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INTERPOLATION IN SOME CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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A criterion is obtained for the solvability of the interpolation problem $f(\lambda_n) = b_n$ in the class of analytic in the unit disk functions f satisfying

$$(\exists c_1 > 0)\ (\forall z:\ |z| < 1)\ :\ |f(z)| \leqslant \exp\left(c_1 \gamma\left(\frac{c_1}{1-|z|}\right)\right),$$

where $\gamma\colon [1; +\infty) \to (0; +\infty)$ is an increasing function continuously differentiable on $[1; +\infty)$ such that $(\exists c_2 > 0)(\forall t > 1) : \dfrac{\gamma(t)\ln\gamma(t)}{t\gamma'(t)} \leqslant c_2$.

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Получен критерий существования решения интерполяционной задачи $f(\lambda_n) = b_n$ в классе целых функций f, для которых

$$(\exists c_1 > 0)\ (\forall z:\ |z| < 1)\ :\ |f(z)| \leqslant \exp\left(c_1 \gamma\left(\frac{c_1}{1-|z|}\right)\right),$$

где $\gamma\colon [1; +\infty) \to (0; +\infty)$ — возрастающая, непрерывно дифференцируемая на $[1; +\infty)$ функция такая, что $(\exists c_2 > 0)(\forall t > 1) : \dfrac{\gamma(t)\ln\gamma(t)}{t\gamma'(t)} \leqslant c_2$.

Let (λ_n) be a sequence of different complex numbers such that $\lim_{n\to\infty} |\lambda_n| = 1$, $\gamma\colon [1, +\infty) \to (0, +\infty)$ is an increasing function convex of $\ln t$. Denote by $A_1[\gamma]$ the class of functions analytic in the unit disk which satisfy the condition

$$(\exists c_1 > 0)\ (\forall z,\ |z| < 1)\ :\ |f(z)| \leqslant \exp\left(c_1 \gamma\left(\frac{c_1}{1-|z|}\right)\right). \tag{1}$$

The well-known Carleson's theorem [1, c. 285] describes interpolation sequences for the class $A_1[\gamma]$, where $\gamma = \text{const}$. We obtain a criterion of the solvability of the interpolation problem

$$f(\lambda_n) = b_n \tag{2}$$

in the class $A_1[\gamma]$, when γ is a fast-growing majorant.

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Let

$$N(r)=\int_0^r \frac{n(x)-n(0)}{x}dt+n(0)\ln r,\qquad n(x)=\sum_{|\lambda_n|\leqslant x}1;$$

$$N_{\lambda_n}(r)=\int_0^r \frac{n_{\lambda_n}(x)-1}{x}dx,\quad n_{\lambda_n}(x)=\sum_{|z-\lambda_n|\leqslant x}1;$$

$$B_n(z)=\prod_{|\lambda_i|\leqslant\frac{1+|\lambda_n|}{2}}\overline{\lambda}_i\ \frac{\lambda_i-z}{1-\overline{\lambda}_i z},\quad G_n(z)=\prod_{|\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}\overline{\lambda}_i\ \frac{\lambda_i-z}{1-\overline{\lambda}_i z},$$

where $\alpha\in(0;1)$ is a fixed number. Denote by $c_1, c_2, \dots$ positive constants. We assume that all $\lambda_n\neq 0$.

The main results of this article are Theorems 1–5.

Theorem 1. *Let γ satisfy the condition*

$$(\exists c_2)(\forall t>1):\frac{\gamma(t)\ln\gamma(t)}{t\gamma'(t)}\leqslant c_2 \tag{3}$$

and

$$(\exists c_3)(\forall r\in(0;1)):N(r)\leqslant c_3\gamma\left(\frac{c_3}{1-r}\right). \tag{4}$$

Then there exists a sequence (p_n) of non-negative integers such that the function

$$L(z)=\prod_{n=1}^{\infty}E\left(\frac{1-|\lambda_n|^2}{1-\overline{\lambda}_n z},\ p_n\right)=\prod_{n=1}^{\infty}\left(1-\frac{1-|\lambda_n|^2}{1-\overline{\lambda}_n z}\right)\exp\sum_{\nu=1}^{p_n}\frac{1}{\nu}\left(\frac{1-|\lambda_n|^2}{1-\overline{\lambda}_n z}\right)^{\nu}$$

is analytic in the disk $\{z:\ |z|<1\}$, it satisfies condition (1) and

$$(\exists c_4):\ \ln|L'(\lambda_n)|=\ln|B_n'(\lambda_n)|+O\left(\gamma\left(\frac{c_4}{1-|\lambda_n|}\right)\right). \tag{5}$$

Proof. Set $p_n=[1/c_2\ln\gamma(2c_3\delta_n)]$, $\delta_n=\frac{1}{1-|\lambda_n|}$. We show that L satisfies condition (1). Let $L=L_1L_2L_3$, $u=\frac{1-|\lambda_n|^2}{1-\overline{\lambda}_n z}$,

$$L_1=\prod_{|u|<1/2}E(u;\ p_n),\ L_2=\prod_{|u|\geqslant 1/2}(1-u),\ L_3=\prod_{|u|\geqslant 1/2}\exp\sum_{\nu=1}^{p_n}\frac{1}{\nu}u^{\nu}.$$

In accordance with [2, c. 24],

$$|\ln|L_1||\leqslant\sum_{|u|<1/2}2|u|^{p_n+1},\ |\ln|L_3||\leqslant\sum_{|u|\geqslant 1/2}(2|u|)^{p_n}.$$

Since $\left|\frac{\overline{\lambda}_n(\lambda_n-z)}{1-\overline{\lambda}_n}\right|<1$ at $|z|<1$, we have $\ln|L_2|<0$. Therefore

$$\ln|L(z)|\leqslant\sum_{|u|<1/2}2|u|^{p_n+1}+\sum_{|u|\geqslant 1/2}(2|u|)^{p_n}\leqslant\sum_{|u|<1/2}2|u|^{[1/c_2\ln\gamma(2c_3\delta_n)]+1}+$$

$$+\sum_{|u|\geqslant 1/2}(2|u|)^{[1/c_2\ln\gamma(2c_3\delta_n)]}\leqslant 2\sum_{n=1}^{\infty}(2|u|)^{1/c_2\ln\gamma(2c_3\delta_n)}\leqslant 2\sum_{n=1}^{\infty}\left(4\frac{1-|\lambda_n|}{1-r}\right)^{1/c_2\ln\gamma(2c_3\delta_n)}\leqslant$$

$$\leqslant 2\sum_{n=1}^{\infty}\left(\frac{4t}{\delta_n}\right)^{1/c_2\ln\gamma(2c_3\delta_n)}=2\sum_{n=1}^{\infty}\left(\frac{4e^{4c_2}t}{\delta_n}\right)^{1/c_2\ln\gamma(2c_3\delta_n)}\gamma^{-4}(2c_3\delta_n),$$

where $t = \frac{1}{1-r}$. As shown in [3, 4],

$$(\exists c_5) : \max\{\left(4e^{4c_2}t/x\right)^{1/c_2 \ln\gamma(2c_3x)} : x \geqslant 1\} \leqslant \gamma(c_5 t),$$
$$(\exists c_6)\ (\exists t_0)\ (\forall t \geqslant t_0)\ :\ \frac{t}{\gamma(t)} \geqslant C_6. \tag{6}$$

From this and from the inequalities

$$N \leqslant n(|\lambda_n|) \leqslant \frac{2}{1-|\lambda_n|} N\left(\frac{1+|\lambda_n|}{2}\right) \leqslant \frac{2c_3}{1-|\lambda_n|}\gamma\left(\frac{2c_3}{1-|\lambda_n|}\right) \tag{7}$$

we have

$$\ln|L(z)| \leqslant 2\gamma\left(\frac{c_5}{1-|z|}\right)\sum_{n=1}^{\infty}\gamma^{-4}\left(\frac{2c_3}{1-|\lambda_n|}\right) \leqslant$$
$$\leqslant 2\gamma\left(\frac{c_5}{1-|z|}\right)\sum_{n=1}^{\infty}\frac{{c_6}^2}{\left(\frac{2c_3}{1-|\lambda_n|}\gamma\left(\frac{2c_3}{1-|\lambda_n|}\right)\right)^2} \leqslant 2\gamma\left(\frac{c_5}{1-|z|}\right)\sum_{n=1}^{\infty}\frac{{c_6}^2}{n^2} \leqslant c_7\gamma\left(\frac{c_7}{1-|z|}\right),$$

Hence $L \in A_1[\gamma]$. Next we prove (5). Let $a_n(z) = L(z)/B_n(z)$. Then $|B_n'(\lambda_n)| = \dfrac{|L'(\lambda_n)|}{|a_n(\lambda_n)|}$, $a_n(\lambda_n) = L_1L_3L_4$, where

$$L_1 = \prod_{\frac{1-|\lambda_i|^2}{|1-\overline{\lambda}_i\lambda_n|}<\frac{1}{2}} E\left(\frac{1-|\lambda_i|^2}{1-\overline{\lambda}_i\lambda_n}, p_i\right), \quad L_3 = \prod_{\frac{1-|\lambda_i|^2}{|1-\overline{\lambda}_i\lambda_n|}\geqslant\frac{1}{2}} \exp\sum_{\nu=1}^{p_i}\frac{1}{\nu}\left(\frac{1-|\lambda_i|^2}{1-\overline{\lambda}_i\lambda_n}\right)^{\nu},$$
$$L_4 = \prod_{\frac{1-|\lambda_i|^2}{|1-\overline{\lambda}_i\lambda_n|}\geqslant\frac{1}{2}, |\lambda_i|>\frac{1+|\lambda_n|}{2}} \overline{\lambda}_i\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}.$$

As before,

$$|\ln|L_1|| + |\ln|L_3|| \leqslant c_7\gamma\left(\frac{c_7}{1-|\lambda_n|}\right). \tag{8}$$

Further, if $|\lambda_i| > \frac{1+|\lambda_n|}{2}$, then

$$\left|\ln\left|\overline{\lambda}_i\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}\right|\right| = \ln\left|\frac{1-\overline{\lambda}_i\lambda_n}{\overline{\lambda}_i(\lambda_i-\lambda_n)}\right| \leqslant \ln\frac{2}{\frac{1+|\lambda_n|}{2}\left(\frac{1+|\lambda_n|}{2}-|\lambda_n|\right)} \leqslant \ln\frac{8}{1-|\lambda_n|} \leqslant \frac{c_8}{1-|\lambda_n|}. \tag{9}$$

Using the equality $\ln\gamma(ct) - \ln\gamma(t) = \int_t^{ct}\dfrac{x\gamma'(x)}{\gamma(x)}\,d\ln x$ and conditions (3), (6), we obtain that

$$(\forall d \in \mathbb{R}_+)\ (\exists c > 1)\ (\exists t_1)\ (\forall t \geqslant t_1)\ :\ t^d\gamma(t) \leqslant \gamma(ct). \tag{10}$$

From the inequality $\frac{1-|\lambda_i|^2}{|1-\overline{\lambda}_i\lambda_n|} \geqslant \frac{1}{2}$ it follows that $|\lambda_i| < \frac{3+|\lambda_n|}{4}$. Therefore, in correspondence with (7), (9), (10), we have

$$|\ln|L_4|| \leqslant \sum_{\frac{1+|\lambda_n|}{2}<|\lambda_i|<\frac{3+|\lambda_n|}{4}}\left|\ln\left|\overline{\lambda}_i\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}\right|\right| \leqslant \frac{c_8}{1-|\lambda_n|}\sum_{|\lambda_i|<\frac{3+|\lambda_n|}{4}} 1 \leqslant$$
$$\leqslant \frac{c_8}{1-|\lambda_n|}n\left(\frac{3+|\lambda_n|}{4}\right) \leqslant c_9\gamma\left(\frac{c_9}{1-|\lambda_n|}\right). \tag{11}$$

Consequently, (5) follows from (8) and (11). $\square$

Theorem 2. *Let conditions (3) and (4) be fulfilled and L be the function from Theorem 1. Then the following statements are equivalent:*

$$1)(\exists c_{10})\ (\forall n\in\mathbb{N})\ :\ 1/|L'(\lambda_n)|\leqslant \exp\left(c_{10}\gamma\left(\frac{c_{10}}{1-|\lambda_n|}\right)\right), \tag{12}$$

$$2)(\exists c_{11})\ (\forall n\in\mathbb{N})\ :\ 1/|G_n'(\lambda_n)|\leqslant \exp\left(c_{11}\gamma\left(\frac{c_{11}}{1-|\lambda_n|}\right)\right), \tag{13}$$

$$3)\ (\exists c_{12})\ (\forall n\in\mathbb{N})\ :\ \prod_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}}\left|\frac{\lambda_n-\lambda_i}{1-\overline{\lambda}_i\lambda_n}\right|\geqslant \exp\left(-c_{12}\gamma\left(\frac{c_{12}}{1-|\lambda_n|}\right)\right),$$

$$4)\ (\exists c_{13})\ (\forall n\in\mathbb{N})\ :\ N_{\lambda_n}(\alpha(1-|\lambda_n|))\leqslant c_{13}\gamma\left(\frac{c_{13}}{1-|\lambda_n|}\right).$$

Proof. It is easy to show that

$$(\exists c_{14}):\ \ln|B_n'(\lambda_n)|=\ln|G_n'(\lambda_n)|+O\left(\gamma\left(\frac{c_{14}}{1-|\lambda_n|}\right)\right). \tag{14}$$

Indeed, let $g_n=B_n/G_n,\ g_n=\pi_1\pi_2$, where

$$\pi_1=\prod_{\substack{|\lambda_i|\leqslant\frac{1+|\lambda_n|}{2}\\ |\lambda_i-\lambda_n|>\alpha(1-|\lambda_n|)}}\overline{\lambda}_i\,\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n},\ \pi_2=\prod_{\substack{|\lambda_i|>\frac{1+|\lambda_n|}{2}\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}}\frac{1-\overline{\lambda}_i\lambda_n}{\overline{\lambda}_i(\lambda_i-\lambda_n)}.$$

If $|\lambda_i|\leqslant\frac{1+|\lambda_n|}{2},\quad |\lambda_i-\lambda_n|>\alpha(1-|\lambda_n|)$, then

$$\left|\ln\left|\overline{\lambda}_i\,\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}\right|\right|=\ln\frac{|1-\overline{\lambda}_i\lambda_n|}{|\lambda_i||\lambda_i-\lambda_n|}\leqslant\ln\frac{2}{\alpha|\lambda_1|(1-|\lambda_n|)}\leqslant\frac{c_{15}}{1-|\lambda_n|}.$$

If $|\lambda_i|>\frac{1+|\lambda_n|}{2}$, then inequalities (9) hold. Therefore, using (7) and (10), we have

$$|\ln|g_n||\leqslant\frac{c_{15}}{1-|\lambda_n|}\sum_{|\lambda_i|\leqslant\frac{1+|\lambda_n|}{2}}1+\frac{c_8}{1-|\lambda_n|}\sum_{|\lambda_i|\leqslant|\lambda_n|+\alpha(1-|\lambda_n|)}1\leqslant$$

$$\leqslant\frac{c_{15}}{1-|\lambda_n|}\,n\left(\frac{1+|\lambda_n|}{2}\right)+\frac{c_8}{1-|\lambda_n|}\,n(|\lambda_n|+\alpha(1-|\lambda_n|))\leqslant c_{16}\gamma\left(\frac{c_{16}}{1-|\lambda_n|}\right).$$

Thus, taking into account that $|B_n'(\lambda_n)|=|G_n'(\lambda_n)||g_n(\lambda_n)|$, we obtain (12). Then the equivalence of conditions 1) and 2) follows from (5). Further, prove equivalence 2) and 4). It is obvious that

$$|G_n'(\lambda_n)|=\frac{|\lambda_n|}{1-|\lambda_n|^2}\prod_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}}|\lambda_i|\left|\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}\right|.$$

From (6) and the inequalities $\ln|\lambda_1|<\ln\frac{|\lambda_n|}{1-|\lambda_n|^2}<\ln\frac{1}{1-|\lambda_n|}<\frac{1}{1-|\lambda_n|}$, $|\lambda_1|\leqslant|\lambda_i|<1$ we obtain

$$\sum_{|\lambda_i|\leqslant|\lambda_n|+\alpha(1-|\lambda_n|)}\ln|\lambda_i|\geqslant -N(|\lambda_n|+\alpha(1-|\lambda_n|))\ln|1/\lambda_1|\geqslant -c_{17}\gamma\left(\frac{c_{17}}{1-|\lambda_n|}\right);$$

$$\ln|G_n'(\lambda_n)| = \ln \prod_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}} \left|\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}\right| + \ln\frac{|\lambda_n|}{1-|\lambda_n|^2} + \sum_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}} \ln|\lambda_i| =$$

$$= \sum_{i\neq n|\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)} \ln\left|\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}\right| + O\left(\gamma\left(\frac{c_{17}}{1-|\lambda_n|}\right)\right),\ n\to\infty. \tag{15}$$

Further, since $\frac{1}{|1-\overline{\lambda}_i\lambda_n|}\leqslant\frac{1}{1-|\lambda_n|}$, then

$$\sum_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}} \ln\left|\frac{\lambda_i-\lambda_n}{1-\overline{\lambda}_i\lambda_n}\right| \leqslant \sum_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}} \ln\frac{|\lambda_i-\lambda_n|}{1-|\lambda_n|} =$$

$$= \int_0^{\alpha(1-|\lambda_n|)} \ln\frac{x}{1-|\lambda_n|} d(n_{\lambda_n}(x)-1) = \ln\alpha(n_{\lambda_n}(\alpha(1-|\lambda_n|))-1)-$$

$$-\int_0^{\alpha(1-|\lambda_n|)} \frac{n_{\lambda_n}(x)-1}{x}dx \leqslant -N_{\lambda_n}(\alpha(1-|\lambda_n|)).$$

On the other hand, $\frac{1}{|1-\overline{\lambda}_i\lambda_n|}>\frac{1}{2}$. Therefore

$$\sum_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}} \ln\frac{|\lambda_i-\lambda_n|}{|1-\overline{\lambda}_i\lambda_n|} \geqslant \sum_{\substack{i\neq n\\ |\lambda_i-\lambda_n|\leqslant\alpha(1-|\lambda_n|)}} \ln(|\lambda_i-\lambda_n|/2) =$$

$$= \int_0^{\alpha(1-|\lambda_n|)} \ln x/2\ d(n_{\lambda_n}(x)-1) = \ln\alpha(1-|\lambda_n|)/2(n_{\lambda_n}(\alpha(1-|\lambda_n|))-1)-$$

$$-N_{\lambda_n}(\alpha(1-|\lambda_n|)) = -\ln 2/(\alpha(1-|\lambda_n|))(n_{\lambda_n}(\alpha(1-|\lambda_n|))-1) - N_{\lambda_n}(\alpha(1-|\lambda_n|)) \geqslant$$

$$\geqslant -\frac{c_{18}}{1-|\lambda_n|}\ln\frac{1}{1-|\lambda_n|}\gamma\left(\frac{c_{18}}{1-|\lambda_n|}\right) - N_{\lambda_n}(\alpha(1-|\lambda_n|)) \geqslant$$

$$\geqslant -\gamma\left(\frac{c_{19}}{1-|\lambda_n|}\right) - N_{\lambda_n}(\alpha(1-|\lambda_n|)).$$

Thus,

$$\ln|G_n'(\lambda_n)| = -N_{\lambda_n}(\alpha(1-|\lambda_n|)) + O\left(\gamma\left(\frac{c_{20}}{1-|\lambda_n|}\right)\right),\ n\to\infty,$$

and the equivalence of conditions 2) and 3) follows from (15). □

Theorem 3. *Let γ satisfy condition (3). For (λ_n) to be the sequence of zeros of some function $L\in A_1[\gamma]$ it is necessary and sufficient that condition (4) hold.*

Indeed, the necessity of (4) follows immediately from the Jensen inequality, and sufficiency follows from Theorem 1.

Theorem 4. *Let γ be convex of $\ln t$ on $[1;+\infty)$ and $\ln t = o(\gamma(t))$ $(t\to\infty)$. Then for each sequence (b_n) of complex numbers with the property*

$$(\exists c_{21})\ (\forall n\in\mathbb{N})\ : |b_n| \leqslant \exp\left(c_{21}\gamma\left(\frac{c_{21}}{1-|\lambda_n|}\right)\right) \tag{16}$$

there exists a function $f \in A_1[\gamma]$ which satisfies (2) if for some function $L \in A_1[\gamma]$ which has simple zeros in the points λ_n condition (12) holds and only if condition (4) and

$$(\exists c_{22})(\forall n \in \mathbb{N}) : 1/|G_n'(\lambda_n)| \leqslant \exp\left(\frac{c_{22}}{1-|\lambda_n|}\ln\frac{1}{1-|\lambda_n|}\gamma\left(\frac{c_{22}}{1-|\lambda_n|}\right)\right), \tag{17}$$

are true.

Proof. Necessity of conditions (4) is proved by standard way (see, for example, [3]). We prove necessity of condition (17). Suppose that (17) is not fulfilled. Then there exists $\{\lambda_{n_m}\} \subset \{\lambda_n\}$ such that

$$1/|G_n'(\lambda_{n_m})| > \exp\left(\frac{m}{1-|\lambda_{n_m}|}\ln\frac{1}{1-|\lambda_{n_m}|}\gamma\left(\frac{m}{1-|\lambda_{n_m}|}\right)\right),$$

The sequence (λ_{n_m}) can be chosen such that for some fixed $\alpha_1 \in (\alpha; 1)$ $|\lambda_{n_{m+1}}| > |\lambda_{n_m}| + \alpha_1(1-|\lambda_{n_m}|)$. Since the sequence $(b_n) : b_n = 1$ at $n = n_m$, $b_n = 0$ at $n \neq n_m$, satisfies condition (16), there exists a function $f \in A_1[\gamma]$ which at the points λ_{n_m} equals 1 and at all the other points λ_n equals 0. Therefore

$$\frac{|\lambda_{n_m}|/(1-|\lambda_{n_m}|^2)}{|G_{n_m}'(\lambda_{n_m})|} = \lim_{z\to\lambda_{n_m}}\left|\frac{(z-\lambda_{n_m})f(z)}{G_{n_m}(z)}\right|$$

Further, using the maximum principle we obtain (here $C_m = \{t : |t-\lambda_{n_m}| = \alpha_1(1-|\lambda_{n_m}|)\}$):

$$\lim_{z\to\lambda_{n_m}}\left|\frac{(z-\lambda_{n_m})f(z)}{G_{n_m}(z)}\right| = \left|\frac{1}{2\pi i}\oint_{C_m}\frac{f(t)}{G_{n_m}(t)}\,dt\right| \leqslant$$
$$\leqslant \alpha_1(1-|\lambda_{n_m}|)\exp\left(c_1\gamma\left(\frac{c_1(1-\alpha_1)}{1-|\lambda_{n_m}|}\right)\right)\max_{|t-\lambda_{n_m}|=\alpha_1(1-|\lambda_{n_m}|)}|G_{n_m}(t)|^{-1}.$$

If $|t| = |\lambda_{n_m}| + \alpha_1(1-|\lambda_{n_m}|)$, then

$$|G_{n_m}(t)| = \prod_{|\lambda_i-\lambda_{n_m}|\leqslant\alpha(1-|\lambda_{n_m}|)}|\lambda_i|\frac{|\lambda_i-t|}{|1-\overline{\lambda}_i t|} \geqslant \prod_{|\lambda_i-\lambda_{n_m}|\leqslant\alpha(1-|\lambda_{n_m}|)}|\lambda_i|\frac{||\lambda_i|-|t||}{1+|\lambda_i||t|} \geqslant$$
$$\geqslant \prod_{\substack{|\lambda_{n_m}|-\alpha(1-|\lambda_{n_m}|)\leqslant|\lambda_i|\leqslant\\ \leqslant|\lambda_{n_m}|+\alpha(1-|\lambda_{n_m}|)}}(1-\alpha)(\alpha_1-\alpha)(1-|\lambda_{n_m}|)/2 \geqslant$$
$$\geqslant \exp\left(-\ln\frac{c_{23}}{1-|\lambda_{n_m}|}n(|\lambda_{n_m}|+\alpha(1-|\lambda_{n_m}|))\right) \geqslant \exp\left(-\ln\frac{c_{23}}{1-|\lambda_{n_m}|}\times\right.$$
$$\left.\times\frac{c_3/(1-\alpha)}{1-|\lambda_{n_m}|}\gamma\left(\frac{2c_3/(1-\alpha)}{1-|\lambda_{n_m}|}\right)\right) \geqslant \exp\left(-\frac{c_{24}}{1-|\lambda_{n_m}|}\ln\frac{1}{1-|\lambda_{n_m}|}\gamma\left(\frac{c_{24}}{1-|\lambda_{n_m}|}\right)\right),$$

and therefore

$$1/|G_{n_m}'(\lambda_{n_m})| \leqslant (1-|\lambda_{n_m}|)^2\exp\left(\frac{c_{25}}{1-|\lambda_{n_m}|}\ln\frac{1}{1-|\lambda_{n_m}|}\gamma\left(\frac{c_{25}}{1-|\lambda_{n_m}|}\right)\right) \leqslant$$
$$\leqslant \exp\left(\frac{c_{26}}{1-|\lambda_{n_m}|}\ln\frac{1}{1-|\lambda_{n_m}|}\gamma\left(\frac{c_{26}}{1-|\lambda_{n_m}|}\right)\right).$$

But it contradicts the assumption.

Sufficiency. If γ is convex of $\ln t$ on $[1,\ +\infty)$ the function $c_0\gamma(c_0t)$, where $c_0\geqslant 4(\Delta c_3+c_{10}+c_{21}),\ \Delta>1$, is also convex of $\ln t$. Thus [5] there exists an entire function θ with θ_n as the Taylor coefficients such that

$$\ln M_\theta(t)=(1+o(1))c_0\gamma(c_0t),\ t\to+\infty, \tag{18}$$

where $M_\theta(r)=\max\{|f(z)|:|z|=r\}$. Set $\mu_\theta(r)=\max\{|\theta_n|r^n:\ n\geqslant 0\}$, $\varkappa_n(\theta)=\left|\frac{\theta_{n-1}}{\theta_n}\right|$. From (18) and the known inequalities $\mu_\theta(t)\leqslant M_\theta(t)\leqslant\left(1+\frac{1}{\varepsilon}\right)\mu_\theta((1+\varepsilon)t)$, $\varepsilon>0$, $t>0$ at large t we obtain

$$\begin{gathered}\mu_\theta(t)\leqslant\exp(2c_0\gamma(c_0t)),\ t\geqslant t_0,\\ \mu_\theta(t)\geqslant 1/2\exp(c_0/2\gamma(c_0/2\ t)),\geqslant\exp(c_0/4\gamma(c_0/2\ t)),\ t\geqslant t_0.\end{gathered}\tag{19}$$

Choose a sequence of natural numbers (s_n) such that $\varkappa_{s_n}(\widehat{\theta})\leqslant\delta_n<\varkappa_{s_n+1}(\widehat{\theta})$, $\delta_n=\frac{1}{1-|\lambda_n|}$, where $\widehat{\theta}$ is the Newton majorant of the function θ. Then [6]

$$\mu_\theta(\delta_n)=\widehat{\theta}_{s_n}\delta_n^{s_n},\ \mu_\theta(r)\geqslant\widehat{\theta}_{s_n}r^{s_n}. \tag{20}$$

Let's prove that the function

$$f(z)=\sum_{n=1}^{\infty}\frac{b_n\ L(z)}{(z-\lambda_n)L'(\lambda_n)}\left(\frac{1-|\lambda_n|^2}{1-\overline{\lambda}_nz}\right)^{s_n}$$

satisfies condition (1). In [7] it is shown that for all $r\in[0,\ 1)$ and $\beta>0$ with $(1+\beta)r<1$ we have $\max_{|z|=r}|L(z)|/|z-\lambda_n)|\leqslant K(\beta)M_L((1+\beta)r)/(r+|\lambda_n|)$, where $K(\beta)\leqslant 2(2+\beta)/\beta$.

Therefore, for $\beta=\frac{1-r}{2r}$ we obtain

$$\max_{|z|=r}\left|\frac{L(z)}{z-\lambda_n}\right|\leqslant\frac{8}{1-r}\frac{M_L\left(\frac{r+1}{2}\right)}{r+|\lambda_n|}\leqslant\frac{c_{13}}{1-r}M_L\left(\frac{r+1}{2}\right). \tag{22}$$

Moreover,

$$(\forall q>0)(\exists t_2)(\forall t\geqslant t_2):\ e^{q\gamma(t)}\geqslant t\gamma(t), \tag{22}$$

because $\ln t+\ln\gamma(t)=o(\gamma(t))+o(\gamma(t))=o(\gamma(t)),\ t\to\infty$. Thus, from inequality (7), (19)–(21) and conditions (12), (16), (22) we obtain that

$$\begin{aligned}|f(z)|&=\sum_{n=1}^{\infty}\left|\frac{L(z)}{z-\lambda_n}\right|\frac{|b_n|}{|L'(\lambda_n)|}\left|\frac{1-|\lambda_n|^2}{1-\overline{\lambda}_nz}\right|^{s_n}\leqslant\frac{c_{27}}{1-r}M_L\left(\frac{r+1}{2}\right)\times\\ &\times\sum_{n=1}^{\infty}\frac{|b_n|}{|L'(\lambda_n)|}\frac{\widehat{\theta}_{s_n}\left(\frac{2}{1-r}\right)^{s_n}}{\widehat{\theta}_{s_n}\left(\frac{1}{1-|\lambda_n|}\right)^{s_n}}\leqslant\frac{c_{27}}{1-r}\exp\left(c_1\gamma\left(\frac{2c_1}{1-r}\right)+2c_0\gamma\left(\frac{2c_0}{1-r}\right)\right)\times\\ &\times\sum_{n=1}^{\infty}\exp\left(c_{21}\gamma\left(\frac{c_{21}}{1-|\lambda_n|}\right)+C_{10}\gamma\left(\frac{c_{10}}{1-|\lambda_n|}\right)-\frac{c_0}{4}\gamma\left(\frac{c_0/2}{1-|\lambda_n|}\right)\right)\leqslant\\ &\leqslant\exp\left(c_{28}\gamma\left(\frac{c_{28}}{1-r}\right)\right)\sum_{n=1}^{\infty}\exp\left(-\Delta c_3\gamma\left(\frac{2c_3}{1-|\lambda_n|}\right)\right)\leqslant\\ &\leqslant\exp\left(c_{28}\gamma\left(\frac{c_{28}}{1-r}\right)\right)\sum_{n=1}^{\infty}\frac{1}{\left(\frac{2c_3}{1-|\lambda_n|}\gamma\left(\frac{2c_3}{1-|\lambda_n|}\right)\right)^{\Delta}}\leqslant\exp\left(c_{29}\gamma\left(\frac{c_{29}}{1-r}\right)\right).\end{aligned}$$

It is clear that this function satisfies condition (2). Thus, the theorem is proved. □

Remark. Using estimations from below for the functions analytic in the unit circle [8, c.705], it is possible to show as in [9] that if (λ_n) is the sequence of zeros of some function $L \in A_1[\gamma]$ and γ satisfies the conditions of Theorem 4, then the condition

$$(\exists c_3)(\forall n \in \mathbb{N}): \ 1/|L'(\lambda_n)| \leqslant \exp\left(\frac{c_3}{1-|\lambda_n|}\ln\frac{1}{1-|\lambda_n|}\gamma\left(\frac{c_3}{1-|\lambda_n|}\right)\right),$$

is necessary for interpolation problem (2) to have a solution in the class $A_1[\gamma]$.

Theorem 5. *Let γ satisfy condition (3). Then for each sequence (b_n) of complex numbers with property (16) there exists a function $f \in A_1[\gamma]$ which satisfies (2) it is necessary and sufficient that condition (4) and one of conditions 1)–4) of Theorem 2 hold.*

The validity of this theorem follows from Theorems 2–4 because if (3) is fulfilled then condition (10), from which an equivalence of conditions (13) and (17) is obtained, holds.

We notice that similar results for entire functions were obtained by T. Abanina [3] and for the functions analytic in a half-plane by V. Sharan [4]

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