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EXCEPTIONAL SETS FOR ENTIRE FUNCTIONS

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In the article we list the differences between the Levin-Pflüger theory of completely regular growth (one-term asymptotics) and the case of polynomial asymptotics. The main of these differences is the existence of points of elevation in the exceptional sets for the polynomial asymptotics.

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В работе приведены некоторые отличия между теорией функций вполне регулярного роста Левина-Пфлюгера (одночленные асимптотики) и случаем многочленных асимптотик. Основным результатом статьи является теорема о существовании точек "повышения"в исключительных множествах для случая многочленных асимптотических представлений.

In the well-known Levin-Pflüger theory of entire functions of completely regular growth (see [4]) there is established a relationship between the regularity of the distribution of zeros of an entire function and the regularity of the function growth. Later Azarin [3] proved that this relationship is valid for the masses and growth of a subharmonic function as well. The connection is in terms of one-term asymptotics, one of a subharmonic function (logarithm of entire function) and the other of the function counting its masses (zeros). In the early seventies, this relationship was studied [5], [6] for the case of multi-term (in other words, polynomial) asymptotics. In the eighties, sharp and in a sense complete results about polynomial asymptotics were stated, first for subharmonic functions with masses on a finite union of rays [1] and then for the general case [2]. In turns out that the multi-term case is essentially different from the one-term case. The aim of this paper is to study the major differences between the traditional theory of functions of completely regular growth and the case of polynomial asymptotics. For these differences already appear in the subclass of entire functions, in what follows we consider only this subclass. First, we need to remind some definitions and facts.

Given an entire function $f(z)$, a function $n(t,\alpha,\beta)$, $\alpha<\beta\leq\alpha+2\pi$, is called the *counting function* for $f(z)$ if $n(t,\alpha,\beta)$ equals the number of zeros of f in the sector

$$\{z\in\mathbb{C}:0\leq t,\alpha<\arg z\leq\beta\}.$$

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Let ρ_1 be a positive noninteger number, and let $[\rho_1] < \rho_n < \ldots < \rho_2 < \rho_1$, let f be of order ρ_1, and let

$$n(t,\alpha,\beta) = \sum_{j=1}^{n} \Delta_j(\alpha,\beta)t^{\rho_j} + \phi(t,\alpha,\beta) \tag{1}$$

where the functions $\Delta_j(\alpha,\beta)$ are locally integrable with respect to any of their two arguments, and $\phi(t,\alpha,\beta)$ is $o(t^{\rho_n})$ as $t \to \infty$. It occurs [2] that the function $\Delta_1(\alpha,\beta) = \Delta_1(0,\beta) - \Delta_1(0,\alpha)$ increases of β and decreases as a function of α as well as in the theory of completely regular growth. The functions $\Delta_j(\alpha,\beta)$, $j = 2,\ldots,n$, also have the representation $\Delta_j(\alpha,\beta) = \Delta_j(0,\beta) - \Delta_j(0,\alpha)$ where the functions $\Delta_j(0,\alpha)$ are locally bounded and have only countable set of points of discontinuity. The example below shows that the variation of $\Delta_j(0,\alpha)$, $j \geq 2$, can be infinite.

Let c be a constant such that

$$2\pi = c\sum_{k=1}^{\infty} k^{-1-(\rho_1-\rho_2)}.$$

Define a continuous function $\Delta_j(\theta) \equiv \Delta_2(0,\theta)$ as follows:

$$\Delta_2(0) = \Delta_2(2\pi) = \Delta_2\Big(c\sum_{k=1}^{n} k^{-1-(\rho_1-\rho_2)}\Big) = 0, \quad n = 1,2,\ldots;$$

$$\Delta_2\Big(c\sum_{k=1}^{n-1} k^{-1-(\rho_1-\rho_2)} + \frac{c}{2}n^{-1-(\rho_1-\rho_2)}\Big) = \frac{1}{n}, \quad n = 1,2,\ldots.$$

On the other parts of the segment $[0,2\pi]$ this function is linear. It is evident that Δ_2 is of unbounded variation on $[0,2\pi]$.

Let $h > c/2$, and let for $\theta \in [0,2\pi]$

$$\Delta_1(0,\theta) = h\theta;$$

$$\phi(t,\theta) = -\Delta_2(\theta)t^{\rho_2}\chi_{\left[c\sum_{k=1}^{[t]} k^{-1-(\rho_1-\rho_2)},2\pi\right]}(\theta);$$

$$m(t,\theta) = \Delta_1(0,\theta)t^{\rho_1} + \Delta_2(\theta)t^{\rho_2} + \phi(t,\theta).$$

Here $\chi_A(\theta)$ is the characteristic function of a set A. $m(t,\theta)$ does not decrease as a function of θ for any fixed t. This is obvious for $t < 1$, because $m(t,\theta) = \Delta_1(0,\theta)t^{\rho_1}$ for such t.

For each fixed t the functions $\phi(t,\theta)$ and $m(t,\theta)$ are continuous with respect to θ, because the points of discontinuity of χ are zeros of $\Delta_2(\theta)$. The derivative $m'_\theta(t,\theta)$ is piecewise continuous, and the inequality

$$m'_\theta(t,\theta) > ht^{\rho_1} - \frac{2[t]^{-1}}{c[t]^{-1-(\rho_1-\rho_2)}}t^{\rho_2} \geq \left(h - \frac{2}{c}\right)t^{\rho_1} > 0$$

holds at each point where the derivative exists. Since $\phi(t,\theta) = O(t^{\rho_1-1})$ as $t \to \infty$, the function $n(t,0,\theta) = [m(t,\theta)]$ is a counting function for some canonical product of order ρ_1 and

$$n(t,0,\theta) = \Delta_1(0,\theta)t^{\rho_1} + \Delta_2(\theta)t^{\rho_2} + o(t^{\rho_2})$$

as $t \to \infty$.

The first dramatic difference between the theory of completely regular growth and the case of polynomial asymptotics is the size of exceptional sets. In [2] it is proved that (1) implies that the corresponding entire function has the following asymptotic representation:

$$\ln|f(re^{i\theta})| = H_1(\theta)r^{\rho_1} + \ldots + H_n(\theta)r^{\rho_n} + \psi(re^{i\theta}) \tag{2}$$

where

$$H_j(\theta) = \frac{\pi}{\sin \pi\rho_j} \int_{\theta-2\pi}^{\theta} \cos \rho_j(\theta - \tau - \pi) d\Delta_j(0, \tau), \quad j = 1, \ldots, n,$$

and $\psi(re^{i\theta}) = o(r^{\rho_n})$ as $z = re^{i\theta} \to \infty$ avoiding some exceptional set, which is relatively small. This representation is a generalization of the formula of the theory of completely regular growth, which is its particular case for $n = 1$. For the convenience of readers, we will remind the corresponding definition.

A set $E \in \mathbb{C}$ (only unbounded sets are of interest) is called a $C_{0,\kappa}$- set if it can be covered by a countable system of circles $\{C(z_j, r_j)\}_{j=1}^{\infty}$ such that

$$\lim_{R\to\infty} \frac{1}{R^\kappa} \sum_{\{j:|z_j|\le R\}} r_j^\kappa = 0.$$

It is a well known fact that exceptional sets which appear in the theory of completely regular growth are $C_{0,1}$-sets. It is still the case when $n > 1$ provided that all zeros belong to a finite union of rays (see [1]). In the general case of polynomial asymptotics, it is proved [2] that these exceptional sets can be wider than any $C_{0,2-\varepsilon}$-sets for each fixed $\varepsilon > 0$.

The most interesting, in our opinion, is the following feature of exceptional sets for the case of polynomial asymptotics. In the theory of completely regular growth, these sets contain only points of decline. It means that if the value of the remainder ψ of asymptotic expansion (2) is comparable with the previous term, this value must be negative. In fact, this property of the remainder is the reason that smallness of exceptional sets is usually proved by means of Cartan-type estimates of analytic functions from below. B. Ja. Levin was the first who predicted that in the case of polynomial asymptotics an exceptional set can contain points of elevation, i.e. points at which the remainder of (2) is not small in comparison with the previous term but positive. By this reason, to estimate smallness of exceptional sets for the polynomial asymptotics, such singular integrals as Hilbert transform, Cauchy transform and Beurling transform are used (see [1] and [2]). The idea of the proof of the following theorem also belongs to B. Ja. Levin and was communicated by him to the second author.

Theorem (B. Ja. Levin) *For any $\rho_1 > \rho_2 > [\rho_1] \ge 0$ and any positive function $L(r) = o(\ln r)$ as $r \to \infty$ there exists a canonical product $f(z)$ with positive zeros and the counting zeros function*

$$n(t) = \Delta_1 t^{\rho_1} + \Delta_2 t^{\rho_2} + o(t^{\rho_2}), \quad t \to \infty,$$

such that

$$\limsup_{r\to\infty} \left\{ \ln|f(r)| - \sum_{j=1}^{2} \pi\Delta_j r^{\rho_j} ctg\pi\rho_j \right\} \{r^{\rho_2} L(r)\}^{-1} = \infty.$$

Proof. By the Hadamard Theorem a canonical product $f(z)$ of noninteger order ρ_1 with positive zeros has the representation

$$\ln|f(z)| = -\mathrm{Re}\left(z^{p+1} \text{v.p.} \int_0^\infty \frac{n(t)\, dt}{t^{p+1}(t-z)} \right). \tag{3}$$

where $p=[\rho_1]$. Let the counting function $n(t)=[m(t)]$, where $m(t)$ is defined as follows: let $R>4$ and $k\in(0,1/4)$ be two arbitrary constants, let a monotone sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ tend to 0 so slowly that $n\varepsilon_n/L(R^n)\to\infty$ and $\varepsilon_n R^{n\rho_2}\nearrow\infty$ as $n\to\infty$. It is also convenient to assume that $2R\varepsilon_0<k\rho_1\Delta_1$.

Let

$$\delta_n=\frac{2}{\rho_1\Delta_1}\varepsilon_n R^{n(1-\rho_1+\rho_2)},\quad n=0,1,\ldots.$$

Define $m(t)$ by the formula

$$m(t)=\Delta_1 t^{\rho_1}+\Delta_2 t^{\rho_2}+\chi(t).$$

Here the continuous function $\chi(t)=-\Delta_1 t^{\rho_1}-\Delta_2 t^{\rho_2}$ on $[0,k)$, $\chi(t)=\varepsilon_n R^{n\rho_2}$ on $[kR^n,R^n-\delta_n]$, and we choose $\chi(t)$ on $[R^n-\delta_n,R^n+\delta_n]$ in a way for $m(t)$ be constant there. Finally, we take $\chi(t)$ constant on $[R^n+\delta_n,(1+k)R^n]$ and linear on $[(1+k)R^n,kR^{n+1}],n\in\mathbb{N}$.

By this definition, $m(t)$ is a non-negative and a non-decreasing function. We claim that $\chi(t)=o(t^{\rho_2})$ as $t\to\infty$. In fact, it is sufficient to verify this equality only on the segments $[R^n-\delta_n,R^n+\delta_n]$:

$$\chi(R^n+\delta_n)=\chi(R^n-\delta_n)-\Delta_1\{(R^n+\delta_n)^{\rho_1}-(R^n-\delta_n)^{\rho_1}\}-$$
$$-\Delta_2\{(R^n+\delta_n)^{\rho_2}-(R^n-\delta_n)^{\rho_2}\}=$$
$$=\varepsilon_n R^{n\rho_2}-2\Delta_1\rho_1 R^{n(\rho_1-1)}(1+O(\frac{\delta_n}{R^n}))\delta_n-$$
$$-2\Delta_2\rho_2 R^{n(\rho_2-1)}(1+O(\frac{\delta_n}{R^n}))\delta_n=\varepsilon_n R^{n\rho_2}-4\varepsilon_n R^{n\rho_2}(1+O(\frac{\delta_n}{R^n}))-$$
$$-4\varepsilon_n\frac{\Delta_2\rho_2}{\Delta_1\rho_1}R^{n(2\rho_2-\rho_1)}(1+O(\frac{\delta_n}{R^n})),\quad n\to\infty.$$

It means that for all n large enough the inequality

$$-6\varepsilon_n R^{n\rho_2}<\chi(R^n+\delta_n)<-2\varepsilon_n R^{n\rho_2}$$

is valid. Taking into account that $\chi(t)$ is decreasing on $[R^n-\delta_n,R^n+\delta_n]$, we deduce that

$$\max\{|\chi(t)|:t\in[R^n-\delta_n,R^n+\delta_n]\}\le 6\varepsilon_n R^{n\rho_2}.$$

For the counting function we have

$$n(t)=\Delta_1 t^{\rho_1}+\Delta_2 t^{\rho_2}+\phi(t),$$

where $\phi(t)=\chi(t)+O(1)=o(t^{\rho_2})$ as $t\to\infty$. Evaluating residues, we rewrite (3) at $t=R^n$ as follows:

$$\ln|f(R^n)|=\sum_{j=1}^{2}\pi\Delta_j R^{n\rho_j}\mathrm{ctg}\pi\rho_j-R^{n(p+1)}\left\{\text{v. p.}\int_0^\infty\frac{\phi(t)dt}{t^{p+1}(t-R^n)}\right\}.$$

Only the remainder

$$\psi(R^n)=-R^{n(p+1)}\left\{\text{v. p.}\int_0^\infty\frac{\phi(t)dt}{t^{p+1}(t-R^n)}\right\}. \tag{4}$$

is of interest for us. To evaluate it, let us partition the domain of integration into five parts

$$\psi(R^n) = -R^{n(p+1)} \left\{ \int_0^{kR^n} + \int_{kR^n}^{R^n-\delta_n} + \text{v. p.} \int_{R^n-\delta_n}^{R^n+\delta_n} + \right.$$

$$\left. + \int_{R^n+\delta_n}^{(1+k)R^n} + \int_{(1+k)R^n}^{\infty} \right) = -R^{n(p+1)} \sum_{j=1}^{5} I_j^{(n)}.$$

The estimate of two extreme summands is straightforward:

$$I_j^{(n)} = o(R^{n(\rho_2-p-1)}), \quad n \to \infty, \ j = 1, 5.$$

Since it is easy to verify that for some constant C and all $n \geq 0$

$$\left| \text{v. p.} \int_{R^n-\delta_n}^{R^n+\delta_n} \frac{dt}{t^{p+1}(t-R^n)} \right| \leq C\varepsilon_n R^{n(\rho_2-\rho_1-p-1)}$$

and

$$\phi(t) - \phi(R^n) = \chi(t) - \chi(R^n)$$

on $[R^n - \delta_n, R^n + \delta_n]$, the summand

$$I_3^{(n)} = \int_{R^n-\delta_n}^{R^n+\delta_n} \frac{\phi(t) - \phi(R^n)}{t^{p+1}(t-R^n)} dt + \phi(R^n) \text{v. p.} \int_{R^n-\delta_n}^{R^n+\delta_n} \frac{dt}{t^{p+1}(t-R^n)}$$

has the following estimate:

$$|I_3^{(n)}| \leq \int_{R^n-\delta_n}^{R^n+\delta_n} \left| \frac{\chi(t) - \chi(R^n)}{t^{p+1}(t-R^n)} \right| dt + o(R^{n(\rho_2-p-1)}), \quad n \to \infty.$$

By the Mean Value Theorem the inequality

$$\left| \frac{\chi(t) - \chi(R^n)}{t - R^n} \right| \leq \max\{\rho_1 \Delta_1 \tau^{\rho_1-1} + \rho_2 \Delta_2 \tau^{\rho_2-1} : \tau \in [R^n - \delta_n, R^n + \delta_n]\}$$

holds for every $t \in [R^n - \delta_n, R^n + \delta_n]$ and $n \in \mathbb{Z}_+$. The two previous inequalities and the definition of δ_n imply that for some constant C

$$|I_3^{(n)}| \leq \frac{C\delta_n}{R^{n(p+1)}} \max\{\rho_1 \Delta_1 \tau^{\rho_1-1} + \rho_2 \Delta_2 \tau^{\rho_2-1} : \tau \in [R^n - \delta_n, R^n + \delta_n]\} +$$

$$+ o(R^{(\rho_2-p-1)}) = o(R^{n(\rho_2-p-1)}), \quad n \to \infty.$$

$I_2^{(n)}$ can be estimated from above quite simply, because $\chi(t)$ is constant on the interval of integration:

$$I_2^{(n)} = \int_{kR^n}^{R^n-\delta_n} \frac{\chi(t) + O(1)}{t^{p+1}(t-R^n)} dt = \varepsilon_n R^{n\rho_2}(1 + o(1)) \int_{kR^n}^{R^n-\delta_n} \frac{dt}{t^{p+1}(t-R^n)} \leq$$

$$\leq -C\varepsilon_n(1 + o(1)) R^{n(\rho_2-p-1)} \ln \frac{(1-k)R^n}{\delta_n} \leq -C\varepsilon_n R^{(\rho_2-p-1)} \ln R^n.$$

Here and in the sequel C are some positive constants that do not depend on n. The similar estimate from above holds for $I_4^{(n)}$. By these estimates, the inequality

$$\psi(R^n) \geq C\varepsilon_n R^{n\rho_2} \ln R^n$$

is true for every natural $n \in \mathbb{N}$. By definition of ε_n, it means that

$$\limsup_{r\to\infty} \frac{\psi(t)}{r^{\rho_2} L(r)} = \infty.$$

The theorem is proved. □

Remark 1. The estimate of the remainder from below given by Levin's theorem is sharp.

For the sake of simplicity, let us consider only the points of the positive ray where zeros are situated. To verify the statement of remark in this case, let us partition the ray of integration in (4) in the following way. Let

$$\varepsilon(x) = \sup\{|\phi(t)|/t^{\rho_2} : t \in [x/2, 3x/2]\}.$$

It is evident that $\varepsilon(x) \to 0$ as $x \to \infty$. Define $\delta = \delta(x) = x^{1-\rho_1+\rho_2}$. We have

$$\psi(x) = -x^{p+1}\left\{\text{v. p.} \int_0^\infty \frac{\phi(t)dt}{t^{p+1}(t-x)}\right\} =$$

$$-x^{p+1}\left\{\int_0^{\frac{x}{2}} + \int_{\frac{x}{2}}^{x-\delta} + \text{v. p.} \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\frac{3x}{2}} + \int_{\frac{3x}{2}}^{\infty}\right\} =$$

$$-x^{p+1}\{I_1 + I_2 + I_3 + I_4 + I_5\}.$$

It is easy to check that both the first and the last summands contribute to the remainder $o(x^{\rho_2})$ as $x \to \infty$.

$$I_3(x) = \text{v. p.} \int_{x-\delta}^{x+\delta} \frac{\phi(t)dt}{t^{p+1}(t-x)} = \phi(x)\,\text{v. p.} \int_{x-\delta}^{x+\delta} \frac{dt}{t^{p+1}(t-x)} +$$

$$+\text{v. p.} \int_{x-\delta}^{x+\delta} \frac{\phi(t)-\phi(x)}{t^{p+1}(t-x)} dt. \tag{5}$$

The first term of the sum is $o(x^{\rho_2-p-1})$ as $x \to \infty$. At all of its points of discontinuity the function ϕ jumps upward. If x is one of these points, then the second integral is equal to ∞. On the complement of the set of discontinuity ϕ is smooth, and because of the monotonicity of the counting function,

$$\phi'(t) \geq -\Delta_1\rho_1 t^{\rho_1-1} - \Delta_2\rho_2 t^{\rho_2-1}.$$

It means that the second term in (5) has the following estimate from below:

$$\text{v. p.} \int_{x-\delta}^{x+\delta} \frac{\phi(t)-\phi(x)}{t^{p+1}(t-x)} dt \geq -Cx^{\rho_1-p-2}\delta = -Cx^{\rho_2-p-1}.$$

Here and below, C are different positive constants that do not depend on x.

Both summands I_2 and I_4 are estimated in the same way. Therefore, we will estimate only I_2,

$$I_2 \geq -C\varepsilon(x)x^{\rho_2-p-1} \int_{\frac{x}{2}}^{x-\delta} \frac{dt}{x-t} = -C\varepsilon(x)x^{\rho_2-p-1}\ln x.$$

By these estimates, we have that

$$\psi(x) \leq C \max\{\varepsilon(x)x^{\rho_2}\ln x, x^{\rho_2}\}.$$

Remark 2. For any $\varepsilon > 0$, the function defined in the proof of Levin's theorem can be modified in such a way that its exceptional set of "elevation points" will not be a $C_{0,1-\varepsilon}$-set.

To verify this statement, it is enough to check that replacing R^n to $x \in [R^n/2, 3R^n/2]$ does not change considerably the estimate of the remainder.

Remark 3. For any $\rho_1 > \rho_2 > \ldots > \rho_n > [\rho_1]$ and any $\varepsilon > 0$ there exists such an entire function $f(z)$ with the counting function $n(t,\alpha,\beta)$ satisfying (1) that its exceptional set of elevation is not a $C_{0,2-\varepsilon}$-set.

The proof of this statement can be found in [2].

Remark 4. All results formulated above are true for entire functions f of entire order ρ_1 provided that $\rho_n > \rho_1 - 1$ and the Lindelöf characteristic of f is finite.

The last distinction of the case of polynomial asymptotics is that for $n \geq 2$ (2) does not yield (1) in contrast with the theory of completely regular growth. It was stated by Lyubarskii and Sodin [7] in 1986. Yulmukhametov [10] showed that under some restrictive conditions on the class of subharmonic functions each one of conditions (1) and (2) implies the other. It is worth mentioning that in the Yulmukhametov's subclass of subharmonic functions the exceptional set of representation (2) again contains only points of decline.

Remark 5. All the results above can be extended on the case of polynomial asymptotics for logarithmic order that was studied in [8, 9].

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