УДК 517.53

I. N. Peresyolkova

ON DISTRIBUTION OF ZEROS OF GENERALIZED FUNCTIONS OF MITTAG-LEFFLER'S TYPE

I. N. Peresyolkova. *On distribution of zeros of generalized functions of Mittag-Leffler's type*, Matematychni Studii, **13** (2000) 157–164.

The following two problems related to the entire function

$$\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu_1+n/\rho_1)\Gamma(\mu_2+n/\rho_2)}\ ,\quad \rho_1,\rho_2,\mu_1,\mu_2>0,$$

are considered.

(i) For $0<\rho_1\le 1/2$, determine the values of μ_1,ρ_2,μ_2 such that all zeros of $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$ are negative.

(ii) For $1/2<\rho_1<1$, determine the values of μ_1,ρ_2,μ_2 such that all zeros of $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$ are situated outside the angle $\{z:|\arg z|\le\pi/(2\rho_1)\}$.

The results point out several values of μ_1 and all values of $\rho_2>0,\mu_2>0$ with the above properties. Moreover, we obtain some results on a-points of the function $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$.

И. Н. Пересёлкова. *О распределении нулей обобщенных функций типа Миттаг-Леффлера* // Математичні Студії. – 2000. – Т.13, №2. – С.157–164.

Рассматриваются следующие две задачи о целой функции

$$\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu_1+n/\rho_1)\Gamma(\mu_2+n/\rho_2)}\ ,\quad \rho_1,\rho_2,\mu_1,\mu_2>0.$$

(а) Найти для $0<\rho_1\le 1/2$ такие значения μ_1,ρ_2,μ_2, что все нули $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$ являются отрицательными.

(б) Найти для $1/2<\rho_1<1$ такие значения μ_1,ρ_2,μ_2, что все нули $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$ лежат вне угла $\{z:|\arg z|\le\pi/(2\rho_1)\}$.

Полученные результаты гарантируют выполнение этих свойств для многих значений μ_1 и любых $\rho_2>0,\mu_2>0$. Кроме того, получены некоторые результаты об a-точках функции $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$.

1. Introduction. Following the definition of M. M. Dzhrbashyan [1], the function

$$\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu_1+n/\rho_1)\Gamma(\mu_2+n/\rho_2)}\ ,\tag{1}$$

2000 *Mathematics Subject Classification:* 30D35.

where ρ_1, μ_1 and ρ_2, μ_2 are arbitrary positive parameters, is said to be a generalized function of Mittag-Leffler's type.

Primary properties and different applications of the function $\Phi_{\rho_1,\rho_2}(z, \mu_1, \mu_2)$ are summarized in [1]. The function $\Phi_{\rho_1,\rho_2}(z, \mu_1, \mu_2)$ is a generalization of the well-known function of Mittag-Leffler's type $E_\rho(z, \mu) \equiv \Phi_{\rho,\infty}(z, \mu, 1) \equiv \Phi_{\infty,\rho}(z, 1, \mu)$ studied in [2] in detail. The purpose of this paper is to extend some results [3] on distribution of zeros of the function of Mittag-Leffler's type $E_\rho(z, \mu)$ to the zeros of generalized functions of Mittag-Leffler's type $\Phi_{\rho_1,\rho_2}(z, \mu_1, \mu_2)$.

To state the first result, we define the following three transformations mapping the set $\{(\rho, \mu) : \ 0 < \rho < 1, \ \mu > 0\}$ into itself:

$$A\colon (\rho,\mu) \to \left(\frac{\rho}{2}, \mu\right) \ ; \quad B\colon (\rho,\mu) \to \left(\frac{\rho}{2}, \mu + \frac{1}{\rho}\right) \ ;$$

$$C\colon (\rho,\mu) \to \begin{cases} (\rho, \mu - 1) & \text{for } \mu > 1; \\ (\rho, \mu) & \text{for } 0 < \mu \le 1. \end{cases}$$

Put

$$W_a = \{(\rho, \mu) : \ 1/2 < \rho < 1, \ \mu \in [1/\rho - 1, 1] \cup [1/\rho, 2] \ \}, \quad W_b = AW_a \cup BW_a.$$

Denote by W_i the least set containing W_b and invariant with respect to A, B, C. Evidently, the set W_i can be represented in the form

$$W_i = \bigcup (A^{k_{11}} B^{k_{12}} C^{k_{13}} \dots A^{k_{n1}} B^{k_{n2}} C^{k_{n3}}) W_b \ ,$$

where the union is taken over all $n = 1, 2, \dots$ and over all $3n$-tuples $(k_{11}, k_{12}, k_{13}, \dots, k_{n1}, k_{n2}, k_{n3})$ of nonnegative integers.

In [3], the following theorem is proved.

Theorem A ([3, Theorem 3]). *If $(\rho, \mu) \in W_i$, then all zeros of $E_\rho(z, \mu)$ are negative and simple.*

The following theorem is a generalization of Theorem A.

Theorem 1. *Assume $(\rho_1, \mu_1) \in W_i \cup W_j$, where W_i is the set described above and*

$$W_j = \bigcup_{n=1}^{\infty} \left\{ (\rho_1, \mu_1) : \rho_1 \in \left[\frac{1}{5 \cdot 2^{n-2}}, \frac{A}{2^{n-1}}\right] \cup \left\{2^{-n}\right\}, \ 0 < \mu_1 < 1 + 1/\rho_1 \right\},$$

where

$$A = \frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{10}{9\pi} \ln \frac{3}{2}\right) \ (\approx 0.45).$$

Then, for any $\rho_2 > 0, \mu_2 > 0$, all zeros of the entire function $\Phi_{\rho_1,\rho_2}(z, \mu_1, \mu_2)$ are real and negative.

In [3], the following theorem was also proved.

Theorem B ([3, Theorem 1]). *Assume that one of the following conditions is satisfied*:

(i) $\rho > 1, \ \mu \in [1, 1 + 1/\rho]$;

(ii) $\rho = 1,\ \mu \in (0, 2)$;

(iii) $1/2 < \rho < 1,\ \mu \in [1/\rho - 1, 1] \cup [1/\rho, 2]$.

Then all zeros of $E_\rho(z,\mu)$ *are situated outside the closed angle* $\left\{z : |\arg z| \le \frac{\pi}{2\rho}\right\}$.

The following theorem is a generalization of Theorem B (iii).

Theorem 2. *Assume* $1/2 < \rho_1 < 1, \mu_1 \in [1/\rho_1 - 1, 1] \cup [1/\rho_1, 2]$. *Then, for any* $\rho_2 > 0$, $\mu_2 > 0$, *all zeros of the entire function* $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$ *are situated outside the closed angle*

$$\left\{z : |\arg z| \le \frac{\pi}{2\rho_1}\right\}. \tag{2}$$

2. Proof of Theorem 1. We use the following well-known Laguerre theorem.

Theorem C ([4, p. 269]). *Let* $\varphi(\omega)$ *be an entire function of genus 0 or 1, real–valued for real values of* ω, *whose all zeros are real and negative. Let* $f(z)$ *be an entire function representable in the form*

$$f(z) = e^{az+b}\prod_{n=1}^{\infty}\left(1+\frac{z}{z_n}\right),$$

where a *and all* z_n *are positive. If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$g(z) = \sum_{n=0}^{\infty} a_n \varphi(n) z^n$$

is an entire function whose all zeros are real and negative.

Set

$$f(z) = E_{\rho_1}(z,\mu_1) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu_1 + n/\rho_1)}, \tag{3}$$

$$\varphi(\omega) = \frac{1}{\Gamma(\mu_2 + \omega/\rho_2)}. \tag{4}$$

Show that if the values of parameters $\rho_1, \mu_1, \rho_2, \mu_2$ are taken from the statement of Theorem 1, then the conditions of Theorem C are satisfied, so, by virtue of formulas (1) and (4),

$$\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2) = \sum_{n=0}^{\infty}\frac{\varphi(n)z^n}{\Gamma(\mu_1 + n/\rho_1)} \tag{5}$$

is an entire function whose all zeros are real and negative.

Consider the function $f(z)$. By Theorem A, if $(\rho_1,\mu_1) \in W_i$, then all zeros of this function are real and negative. By [3, Theorem 2] and [5, Theorem 1], if $(\rho_1,\mu_1) \in W_j$, then all zeros of this function are also real and negative. Since $E_{\rho_1}(z,\mu_1)$ is an entire function of order ρ_1 (but W_i and W_j have only $\rho_1 \le 1/2$), by the Hadamard theorem, we have

$$E_{\rho_1}(z,\mu_1) = \frac{1}{\Gamma(\mu_1)} \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right),$$

where all z_n are positive.

Thus, the function $f(z)$ satisfies the conditions of Theorem C.

The function $\varphi(\omega)$ is an entire function of genus 1 vanishing at the points $\omega_k = -\rho_2(k+\mu_2), k \in \mathbb{Z}_+$. All of them are real and negative. Moreover, since $\varphi(\mathbb{R}) \subseteq \mathbb{R}$, the function $\varphi(\omega)$ satisfies the conditions of Theorem C. This completes the proof of Theorem 1.

3. Proof of Theorem 2. We use the following two lemmas. The first one was also used in [3].

Lemma A (Gauss's lemma). *Let $f(z) \not\equiv 0$ be a meromorphic function representable in the form*

$$f(z) = \frac{\gamma_0}{z} + \sum_{k=1}^{\infty} \frac{\gamma_k}{z - a_k},$$

where

$$\gamma_k \ge 0 \quad (k = 0, 1, 2, ...), \quad \sum_{k=1}^{\infty} \frac{\gamma_k}{|a_k|} < \infty .$$

If the sequence $\{a_k\}_{k=1}^{\infty}$ is contained in the open angle

$$A_{\eta,\delta} = \{z : \eta < \arg z < \eta + \delta\}, \quad \eta \in [0, 2\pi), \quad 0 < \delta \le \pi, \tag{6}$$

then $f(z)$ does not vanish outside $A_{\eta,\delta}$.

The following lemma is a slight generalization of the Laguerre theorem mentioned above.

Lemma 1. *Let $f(z)$ be an entire function of order $\rho < 1$, nonvanishing outside the open angle $A_{\eta,\delta}$ defined by formula (6). Let $\varphi(\omega)$ be an entire function of genus 0 or 1, real-valued for real values of ω, whose all zeros are real and negative. If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$g(z) = \sum_{n=0}^{\infty} a_n \varphi(n) z^n$$

is an entire function whose all zeros are contained in the open angle $A_{\eta,\delta}$.

Proof. First of all, $g(z)$ is an entire function. Indeed, by the Hadamard theorem

$$\varphi(\omega) = a e^{\omega b} \prod_{n=1}^{\infty} \left(1 + \frac{\omega}{\alpha_n}\right) e^{-\frac{\omega}{\alpha_n}},$$

where $\alpha_n > 0$ for all n.

Since $(1+x)e^{-x} \leq 1$ for $x \geq 0$, we have $|\varphi(n)| \leq |a|e^{nb}$, and hence the series for $g(z)$ converges everywhere.

Consider the function

$$g_1(z) = f(z) + \frac{z}{\alpha_1} f'(z) = \frac{f(z)z}{\alpha_1} \left(\frac{\alpha_1}{z} + \frac{f'(z)}{f(z)} \right).$$

Show that all zeros of $g_1(z)$ are contained in $A_{\eta,\delta}$.

By the Hadamard theorem

$$f(z) = c \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k} \right),$$

where

$$a_k \in A_{\eta,\delta} \ (k = 1, 2, ...) \ , \quad \sum_{k=1}^{\infty} \frac{1}{|a_k|} < \infty \ .$$

Therefore

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{\infty} \frac{1}{z - a_k} \ .$$

By Lemma A, the function $\frac{\alpha_1}{z} + \frac{f'(z)}{f(z)}$ does not vanish outside $A_{\eta,\delta}$. Since, moreover, $g_1(0) \neq 0$, we conclude that $g_1(z)$ does not vanish outside $A_{\eta,\delta}$, as well.

Iterating this argument we establish that for any $n = 2, 3, ...$

$$g_n(z) = g_{n-1}(z) + \frac{z}{\alpha_n} g'_{n-1}(z)$$

does not vanish outside $A_{\eta,\delta}$.

On the other hand,

$$g_1(z) = a_0 + a_1 \left(1 + \frac{1}{\alpha_1} \right) z + ... + a_p \left(1 + \frac{p}{\alpha_1} \right) z^p + ... \ ,$$

$$g_2(z) = a_0 + a_1 \left(1 + \frac{1}{\alpha_1} \right) \left(1 + \frac{1}{\alpha_2} \right) z + ... + a_p \left(1 + \frac{p}{\alpha_1} \right) \left(1 + \frac{p}{\alpha_2} \right) z^p + ... \ ,$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$g_n(z) = a_0 + a_1 \left(1 + \frac{1}{\alpha_1} \right) ... \left(1 + \frac{1}{\alpha_n} \right) z + ... + a_p \left(1 + \frac{p}{\alpha_1} \right) ... \left(1 + \frac{p}{\alpha_n} \right) z^p + ... \ .$$

Therefore

$$g_n\Big(\exp\Big(b - \sum_1^n {\alpha_i}^{-1}\Big)\tilde{z}\Big) =$$

$$= a_0 + a_1 e^b \prod_{i=1}^{n} \left(\left(1 + \frac{1}{\alpha_i} \right) e^{-\frac{1}{\alpha_i}} \right) \tilde{z} + ... + a_p e^{pb} \prod_{i=1}^{n} \left(\left(1 + \frac{p}{\alpha_i} \right) e^{-\frac{p}{\alpha_i}} \right) \tilde{z}^p + ... \ ,$$

and also all zeros of this function are situated inside the angle $A_{\eta,\delta}$.

Let

$$G_n(z) = e^{bz} \prod_{i=1}^{n} \left(1 + \frac{z}{\alpha_i} \right) e^{-\frac{z}{\alpha_i}} \ .$$

Then

$$g_n\Big(\exp\big(b-\sum_1^n \alpha_i^{-1}\big)\tilde z\Big)=a_0+a_1G_n(1)\tilde z+...+a_pG_n(p)\tilde z^p+...\ .$$

Since

$$G_n(j)\to\frac{\varphi(j)}{a}\quad(n\to\infty)\ ,\quad j\in\mathbb{N}\ ,$$

we have

$$g_n\Big(\exp\Big(b-\sum_1^n\frac{1}{\alpha_i}\Big)\tilde z\Big)\to\frac{g(\tilde z)}{a}$$

uniformly in any finite domain.

In virtue of the Hurwitz theorem, all zeros of the function $g(\tilde z)/a$ are exactly limits of the zeros of the function $g_n\left(\exp\big(b-\sum_1^n\alpha_i^{-1}\big)\tilde z\right)$. Hence all zeros of $g(z)$ are contained in the open angle $A_{\eta,\delta}$. □

Let us immediately prove Theorem 2. Let $f(z)$ and $\varphi(\omega)$ be the functions defined by formulas (3) and (4), respectively. Show that if the values of parameters $\rho_1,\mu_1,\rho_2,\mu_2$ are taken from the statement of Theorem 2, then the conditions of Lemma 1 are satisfied, hence, in virtue of formula (5), $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$ is an entire function whose all zeros are situated in the same domain where the zeros of $f(z)$ are.

Since $1/2<\rho_1<1$, $\mu_1\in[1/\rho_1-1,1]\cup[1/\rho_1,2]$, by Theorem B(iii), we have that all zeros of $E_{\rho_1}(z,\mu_1)$ are situated outside the closed angle (2). Therefore $f(z)$ is an entire function of order $\rho_1<1$ nonvanishing outside the angle $A_{\eta,\delta}$, where

$$\eta=\pi/(2\rho_1)\in[0,2\pi)\ ,\quad 0<\delta=2\pi-\pi/\rho_1\le\pi\ .$$

The function $\varphi(\omega)$ also satisfies the conditions of Lemma 1. These conditions can be verified by reasoning as in the proof of Theorem 1. Thus Theorem 2 is proved.

4. In conclusion we obtain some results on a-points of the function $\Phi_{\rho_1,\rho_2}(z,\mu_1,\mu_2)$, which are more general than the results on a-points of the function $E_\rho(z,\mu)$ obtained in [5]. Namely, we obtain analogues of the following theorems.

Theorem D ([5, Theorem 3]). *Assume $\rho>1/2, a\in\{z\in\mathbb{C}:\ |z|\le R_\rho\}$, where $R_\rho=\min\{2\rho-1,1\}$. Then all a-points of the function $E_\rho(z,1)$ are situated outside the closed angle*

$$\{z:|\arg z|\le\pi/(2\rho)\}$$

(if $a=1$, then the point 0 is excluded from the angle).

Corollary A ([5, Corollary 1]). *Assume $a\in(-1,1)$. If $(1+|a|)\cdot 2^{-n-1}\le\rho<2^{-n}$ $(n\in\mathbb{N})$, then all a-points of the function $E_\rho(z,1)$ are strictly negative.*

Theorem E ([5, Theorem 4]). *Assume $1/2<\rho<1$, $a\in\{z\in\mathbb{C}:\ |z|\le\rho\}$. Then all a-points of the function $E_\rho(z,2)$ are situated outside the closed angle*

$$\Big\{z:\ |\arg z|\le\frac{1}{\rho}\arccos\frac{|a|}{\rho}\Big\}.$$

We need the following lemma, which is a generalization of Lemma 1.

Lemma 1. *Let $f(z)$ be an entire function of order $\rho < 1$ whose all a-points are contained in the open angle $A_{\eta,\delta}$ defined by formula (6). Let $\varphi(\omega)$ be an entire function of genus 0 or 1, real-valued for real values of ω, whose all zeros are real and negative. If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$g(z) = \sum_{n=0}^{\infty} a_n \varphi(n) z^n$$

is an entire function whose all $a\varphi(0)$-points are contained in the open angle $A_{\eta,\delta}$.

Proof. Consider the function $f(z) - a$. This function satisfies the conditions of Lemma 1. Since

$$g(z) - a\varphi(0) = \sum_{n=1}^{\infty} a_n \varphi(n) z^n + (a_0 - a)\varphi(0),$$

in virtue of Lemma 1 we have that all zeros of the function $g(z) - a\varphi(0)$ are contained in the same domain where the zeros of the function $f(z) - a$ are, i. e. in the open angle $A_{\eta,\delta}$. □

Theorem 3 (Analogue of Theorem D). *Assume $1/2 < \rho_1 < 1$, $\rho_2 > 0$, $\mu_2 > 0$, $a \in \{z \in \mathbb{C} : |z| \le R_{\rho_1}\}$, where $R_{\rho_1} = 2\rho_1 - 1$. Then all $a/\Gamma(\mu_2)$-points of the function $\Phi_{\rho_1,\rho_2}(z, 1, \mu_2)$ are situated outside the closed angle*

$$\{z : |\arg z| \le \pi/(2\rho_1)\}$$

(if $a = 1$, then the point 0 is excluded from the angle).

Corollary 1 (Analogue of Corollary A). *Assume $a \in (-1, 1)$. If $(1 + |a|) \cdot 2^{-n-1} \le \rho_1 < 2^{-n}$ $(n \in \mathbb{N})$, $\rho_2 > 0$, $\mu_2 > 0$, then all $a/\Gamma(\mu_2)$-points of the function $\Phi_{\rho_1,\rho_2}(z, 1, \mu_2)$ are strictly negative.*

Theorem 4 (Analogue of Theorem E). *Assume $1/2 < \rho_1 < 1$, $\rho_2 > 0$, $\mu_2 > 0$, $a \in \{z \in \mathbb{C} : |z| \le \rho_1\}$. Then all $a/\Gamma(\mu_2)$-points of the function $\Phi_{\rho_1,\rho_2}(z, 2, \mu_2)$ are situated outside the closed angle*

$$\left\{z : |\arg z| \le \frac{1}{\rho_1} \arccos \frac{|a|}{\rho_1}\right\}.$$

To prove these Theorems, it suffices to apply Lemma 2 to the functions

$$f(z) = E_{\rho_1}(z, \mu_1), \quad \varphi(\omega) = \frac{1}{\Gamma(\mu_2 + \omega/\rho_2)}.$$

Reasoning as in the proof of Theorem 1, we can prove that the function $\varphi(\omega)$ satisfies the conditions of Lemma 2 for any $\rho_2 > 0, \mu_2 > 0$. If the values of parameters ρ_1, μ_1 are taken from the statements of Theorems 3 and 4, then the function $f(z)$ also satisfies the conditions of Lemma 2, as, by Theorems D and E, all its a-points are situated in the corresponding open angles. The author is deeply grateful to Prof. I. V. Ostrovskii for formulating the problem.

REFERENCES

1. Джрбашян М. М. *Об интегральных преобразованиях, порождаемых обобщенной функцией типа Миттаг-Леффлерра,* АН АрмССР **13** (1960), № 3, 21–63.
2. Джрбашян М. М. *Интегральные преобразования и представления функций в комплексной области,* Москва, Наука, 1966, с. 672.
3. Ostrovskii I. V., Peresyolkova I. N. *Nonasymptotic results on distribution of zeros of the function* $E_\rho(z,\mu)$, Analysis Math. **23** (1997), 283–296.
4. Титчмарш Е. *Теория функций,* Москва, Наука, 1980, с. 464.
5. Переселкова И. Н. *Неасимптотические результаты о распределении a-точек функции* $E_\rho(z,\mu)$, Доп. НАН України (1999), №2, 18–21.

Department of Mechanics and Mathematics, Kharkov National University,
Svobody Sq., 4, 310077, Kharkov, Ukraine
peresyolkova@mail.ru ; Irina.N.Peresyolkova@univer.kharkov.ua

Received 13.10.1998