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AN EXAMPLE OF ENTIRE FUNCTION OF STRONGLY REGULAR GROWTH

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We construct an entire function of zero order of strongly regular growth such that its zeros do not have any angular density with respect to the function $v(r) = r^{\lambda(r)}$, where $\lambda(r)$ is a proximate order of the counting function $n(r)$.

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Построена целая функция нулевого порядка сильно регулярного роста такая, что её нули не имеют угловой плотности относительно функции $v(r) = r^{\lambda(r)}$, где $\lambda(r)$ — уточненный порядок считающей функции $n(r)$.

Let f be an entire function of positive order and completely regular growth (c.r.g.) by Levin-Pfluger [1, p. 182]. The necessary and sufficient condition of c.r.g. of functions f of non-entire order is the existence of the angular density $\Delta(\psi)$ of its zeros relatively the comparison function $V(r) = r^{\rho(r)}$, $\rho(r)$ be the proximate order of function f.

In the case of zero order the proximate order $\rho(r)$ of function f will not be the proximate order of the counting function $n(r)$ of its zeros. Then $n(r) = o(V(r))$ as $r \to \infty$. Hence zeros of an entire function of zero order always have the angular density $\Delta(\psi) = 0$. Therefore in case of zero order we consider the comparison function $v(r) = r^{\lambda(r)}$, where $\lambda(r)$ is the proximate order of the function $n(r)$.

In [2] it was introduced a new notion of regular growth for entire functions of zero order. Let f be an entire function, $f(0) = 1$, $n(r, \alpha, \beta)$ be the number of zeros $a_n = |a_n|e^{i\alpha_n}$ of the function f in the sector $\{z : |z| \le r, \alpha < \arg z \le \beta\}$, $\ln f$ is a univalent branch of $\operatorname{Ln} f$ in the domain $D = \mathbb{C} \setminus \bigcup_{n=1}^{\infty}\{re^{i\alpha_n} : r \ge |a_n|\}$ such that $\ln f(0) = 0$. The ray $\arg z = \theta$ is ordinary, if

$$\lim_{h\to 0} \overline{\lim_{r\to\infty}}\, n(r, \theta - h, \theta + h)/v(r) = 0.$$

We shall say that a function f has a strongly regular growth (s.r.g.) if for all ordinary rays $\arg z = \theta$ we have

$$\lim_{r\to\infty, r\notin E} \frac{\ln f(re^{i\theta}) - N(r)}{v(r)} = H_f(\theta),$$

where E is some set of zero relative measure, i.e., $\operatorname{mes}(E \cap [0, r]) = o(r)$ $(r \to \infty)$, and $N(r) = \int_0^r n(t)/t\,dt$.

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Without loss of generality suppose that the ray $\arg z = -\pi$ is ordinary for zeros of the functions. We say that the set of roots of an entire function has angular density if the limit $\Delta(\psi) = \lim_{r\to\infty} n(r, -\pi, \psi)/v(r)$ exists for all ψ with the exception, perhaps, of ψ belonging to some countable set.

Theorem A [2]. *Let f be an entire function which zeros have angular density $\Delta(\psi)$. Then f has a s.r.g., moreover for all ordinary rays arg $z = \theta$ we have*

$$H_f(\theta) = i\int_{-\pi}^{\pi} \arg_{-\pi} e^{i(\theta-\psi-\pi)} d\Delta(\psi), \quad -\pi \le \theta < \pi,$$

where $\arg_{-\pi} z$ *is the value of* $\arg z \in [-\pi, \pi)$. *On the contrary, let f be an entire function whose zeros lie on a finite system of rays arg $z = \theta_j, j = 1, ..., k$. and f has a s.r.g. Then zeros f have angular density $\Delta(\psi)$, moreover*

$$\Delta(\psi) = \frac{1}{2\pi}\, Im \sum_{-\pi\le\theta_j<\psi} \left(H_f(\theta_j - 0) - H_f(\theta_j + 0)\right).$$

The following theorem implies that s.r.g. of entire function f is not a sufficient condition for existence of angular density of its zeros. Thus, this indicates that the condition of zeros placement on a finite system of rays is essential in Theorem A.

Theorem. *Let $\alpha(r)$ be a continuous positive increasing to $+\infty$ on $[0, +\infty)$ function. Then for arbitrary increasing to $+\infty$ on $[0, +\infty)$ function $\beta(r)$ there exist an entire function $f, f(0) = 1$, and a set E of zero relative measure such that for all $\theta, 0 \le \theta < 2\pi$, with the exception of θ belonging to a countable set, the following holds*

$$\ln f(re^{i\theta}) = N(r) + o(\beta(r)), \;\; r \to \infty, \;\; r \notin E, \tag{1}$$

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{\alpha(r)} = 1 \tag{2}$$

and zeros of f do not have any angular density with respect to the function $\alpha(r)$.

Proof. Without loss of generality, we shall assume that $\beta(r) \ge 0$ and $\beta(r) < \alpha(r)$ for all $r > 0$. We choose two increasing sequences (r_k) of positive numbers and n_k of positive integers by the induction. Let $r_1 \ge 1$ be such that $\alpha(r_1) \in \mathbb{N}, n_1 = \alpha(r_1)$, and assume that $r_2, r_3, ..., r_m$ and $n_2, n_3, ..., n_m$ are already chosen. Then we choose r_{m+1} such that

$$r_{m+1} > 4r_m, \;\; \frac{\alpha(r_m)}{\alpha(r_{m+1})} < \frac{1}{m}, \tag{3}$$

$$\alpha(r_{m+1}) \in \mathbb{N}, \;\; \beta(r_{m+1}) > \pi(m + 2)^2, \tag{4}$$

and $n_{m+1} = \alpha(r_{m+1}) - \alpha(r_m)$.

We consider the entire function $f(z) = \prod_{k=1}^{+\infty}\Big(1 - (z/r_k)^{n_k}\Big)$. Let $r_m \le r < r_{m+1}$. Then

$$n(r) = \sum_{r_k\le r} n_k = \alpha(r_1) + \sum_{r_k\le r} \big(\alpha(r_k) - \alpha(r_{k-1})\big) = \alpha(r_m) \le \alpha(r),$$

whence (2) follows. Further, if $0 \le \alpha < \beta < 2\pi$, then

$$n(r, \alpha, \beta) = (1 + o(1))\frac{\beta - \alpha}{2\pi} \sum_{r_k\le r} n_k = (1 + o(1))\frac{\beta - \alpha}{2\pi}\alpha(r_m), \quad r \to \infty,$$

and from (3) we have that the limit $\lim_{r\to\infty} n(r)/\alpha(r)$ does not exist.

Let's show that f satisfies condition (1). Let $z = re^{i\theta}, \theta \neq 2\pi j/n_k, k \in \mathbb{N}, j = 0, 1, \ldots, n_k$. Then

$$\ln f(re^{i\theta}) = \sum_{k=1}^{+\infty} \ln\left(1 - \left(\frac{z}{r_k}\right)^{n_k}\right) = \int_0^z \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{k=1}^{+\infty} n_k \int_0^z \frac{-\zeta^{n_k-1} d\zeta}{r_k^{n_k} - \zeta^{n_k}} =$$
$$= \sum_{k=1}^{+\infty} n_k \int_0^r \frac{t^{n_k-1} dt}{t^{n_k} - r_k^{n_k} e^{-in_k\theta}} = \sum_{k=1}^{+\infty} J_k. \tag{5}$$

Set $\tau = t^{n_k}$, $\varphi_k = -n_k\theta$, $R_k = r_k^{n_k}$, $R = r^{n_k}$. We have

$$|\mathrm{Im}\ \ J_k| = \left|\mathrm{Im} \int_0^R \frac{d\tau}{\tau - R_k e^{i\varphi_k}}\right| = \left|\int_0^R \frac{R_k \sin\varphi_k \, d\tau}{|\tau - R_k e^{i\varphi_k}|^2}\right| =$$
$$= \left|R_k \sin\varphi_k \int_0^R \frac{d\tau}{(\tau - R_k\cos\varphi_k)^2 + R_k^2 \sin^2\varphi_k}\right| = \left|\operatorname{arctg} \frac{\tau - R_k\cos\varphi_k}{R_k \sin\varphi_k}\Big|_0^R\right| < \pi,$$

and owing to (4)

$$\sum_{k=1}^{m+1} |\mathrm{Im}\ \ J_k| \leq (m+1)\pi < \frac{\beta(r_m)}{m+1}. \tag{6}$$

Let $r_m \leq r < r_{m+1}$. Then from (3) for $k \geq m+2$ we obtain $\dfrac{r}{r_k} \leq 4^{m+1-k}$. Since $-\ln(1-x) \leq 2x$ for $0 \leq x \leq 1/2$, we have

$$|J_k| \leq \int_0^r \frac{n_k t^{n_k-1} dt}{r_k^{n_k} - t^{n_k}} = -\ln\left(1 - \left(\frac{r}{r_k}\right)^{n_k}\right) \leq 2\left(\frac{r}{r_k}\right)^{n_k} \leq 2\left(\frac{1}{4}\right)^{k-m-1}.$$

Therefore

$$\sum_{k=m+2}^{+\infty} |J_k| \leq \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1, \tag{7}$$

and from (5)–(7) we have

$$\left|\arg f(re^{i\theta})\right| \leq \frac{\beta(r)}{m+1} + 1 = o(\beta(r)), \quad r \to \infty. \tag{8}$$

Find the asymptotics of $\ln|f(re^{i\theta})|$. We have ($R = r^{n_k}$)

$$\mathrm{Re}\ J_k = \ln\frac{R}{R_k} + \ln\left|1 - \frac{R_k e^{-i\varphi_k}}{R}\right| = n_k \ln\frac{r}{r_k} + \ln\left|1 - \left(\frac{r_k}{r}\right)^{n_k} e^{in_k\theta}\right|. \tag{9}$$

Let $r_m + \frac{r_m}{n_m} \leq r \leq r_{m+1} + \frac{r_{m+1}}{n_{m+1}}$. Then for $k \leq m-1$ it holds

$$-2\left(\frac{r_k}{r}\right)^{n_k} \leq \ln\left(1 - \left(\frac{r_k}{r}\right)^{n_k}\right) \leq \ln\left|1 - \left(\frac{r_k}{r}\right)^{n_k} e^{in_k\theta}\right| \leq \ln\left(1 + \left(\frac{r_k}{r}\right)^{n_k}\right) \leq \left(\frac{r_k}{r}\right)^{n_k}. \tag{10}$$

Further,

$$\ln\left(1 - \left(\frac{r_m}{r}\right)^{n_m}\right) \leq \mathrm{Re}\ J_m = \ln\left|1 - \left(\frac{r_m}{r}\right)^{n_m} e^{in_m\theta}\right| \leq \ln\left(1 + \left(\frac{r_m}{r}\right)^{n_m}\right) \leq \ln 2. \tag{11}$$

and

$$\ln\left(1-\left(\frac{r_m}{r}\right)^{n_m}\right)\geq \ln\left(1-\left(\frac{r_m}{r_m+r_m/n_m}\right)^{n_m}\right)\geq$$
$$\geq \ln\left(1-\left(1+\frac{1}{n_m}\right)^{-n_m}\right)\geq \ln\left(1-\frac{2}{e}\right). \quad (12)$$

Analogously, we obtain

$$\text{Re } J_{m+1}\leq \ln\left|1-\left(\frac{r}{r_{m+1}}\right)^{n_{m+1}}e^{in_{m+1}\theta}\right|\leq \ln\left(1+\left(\frac{r}{r_{m+1}}\right)^{n_{m+1}}\right)\leq \ln 2, \quad (13)$$

and

$$\text{Re } J_{m+1}\geq \ln\left(1-\left(\frac{r}{r_{m+1}}\right)^{n_{m+1}}\right)\geq \ln\left(1-\left(1-\frac{1}{n_{m+1}}\right)^{n_{m+1}}\right)\geq \ln\left(1-\frac{2}{e}\right). \quad (14)$$

From (7) and (9)–(14) for $r\notin E$, $E=\bigcup_{k=1}^{+\infty}(r_k-r_k/n_k, r_k+r_k/n_k)$, we obtain

$$\left|\ln|f(re^{i\theta})|-\sum_{k=1}^{m}n_k\ln\frac{r}{r_k}\right|\leq \sum_{r=1}^{m-1}|\text{Re } J_k|+|\text{Re } J_m|+$$
$$+|\text{Re } J_{m+1}|+\sum_{k=m+2}^{+\infty}|J_k|\leq 2\sum_{k=1}^{m-1}\frac{1}{4^{n_k}}+O(1)=O(1),\quad r\to\infty. \quad (15)$$

From (8) and (15) we have (1), since the relative measure of the set E is zero. Indeed, for $r_m\leq r\leq r_{m+1}$ we obtain

$$\text{mes }(E\cap[0,r])\leq \frac{2r_1}{n_1}+\frac{2r_2}{n_2}+...+\frac{2r_m}{n_m}\leq$$

$$\leq \frac{2r_m}{4^{m-1}n_1}+\frac{2r_m}{4^{m-2}n_2}+...+\frac{2r_m}{4n_{m-1}}+\frac{2r_m}{n_m}=2r_m\sum_{k=1}^{m}\frac{1}{4^{m-k}n_k},$$

and by the Töplitz theorem, we have

$$\text{mes }(E\cap[0,r])=o(r_m)=o(r),\quad r\to\infty.$$

□

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