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A. A. Kondratyuk, Ya. V. Vasyl'kiv

CONJUGATE OF SUBHARMONIC FUNCTION

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For a subharmonic function u on a star domain the notion of its conjugate is introduced. It is shown that in the case $u = \log|f|$, f is holomorphic, the conjugate of u is a branch of $\operatorname{Arg} f$. Some properties of the conjugate and its representations are established. As an example, the growth of square means of Green potentials and their conjugates is studied.

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Введено понятие сопряжения для субгармонической в звездной области функции u. Показано, что в случае $u = \log|f|$, f голоморфна, сопряжение u является ветвью $\operatorname{Arg}\ f$. Установлены представления и некоторые свойства сопряжения. В качестве примера исследуется рост средних квадратических потенциала Грина и их сопряжений.

1. Introduction. Definition

For $z \in \mathbb{C}$, $a \neq 0$ we set

$$\log\left(1-\frac{z}{a}\right) = \begin{cases} \displaystyle\int_0^z \frac{d\zeta}{\zeta - a}, & z \neq ta,\ t \geq 1 \\ \log\left|1-\dfrac{z}{a}\right|, & z = ta,\ t \geq 1. \end{cases}$$

A set G in $\mathbb{C}$ is called a star set if $z \in G$ implies $[0, z] \subset G$, where $[0, z]$ is the interval in $\mathbb{C}$. By $\overset{\circ}{E}$ the interior of a set E is denoted.

Let u be a subharmonic function on a domain G. By the Riesz theorem [1, p. 123] for each compact subset K of G, $\overset{\circ}{K} \neq \varnothing$, and for each point $z \in \overset{\circ}{K}$ we have

$$u(z) = h_K(z) + \int_K \log\left|1-\frac{z}{a}\right| d\mu_a,$$

where the function h_K is harmonic on $\overset{\circ}{K}$, μ is the Riesz measure of u.

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Definition 1. *Let G be a star domain in $\mathbb{C}$, u a subharmonic function on G, $u(0)=0$, $0\notin \operatorname{supp}\mu$. The conjugate $\breve{u}$ of u on G is defined by the relation*

$$u(z)+i\breve{u}(z)=h_K(z)+i\widetilde{h}_K(z)+\int_K \log\left(1-\frac{z}{a}\right)d\mu_a,$$

where K is a compact subset of G with star interior, $z\in\overset{\circ}{K}$, $\widetilde{h}_K$ is the harmonic conjugate of h_K on $\overset{\circ}{K}$, $\widetilde{h}_K(0)=0$.

The function

$$\psi_z(a)=\operatorname{Im}\log\left(1-\frac{z}{a}\right)=\begin{cases}-\arctan\dfrac{|z|^2-\operatorname{Re}(z\bar{a})}{\operatorname{Im}(z\bar{a})}-\arctan\dfrac{\operatorname{Re}(z\bar{a})}{\operatorname{Im}(z\bar{a})}, & z\bar{a}\notin\mathbb{R},\\ 0, & z\bar{a}\in\mathbb{R},\end{cases}$$

is bounded and μ-integrable on K. Consequently, $\breve{u}(z)$ exists.

The following lemma shows that the conjugate $\breve{u}$ of u is independent of the choice of K.

Lemma 1. *Let K_1, K_2 be a couple of compact subset in a star domain G with star interiors and let*

$$F_j(z)=h_{K_j}(z)+i\widetilde{h}_{K_j}(z)+\int_{K_j}\log\left(1-\frac{z}{a}\right)d\mu_a,\quad j=1,2,$$

h_{K_j} is a harmonic function on $\overset{\circ}{K}_j$, $h_{K_j}(0)=0$, $\widetilde{h}_{K_j}$ its harmonic conjugate on $\overset{\circ}{K}_j$, $\widetilde{h}_{K_j}(0)=0$, μ a Borel measure on G. If $\operatorname{Re}F_1=\operatorname{Re}F_2$ on $\overset{\circ}{K}_1\cap\overset{\circ}{K}_2$ then $F_1=F_2$ on $\overset{\circ}{K}_1\cap\overset{\circ}{K}_2$.

Proof. We have

$$F_1(z)-F_2(z)=f(z)+\int_{K_1\setminus K_2}\log\left(1-\frac{z}{a}\right)d\mu_a-\int_{K_2\setminus K_1}\log\left(1-\frac{z}{a}\right)d\mu_a,\tag{1}$$

where f is a holomorphic function on $\overset{\circ}{K}_1\cap\overset{\circ}{K}_2$, $f(0)=0$. Both the integrals in (1) are also holomorphic functions on $\overset{\circ}{K}_1\cap\overset{\circ}{K}_2$. Since $\operatorname{Re}(F_1-F_2)=0$ on $\overset{\circ}{K}_1\cap\overset{\circ}{K}_2$, $F_1(0)=F_2(0)=0$, we obtain $F_1=F_2$ on $\overset{\circ}{K}_1\cap\overset{\circ}{K}_2$. □

2. Representations

Theorem 1. *Let G be a star domain in $\mathbb{C}$, f a holomorphic function on G, $f(0)=1$, and $\mathcal{Z}$ the set of their zeros in G, $u=\log|f|$. Then*

$$u(z)+i\breve{u}(z)=\int_0^z\frac{f'(\zeta)}{f(\zeta)}d\zeta,\quad z\neq ta,\quad t\geq 1,\quad a\in\mathcal{Z}.$$

Proof. For a compact subset K of G with star interior we set

$$F_K(z)=f(z)\Big/\prod_j\left(1-\frac{z}{a_j}\right),\quad a_j\in K\cap\mathcal{Z}.$$

The function $F_K(z)$ is holomorphic in G and $F_K(z) \neq 0$ in K, $F_K(0) = 1$. Then $F_K(z) = \exp g_K(z)$ on $\overset{\circ}{K}$, where $g_K = h_K + i\widetilde{h}_K$ is holomorphic in $\overset{\circ}{K}$, $g_K(0) = 0$. Then

$$u(z) = \log|f(z)| = h_K(z) + \sum_j \log\left|1 - \frac{z}{a_j}\right|, \quad z \in \overset{\circ}{K}, \quad a_j \in K \cap \mathcal{Z}.$$

For $z \in \overset{\circ}{K}$ we have

$$\int_0^z \frac{f'(\zeta)}{f(\zeta)} d\zeta = g_K(z) + \sum_j \int_0^z \frac{d\zeta}{\zeta - a_j} = h_K(z) + i\widetilde{h}_K(z) + \sum_j \log\left(1 - \frac{z}{a_j}\right) =$$
$$= u(z) + i\breve{u}(z), \quad z \neq ta_j, \ t \geq 1, \ a_j \in K \cap \mathcal{Z}.$$

□

In the sequel $G = \mathbb{D}_{R_0} = \{z : |z| < R_0\}$, $R_0 \leq +\infty$, $\mathbb{T} = \{\zeta : |\zeta| = 1\}$, $dm(\zeta) = d\zeta/(2\pi i\zeta)$, $\zeta \in \mathbb{T}$. We set

$$\log \frac{R^2(a - z)}{a(R^2 - \bar{a}z)} = \log\left(1 - \frac{z}{a}\right) - \log\left(1 - \frac{\bar{a}z}{R^2}\right).$$

Theorem 2 (Generalized Schwarz formula). *Let u be a subharmonic function on $\mathbb{D}_{R_0}$, $u(0) = 0$, $0 \notin \operatorname{supp}\mu$, $0 < R < R_0$. Then for $z \in \mathbb{D}_R$*

$$u(z) + i\breve{u}(z) = \int_{\mathbb{T}} u(R\zeta)\frac{R\zeta + z}{R\zeta - z} dm(\zeta) + \int_{|a|\leq R} \log \frac{R^2(a - z)}{a(R^2 - \bar{a}z)} d\mu_a + \int_{|a|\leq R} \log \frac{|a|}{R} d\mu_a. \quad (2)$$

Proof. The function

$$g_R(z) = \int_{\mathbb{T}} u(R\zeta)\frac{R\zeta + z}{R\zeta - z} dm(\zeta) - \int_{|a|\leq R} \log\left(1 - \frac{\bar{a}z}{R^2}\right) d\mu_a + \int_{|a|\leq R} \log \frac{|a|}{R} d\mu_a \quad (3)$$

is holomorphic in $\mathbb{D}_R$, $g_R(0) = 0$.

It follows from Poisson-Jensen formula [1, p. 139] that

$$u(z) = h_R(z) + \int_{|a|\leq R} \log\left|1 - \frac{z}{a}\right| d\mu_a,$$

where $h_R(z) = \operatorname{Re} g_R(z)$. Then

$$u(z) + i\breve{u}(z) = g_R(z) + \int_{|a|\leq R} \log\left(1 - \frac{z}{a}\right) d\mu_a, \quad z \in \mathbb{D}_R. \quad (4)$$

From (3) and (4) we get (2). □

Let $P_r(w)$ be the Poisson kernel, $P_r(w) = \operatorname{Re}\left[(r + w)(r - w)^{-1}\right]$, μ a Borel measure on $\mathbb{D}_{R_0}$, $0 \notin \operatorname{supp}\mu$, $N(r) = \int_0^r \mu(\mathbb{D}_{R_0})/t \, dt$.

Definition 2. *The function*

$$p(z) = \int_0^{|z|} \frac{dt}{t} \int_{|a|\le t} P_{|z|}\left(t\frac{\bar{a}z}{|az|}\right) d\mu_a, \quad |z| < R_0,$$

is called the distribution function of μ*.*

Since $p(z) \ge 0$ and $\int_{\mathbb{T}} P_{|z|}(t\zeta)dm(\zeta) = 1,\ 0 < t < |z|$, the Fubini theorem implies that for arbitrary $R \in (0, R_0)$ we have

$$\int_{\mathbb{T}} p(r\zeta)dm(\zeta) = \int_0^r \frac{dt}{t} \int_{|a|\le t} d\mu_a \int_{\mathbb{T}} P_{|z|}\left(t\zeta\frac{\bar{a}}{|a|}\right) dm(\zeta) = N(r) < +\infty,$$

where $N(r) = \int_0^r \mu(\{|a| \le t\})t^{-1}dt$. Hence, the function $p(z)$ is bounded a. e. on $\mathbb{D}_{R_0}$ and for arbitrary $r \in (0, R_0)$ the function $p(r\zeta),\ \zeta \in \mathbb{T}$, belongs to $L^1(\mathbb{T})$.

We denote the linear operator of conjugation [2, p. 108] in $L^1(\mathbb{T})$ by $\sim$. For a subharmonic in $\mathbb{D}_{R_0}$ function u we put $F = u + i\breve{u}$.

Theorem 3. *Let* u *be a subharmonic function on* $\mathbb{D}_{R_0},\ u(0) = 0,\ 0 \notin \operatorname{supp}\mu,\ q \in [1, +\infty)$*. Then*

(a) for arbitrary $r \in (0, R_0)$ *the function* $F(r\zeta),\ \zeta \in \mathbb{T}$*, belongs to* $L^q(\mathbb{T})$*;*

(b) for arbitrary $r \in (0, R_0)$

$$\breve{u}(r\zeta) = \widetilde{u}(r\zeta) - \widetilde{p}(r\zeta) \quad a.\ e. \tag{5}$$

Let $k \in \mathbb{Z},\ 0 < r < R_0$,

$$l_k(r) = \int_{\mathbb{T}} F(r\zeta)\bar{\zeta}^k\, dm(\zeta), \quad n_k(r) = \int_{|a|\le r} (\bar{a}/|a|)^k\, d\mu_a, \quad n(r) = n_0(r).$$

Since $0 \notin \operatorname{supp}\mu$, we may write

$$F(z) = \sum_{k\in\mathbb{N}} \gamma_k z^k, \quad \gamma_k = \frac{1}{k!}\frac{d^k}{dz^k}F(z)\Big|_{z=0}, \quad k \in \mathbb{N}. \tag{6}$$

in some neighbourhood of the origin.

We will need the following lemma.

Lemma 2. *Let* u *be a subharmonic function on* $\mathbb{D}_{R_0},\ u(0) = 0,\ 0 \notin \operatorname{supp}\mu,\ 0 < r < R_0$*. Then*

$$l_0(r) = N(r); \tag{7}$$

$$l_k(r) = \gamma_k r^k + r^k \int_0^r t^{-k-1} n_k(t)\, dt, \quad k \in \mathbb{N}; \tag{7'}$$

$$l_{-k}(r) = r^{-k} \int_0^r t^{k-1} n_{-k}(t)\, dt, \quad k \in \mathbb{N}. \tag{7''}$$

Proof. We may write (4) in the form

$$F(z) = g_R(z) + \int_{|a|\le r} \log\left(1 - \frac{z}{a}\right) d\mu_a + \int_{r<|a|\le R} \log\left(1 - \frac{z}{a}\right) d\mu_a. \tag{8}$$

If $z = r\zeta,\ \zeta \in \mathbb{T}$, then the Fourier series of the last summand in (8), as a function of ζ, does not contain the terms with $\bar\zeta^k,\ k \in \mathbb{N}$. Since

$$\int_{\mathbb{T}} \log\left(1 - \frac{r\zeta}{a}\right)\zeta^k dm(\zeta) = \int_0^{|a|} dt \int_{\mathbb{T}} \frac{\zeta^{k+1}}{t\zeta - a} dm(\zeta) +$$
$$+ \int_{|a|}^{r} dt \int_{\mathbb{T}} \frac{\zeta^{k+1}}{t\zeta - a} dm(\zeta) = \begin{cases} \log(r/|a|), & k = 0; \\ k^{-1}\left[(a/|a|)^k - (a/r)^k\right], & k \in \mathbb{N}, \end{cases}$$

we get

$$l_0(r) = \int_{|a|\le r} \log\frac{r}{|a|} d\mu_a, \tag{9}$$

$$l_{-k}(r) = \frac{1}{k} n_{-k}(r) - \frac{1}{k}\int_{|a|\le r}\left(\frac{a}{r}\right)^k d\mu_a, \quad k \in \mathbb{N}. \tag{10}$$

Integrating by parts we obtain (7) and (7″).

We next observe that $u = (F + \overline{F})/2$,

$$\frac{R\zeta + z}{R\zeta - z} = 1 + \frac{2z}{R\zeta - z}.$$

Then (2) and (6) imply

$$\gamma_k r^k = \left(\frac{r}{R}\right)^k \left\{ l_k(R) + \bar l_{-k}(R) + \frac{1}{k}\int_{|a|\le R}\left[\left(\frac{\bar a}{R}\right)^k - \left(\frac{R}{a}\right)^k\right] d\mu_a \right\}, \quad k \in \mathbb{N}.$$

Using (9), where r is replaced by R, we find

$$l_k(R) = \gamma_k R^k + \frac{1}{k}\int_{|a|\le R}\left(\frac{R}{a}\right)^k d\mu_a - \frac{n_k(R)}{k}, \quad k \in \mathbb{N}.$$

Integrating by parts we get (7′). □

Proof of Theorem 3. Let $r \in (0, R_0)$,

$$T(r) = \int_{\mathbb{T}} u^+(r\zeta)\, dm(\zeta), \quad u^+ = \max(u, 0),$$
$$c_k(r) = \int_{\mathbb{T}} u(r\zeta)\bar\zeta^k dm(\zeta), \quad k \in \mathbb{Z}.$$

Since $u = \operatorname{Re} F$, we have

$$l_k(r) = 2c_k(r) - \bar l_{-k}(r) = 2c_k(r) - \int_0^r \left(\frac{t}{r}\right)^k \frac{n_k(t)}{t} dt, \quad k \in \mathbb{N}.$$

Using the identity $|u| = 2u^+ - u$ and (7) we conclude that

$$|l_k(r)| \le 2\int_{\mathbb{T}} |u(r\zeta)|dm(\zeta) + N(r) \le 4T(r), \quad k \in \mathbb{N}.$$

If $\sqrt[k]{k}r < R < R_0$, (7′) implies

$$l_k(r) = \frac{l_k(\sqrt[k]{k}r)}{k} - r^k \int_r^{\sqrt[k]{k}r} \frac{n_k(t)}{t^{k+1}}dt.$$

Consequently

$$|l_k(r)| \le \frac{4T(R) + n(R)}{k}, \quad \sqrt[k]{k}r < R < R_0.$$

From (7″) it follows that

$$|l_{-k}(r)| \le \int_0^r \left(\frac{t}{r}\right)^k \frac{n(t)}{t}dt \le \frac{n(r)}{k}, \quad k \in \mathbb{N}.$$

Applying the Hausdorff-Young theorem, we obtain statement (a) of Theorem 3.

Now we proceed to obtain representation (5). A routine verification shows that

$$p(r\zeta) = \sum_{k\in\mathbb{Z}} \left(\int_0^r \left(\frac{t}{r}\right)^{|k|} \frac{n_k(t)}{t}dt\right)\zeta^k, \quad 0 < r < R_0, \quad \zeta \in \mathbb{T}.$$

For fixed r, (5) may be written in the form

$$2\widetilde{p} = (F + \overline{F})^{\sim} + i(F - \overline{F}). \tag{11}$$

By properties of the operator $\sim$ (see [2, p. 112]), (10) is equivalent to

$$-2i\operatorname{sgn} k \left(\int_0^r \left(\frac{t}{r}\right)^{|k|} \frac{n_k(t)}{t}dt\right) = -i\operatorname{sgn} k \left(l_k(r) + \bar{l}_{-k}(r)\right) + i\left(l_k(r) - \bar{l}_{-k}(r)\right),$$

$k \ne 0$, where $\operatorname{sgn} k = k/|k|$. We obtain this from (7″). □

3. Growth of square means of Green potentials and their conjugates

Let μ be a Borel measure in $\mathbb{D} = \mathbb{D}_1$ such that $0 \notin \operatorname{supp}\mu$,

$$\int_{\mathbb{D}} (1 - |a|)\, d\mu_a < +\infty.$$

By $G_\mu(z)$ we denote the Green potential of μ,

$$G_\mu(z) = \int_{\mathbb{D}} \log\left|\frac{a - z}{1 - \bar{a}z}\right| d\mu_a, \quad z \in \mathbb{D},$$

$n(r) = \mu\left(\overline{\mathbb{D}}_r\right), \ 0 < r < 1.$

The function $G_\mu(z)$ is subharmonic on $\mathbb{D}$. We denote $u(z) = G_\mu(z) - G_\mu(0)$, $F = u + i\breve{u}$,

$$m_2(r, F) = \left\{ \int_{\mathbb{T}} |F(r\zeta)|^2 \, dm(\zeta) \right\}^{1/2}, \quad 0 < r < 1.$$

If $\mu = \sum_{j\in\mathbb{N}} \delta_{a_j}$, where δ_a is the Dirac measure concentrated at a point a, then $G_\mu = \log|B|$, where B is the Blaschke product.

The problem of the boundedness of $m_2(r, \log|B|)$ was posed by A. Zygmund (see in [3]).

It was shown by G. MacLane and L. Rubel [3] that the condition

$$n(r) = O\left(\frac{1}{\sqrt{1-r}}\right), \quad r \to 1, \tag{12}$$

is sufficient for boundedness of $m_2(r, \log|B|)$. If the zeros of B are positive and

$$m_2(r, \log|B|) = O(1), \quad t \to 1,$$

then (11) is fulfilled [3].

It was observed [4, 5] that condition (11) is not sufficient for the boundedness of $m_2(r, \breve{u})$ in the case $u = \log|B|$. In this context we are going to prove the following theorem.

Theorem 4. *Let $u(z) = G_\mu(z) - G_\mu(0)$, $F = u + i\breve{u}$. If*

$$\int_0^1 \frac{n(t)}{\sqrt{1-t}} \frac{dt}{t} < +\infty, \tag{13}$$

then

$$m_2(r, F) = O(1), \quad r \to 1. \tag{14}$$

Proof. From representation (5) we obtain

$$m_2^2(r, F) = m_2^2(r, u) + m_2^2(r, \widetilde{u} - \widetilde{p}).$$

Using the Parseval identity and the Minkowski inequality we have

$$m_2(r, F) \le 2m_2(r, u) + m_2(r, p). \tag{15}$$

As in [6] we find

$$m_2(r, u) \le \frac{C_1}{\sqrt{1-r}} \int_r^1 \frac{n(t)}{t} dt + C_2 \tag{16}$$

for $r \in (0, 1)$ and for some positive constants C_1, C_2.

We now obtain an estimate of $m_2(r, p)$. Putting $t = rx$ and applying twice the Minkowski integral inequality we obtain

$$m_2(r, p) = \left\{ \int_{\mathbb{T}} \left(\int_0^r \frac{dt}{t} \int_{|a|\le t} P_r\left(t\zeta \frac{\bar{a}}{|a|}\right) d\mu_a \right)^2 dm(\zeta) \right\}^{1/2} \le$$
$$\le \int_0^1 \frac{dx}{x} \int_{|a|\le rx} \left\{ \int_{\mathbb{T}} P_1^2 \left(x\zeta \frac{\bar{a}}{|a|}\right) dm(\zeta) \right\}^{1/2} d\mu_a.$$

Since

$$\left\{ \int_{\mathbb{T}} P_1^2(t\zeta) dm(\zeta) \right\}^{1/2} = O\left(\frac{1}{\sqrt{1-t}} \right), \quad t \to 1,$$

we find

$$m_2(r,p) \le C \int_0^1 \frac{n(rt)}{\sqrt{1-t}} \frac{dt}{t}, \quad 0 < r < 1, \quad C = \text{const}. \tag{17}$$

Thus, from (12),

$$\int_r^1 \frac{n(\tau)}{\sqrt{1-\tau}} \frac{d\tau}{\tau} = o(1), \quad r \to 1.$$

Since

$$\int_r^1 \frac{n(\tau)}{\sqrt{1-\tau}} \frac{d\tau}{\tau} \ge \frac{1}{\sqrt{1-r}} \int_r^1 \frac{n(\tau)}{\tau} d\tau,$$

from (14),(15) and (16) we obtain (13). □

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Lviv National University, Faculty of Mechanics and Mathematics

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