarrow$(i$_2$) is evident, we complete the proof by showing that (iii$_2$) implies (ii$_2$). Let γ satisfy (iii$_2$), and suppose, contrary to our claim, that $w_1(\mathcal{A})=0$. Then Lemma 4 yields $\gamma=0$, which is impossible. □

Theorem 3. *If γ is a measure of the class $\mathfrak{M}_1'(\mathcal{A})$ with $\|\gamma\|^2=w_1(\mathcal{A})$, then*

$$\kappa(x,\gamma)=\begin{cases} w_1(\mathcal{A}) & \text{n. e. in } A_1\,,\\ 0 & \text{n. e. in } A_0\,,\end{cases} \tag{15}$$

$$0\leqslant \kappa(x,\gamma)\leqslant w_1(\mathcal{A})\quad \forall\, x\in\mathbf{X}\,. \tag{16}$$

Proof. We begin by observing that both the second relation in (15) and the left part of (16) follow from Lemma 9, because γ^0 is a projection of γ^1 onto $\mathcal{E}^+(A_0)$. Since γ^0 is bounded, we conclude by integrating that $\kappa(\gamma^0,\gamma)=0$, and hence that

$$\kappa(\gamma^1,\gamma)=w_1(\mathcal{A})\,. \tag{17}$$

To prove the other statements of the theorem, let us first show that

$$\kappa(x,\gamma)\geqslant \kappa(\gamma^1,\gamma)\quad \text{n. e. in } A_1\,. \tag{18}$$

Suppose relation (18) is false. According to [1, Lemmas 2.2.2 and 2.3.3], we can find a compact set $K\subset A_1$ with $\operatorname{cap} K>0$ and a positive number η such that $\kappa(x,\gamma)\leqslant \kappa(\gamma^1,\gamma)-\eta$ $\forall\, x\in K$. Fix $\mu\in\mathfrak{M}^+(K,1)$ with $\|\mu\|<+\infty$. The latter inequality gives

$$\kappa(\gamma,\mu)\leqslant \kappa(\gamma^1,\gamma)-\eta<\kappa(\gamma^1,\gamma)\,. \tag{19}$$

Observing that $\gamma-h\gamma^1+h\mu\in\mathfrak{M}_1(\mathcal{A})$ for every $h\in(0,1]$, we conclude that $\|\gamma\|^2\leqslant \|\gamma-h\gamma^1+h\mu\|^2$, and hence that $h\,\|\gamma^1-\mu\|^2\geqslant 2\,\kappa(\gamma,\gamma^1-\mu)$. Letting $h\to 0$, we obtain $\kappa(\gamma,\gamma^1)\leqslant\kappa(\gamma,\mu)$, which contradicts (19).

Having thus proved (18), we conclude that $\kappa(x,\gamma)\geqslant\kappa(\gamma^1,\gamma)$ γ^1-a. e. Integrating with respect to γ^1, we obtain $\kappa(\gamma^1,\gamma)=\int\kappa(x,\gamma)\,d\gamma^1(x)\geqslant\kappa(\gamma^1,\gamma)$, so that, actually, $\kappa(x,\gamma)=\kappa(\gamma^1,\gamma)$ γ^1–a.e. As in the proof of Lemma 9, from this we deduce that $\kappa(x,\gamma)\leqslant\kappa(\gamma^1,\gamma)$ $\forall\, x\in S(\gamma^1)$, and hence that $\kappa(x,\gamma)\leqslant\kappa(\gamma^1,\gamma)$ $\forall\, x\in\mathbf{X}$, by the complete maximum principle. Combining the latter inequality with relations (17) and (18), we complete the proof. □

6. Proof of Theorem 1

It follows from Theorems 2 and 3 that Theorem 1 will be proved if we show that (ii_1) implies (ii_2). If $\sigma \in \mathcal{E}_b$ is the measure of the condenser $\mathcal{A}$, then for every $\nu \in \mathfrak{M}'_1(\mathcal{A})$ of finite energy it holds $\kappa(\sigma, \nu) = \int \kappa(x, \sigma)\, d\nu(x) = 1$. Applying the Cauchy-Schwarz inequality, we obtain (ii_2).

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Institute of Mathematics, Department of Complex Analysis,
Tereshchenkivs'ka 3, Kyiv, Ukraine

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