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TO THE SHEREMETA THEOREM CONCERNING RELATIONS BETWEEN THE MAXIMAL TERM AND THE MAXIMUM MODULUS OF ENTIRE DIRICHLET SERIES

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For an entire Dirichlet series $F(s)=\sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s=\sigma+it$, $0\le\lambda_n\uparrow\infty$, necessary and sufficient condition on a_n is established in order that $M(\sigma,F)<\mu(\sigma,F)h(\ln\mu(\sigma,F))$, $\sigma\ge\sigma_0$, where $M(\sigma,F)=\sup\{|F(\sigma+it)|:t\in\mathbb{R}\}$, $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}:n\ge 0\}$ and h is a continuous function on $[0;+\infty)$ such that $h(r)\to+\infty$ $(r\to+\infty)$.

П. В. Филевич. *К теореме Шереметы о соотношениях между максимальным членом и максимумом модуля целого ряда Дирихле* // Математичні Студії. – 2000. – Т.13, №2. – С.139–144.

Для целых рядов Дирихле $F(s)=\sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s=\sigma+it$, $0\le\lambda_n\uparrow\infty$, получены необходимые и достаточные условия на a_n, при которых выполняется соотношение $M(\sigma,F)<\mu(\sigma,F)h(\ln\mu(\sigma,F))$, $\sigma\ge\sigma_0$, где $M(\sigma,F)=\sup\{|F(\sigma+it)|:t\in\mathbb{R}\}$, $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}:n\ge 0\}$, а h — непрерывная на $[0;+\infty)$ функция, стремящаяся к $+\infty$ при $r\to+\infty$.

Introduction

Let $\Lambda=\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers increasing to $+\infty$ and $S(\Lambda)$ the class of entire (i.e., of absolutely convergent in $\mathbb{C}$) Dirichlet series

$$F(s)=\sum_{n=0}^{\infty} a_n e^{s\lambda_n},\qquad s=\sigma+it. \tag{1}$$

For $F\in S(\Lambda)$ we define by $M(\sigma,F)=\sup\{|F(\sigma+it)|:t\in\mathbb{R}\}$ the maximum modulus and $\mu(\sigma,F)=\max\{|a_n|\exp\{\sigma\lambda_n\}:n\ge 0\}$ the maximal term.

The class of nonnegative continuous on $[0;+\infty)$ functions tending to $+\infty$ is denoted by L_0, the subclass of functions $l\in L_0$ such that $l(x)\nearrow+\infty$ as $x\uparrow+\infty$ is denoted by L. We put $L_{SG}=\{\varphi\in L:\varphi(2x)\sim\varphi(x),\ x\to+\infty\}$. For $\psi\in L$ let $S_\psi(\Lambda)$ be the class of entire Dirichlet series (1) such that

$$|a_n|\le\exp\{-\lambda_n\psi(\lambda_n)\},\qquad n\ge n_0. \tag{2}$$

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M. M. Sheremeta [1] proved the following result.

Theorem A. *Let* $\psi \in L$,

$$\varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n \psi(\lambda_n)} = q < 1, \tag{3}$$

a function $\varphi \in L$ *be such that* $\varphi(\ln x) \in L_{SG}$, *and* $\ln \varphi(x)$ *be a concave function. In order that*

$$\forall F \in S_\psi(\Lambda): \qquad \varphi(\ln M(\sigma, F)) \sim \varphi(\ln \mu(\sigma, F)), \quad \sigma \to +\infty, \tag{4}$$

it is necessary and sufficient that

$$\forall \gamma \in L_{SG}: \qquad \lim_{n\to\infty} \frac{\varphi(\ln n + \psi(\lambda_n)\gamma(\psi(\lambda_n)))}{\varphi(\psi(\lambda_n)\gamma(\psi(\lambda_n)))} = 1. \tag{5}$$

Remark 1. Let $\psi \in L$, and condition (3) is fulfilled. Then Dirichlet series (1) with the coefficients satisfying condition (2) defines an entire function $F \in S_\psi(\Lambda)$. Indeed, if $n_1(\sigma) = \min\{n \ge 0 : \psi(\lambda_n) \ge 3\sigma/(1-q)\}$ then [1, p. 144]

$$\sum_{n \ge n_1(\sigma)} |a_n| e^{\sigma\lambda_n} \le K, \qquad \sigma \ge \sigma_0, \tag{6}$$

where K is some positive constant. Consequently, the series (1) is absolutely convergent in $\mathbb{C}$.

Remark 2. The condition $\varphi(\ln x) \in L_{SG}$ is essential [1, p. 147]. For every functions $\psi \in L$, $\varphi(\ln x) \in L \backslash L_{SG}$, and a sequence Λ satisfying condition (3) there exists a function $F \in S_\psi(\Lambda)$ such that relation (4) is not fulfilled.

Remark 3. The proof of Theorem A is based on the method which expects that the function $\ln \varphi(x)$ is concave. In connection with this, M. M. Sheremeta [1, p. 148] formulated the question on the essentiality of above condition in Theorem A.

We will prove that the concavity condition of the function $\ln \varphi(x)$ is essential in general, but we can drop this condition if we change the necessary and sufficient condition (5). We have the following theorem.

Theorem 1. *Let* $\psi \in L$, *condition (3) be valid, and a function* $\varphi \in L$ *be such that* $\varphi(\ln x) \in L_{SG}$. *Then, in order that relation (4) holds, it is necessary and sufficient that*

$$\forall l \in L: \qquad \lim_{n\to\infty} \frac{\varphi\big(\ln n + \psi(\lambda_n) l(\psi(\lambda_n))\big)}{\varphi\big(\psi(\lambda_n) l(\psi(\lambda_n))\big)} = 1. \tag{7}$$

First of all we point that conditions (5) and (7) are not equivalent in general. For that we construct an example of function $\varphi \in L$ with the following properties:

(i) $\varphi(\ln x) \in L_{SG}$;

(ii) for every sequence Λ there exists a function $\psi \in L$ such that (3) and (5) hold, and condition (7) is not valid.

Let $\{x_n\}_{n=2}^{\infty}$ be a sequence such that $x_2 = 2$ and $x_n^3 + \ln n < x_{n+1}$, $n \ge 2$. Put

$$\varphi(x) = \begin{cases} 4, & x \in [0; 8), \\ 2^n \frac{x - x_n^3}{\ln n} + 2^n, & x \in [x_n^3; x_n^3 + \ln n), \\ 2^{n+1}, & x \in [x_n^3 + \ln n; x_{n+1}^3), \end{cases}$$

for $n \ge 2$. It is clear that $\varphi \in L$.

Let us prove (i). If $x \in [x_n^3; x_n^3 + \ln n)$ then

$$\frac{\varphi(x+1)}{\varphi(x)} \le \Big(2^n \frac{x+1-x_n^3}{\ln n} + 2^n\Big)\Big(2^n \frac{x-x_n^3}{\ln n} + 2^n\Big)^{-1} = 1 + \frac{1}{x - x_n^3 + \ln n} \to 1, \quad n \to \infty.$$

If $x \in [x_n^3 + \ln n; x_{n+1}^3)$ we obtain

$$\frac{\varphi(x+1)}{\varphi(x)} \le \frac{\varphi(x_{n+1}^3 + 1)}{2^{n+1}} = \frac{1}{\ln(n+1)} + 1 \to 1, \qquad n \to \infty.$$

Hence, $\varphi(\ln x) \in L_{SG}$.

Now, we prove (ii). Let $\psi \in L$ such that $\psi(\lambda_n) = x_n$, $n \ge 2$. It is clear that the function ψ satisfies condition (3). Further, because $\gamma(t) = o(t)$ $(t \to +\infty)$ for every $\gamma \in L_{SG}$, we get

$$\overline{\lim_{n\to\infty}} \frac{\varphi(\ln n + \psi(\lambda_n)\gamma(\psi(\lambda_n)))}{\varphi(\psi(\lambda_n)\gamma(\psi(\lambda_n)))} \le \overline{\lim_{n\to\infty}} \frac{\varphi(\ln n + x_n^2)}{\varphi(x_n)} \le \overline{\lim_{n\to\infty}} \frac{\varphi(x_n^3)}{\varphi(x_{n-1}^3 + \ln(n-1))} = 1,$$

i.e., condition (5) holds. Let us now take $l(x) = x^2$, $x \ge 0$, then

$$\overline{\lim_{n\to\infty}} \frac{\varphi(\ln n + \psi(\lambda_n)l(\psi(\lambda_n)))}{\varphi(\psi(\lambda_n)l(\psi(\lambda_n)))} = \overline{\lim_{n\to\infty}} \frac{\varphi(\ln n + x_n^3)}{\varphi(x_n^3)} = 2.$$

Hence, condition (7) is not fulfilled. So, we may not decline the concavity condition of the function $\ln\varphi(x)$ in Theorem A in general.

The validity of Theorem 1 follows from the following assertion.

Theorem 2. *Let $\psi \in L$, condition (3) holds, and $h \in L_0$. Then, in order that*

$$\forall F \in S_\psi(\Lambda): \qquad M(\sigma, F) < \mu(\sigma, F)h(\ln\mu(\sigma, F)), \quad \sigma \ge \sigma_0, \tag{8}$$

it is necessary and sufficient that

$$\forall l_1, l_2 \in L: \qquad n < l_1(n) + h(l_2(n)\psi(\lambda_n)), \quad n \ge n_0. \tag{9}$$

Proof of Theorem 2

Sufficiency. Let $\psi \in L$, condition (3) is fulfilled, and $h \in L_0$.

First of all we prove sufficiency. We assume that condition (9) is satisfied. Suppose that condition (8) is not fulfilled, i.e., there exist $F \in S_\psi(\Lambda)$, and a sequence $\{x_p\}_{p=0}^{\infty}$, $x_p \to +\infty$ as $p \to \infty$, such that

$$M(x_p, F) \ge \mu(x_p, F)h(\ln\mu(x_p, F)), \qquad p \ge 0. \tag{10}$$

Since

$$\sigma = o(\ln\mu(\sigma, F)), \qquad \sigma \to +\infty, \tag{11}$$

for every entire Dirichlet series which is not an exponential polynomial, then there exists a function $n_2 \in L$ such that

$$\sum_{n \le n_2(\sigma)} |a_n| e^{\sigma\lambda_n} \le \mu(\sigma, F)/2, \qquad \sigma > \sigma_1. \tag{12}$$

Let $n_1(\sigma)=\min\{n\ge 0:\psi(\lambda_n)\ge 3\sigma/(1-q)\}$. From (6) we obtain

$$\sum_{n\ge n_1(\sigma)}|a_n|e^{\sigma\lambda_n}\le \mu(\sigma,F)/2,\qquad \sigma\ge\sigma_2. \tag{13}$$

Then, from (12) and (13), we get

$$M(\sigma,F)\le\mu(\sigma,F)+\sum_{n=[n_2(\sigma)]+1}^{n_1(\sigma)-1}|a_n|e^{\sigma\lambda_n}\le(n_1(\sigma)-n_2(\sigma)+1)\mu(\sigma,F),\qquad\sigma\ge\sigma_3. \tag{14}$$

We can choose a function $l_1\in L$ such that $l_1(n_1(\sigma)-1)\le n_2(\sigma)-2$, $\sigma\ge\sigma_4$, and let $\sigma_5=\max\{\sigma_3;\sigma_4\}$.

Since $\psi(\lambda_{n_1(\sigma)-1})<3\sigma/(1-q)$, it follows from (11) that

$$\ln\mu(\sigma,F)/\psi(\lambda_{n_1(\sigma)-1})\to+\infty,\qquad \sigma\to+\infty.$$

We may assume that the sequence $\{x_p\}_{p=0}^{\infty}$ satisfies the condition $x_0\ge\sigma_5$, and the sequence $\{\ln\mu(x_p,F)/\psi(\lambda_{n_1(x_p)-1})\}_{p=0}^{\infty}$ is nonnegative and increasing. Consider the function $l_2\in L$ such that $l_2(n_1(x_p)-1)=\ln\mu(x_p,F)/\psi(\lambda_{n_1(x_p)-1})$, $p\ge 0$. Then, from (14) and (9), we have

$$M(x_p,F)\le(n_1(x_p)-1-l_1(n_1(x_p)-1))\mu(x_p,F)<$$
$$<\mu(x_p,F)h(l_2(n_1(x_p)-1)\psi(\lambda_{n_1(x_p)-1}))=\mu(x_p,F)h(\ln\mu(x_p,F)),\quad p\ge p_0,$$

that contradicts to (10). The sufficiency of condition (9) is proved.

Necessity. Suppose that condition (9) is not fulfilled, i.e., there exist $l_1,l_2\in L$ and a sequence $\{n_k\}_{k=0}^{\infty}$, $n_k\to+\infty$ as $k\to\infty$, such that

$$n_k\ge l_1(n_k)+h(l_2(n_k)\psi(\lambda_{n_k})),\qquad k\ge 0. \tag{15}$$

We will prove that in this case there exist a function $F\in S_\psi(\Lambda)$ and a sequence $\{\varkappa_p\}_{p=0}^{\infty}$, $\varkappa_p\to+\infty$ as $p\to\infty$, such that

$$M(\varkappa_p,F)\ge\mu(\varkappa_p,F)h(\ln\mu(\varkappa_p,F)),\qquad p\ge 0.$$

Put $m_0=1$. We may assume that $\psi(\lambda_{n_0})>0$ and the sequence $\{n_k\}_{k=0}^{\infty}$ satisfies the following conditions

$$l_2(n_{k+1})\ge 2\lambda_{n_k}, \tag{16}$$
$$m_{k+1}\overset{\text{def}}{=\!=}\frac{l_2(n_{k+1})(\lambda_{n_{k+1}}-\lambda_{n_k})}{\lambda_{n_{k+1}}\lambda_{n_k}}+m_k\frac{\psi(\lambda_{n_k})}{\psi(\lambda_{n_{k+1}})}\ge 1, \tag{17}$$
$$\lambda_{[l_1(n_{k+1})]}\ge 2\lambda_{n_k}, \tag{18}$$
$$\varkappa_k\overset{\text{def}}{=\!=}\frac{l_2(n_{k+1})\psi(\lambda_{n_{k+1}})}{\lambda_{n_k}}+m_k\psi(\lambda_{n_k})\uparrow+\infty,\quad k\to\infty, \tag{19}$$

for $k\ge 0$.

Let, for every $k\ge 0$, $a_{n_k}=\exp\{-m_k\lambda_{n_k}\psi(\lambda_{n_k})\}$ and $a_{n_k+p}=a_{n_k}\exp\{-\varkappa_k(\lambda_{n_k+p}-\lambda_{n_k})\}$ whenever $[l_1(n_{k+1})]-n_k\le p<n_{k+1}-n_k$. If $[l_1(n_{k+1})]-n_k\le p<n_{k+1}-n_k$ and $k\ge 0$,

then, by (16), (18) and (19), we obtain

$$
\begin{aligned}
-\ln a_{n_k+p} &= \varkappa_k(\lambda_{n_k+p}-\lambda_{n_k}) + m_k\lambda_{n_k}\psi(\lambda_{n_k}) \geq \varkappa_k(\lambda_{n_k+p}-\lambda_{n_k}) \geq \\
&\geq \frac{l_2(n_{k+1})\psi(\lambda_{n_{k+1}})}{\lambda_{n_k}}\left(\frac{\lambda_{n_k+p}}{2}+\frac{\lambda_{[l_1(n_{k+1})]}}{2}-\lambda_{n_k}\right) \geq \\
&\geq \frac{2\lambda_{n_k}\psi(\lambda_{n_{k+1}})}{\lambda_{n_k}}\frac{\lambda_{n_k+p}}{2} \geq \psi(\lambda_{n_{k+1}})\lambda_{n_k+p} \geq \psi(\lambda_{n_{k+1}})\lambda_{n_k+p},
\end{aligned}
$$

and, therefore, $a_{n_k+p} \leq \exp\{-\psi(\lambda_{n_{k+1}})\lambda_{n_k+p}\}$. Let us consider the Dirichlet series

$$
F(s) = \sum_{k=0}^{\infty}\left(a_{n_k}e^{s\lambda_{n_k}} + \sum_{p=[l_1(n_{k+1})]-n_k}^{n_{k+1}-n_k-1} a_{n_k+p}e^{s\lambda_{n_k+p}}\right).
$$

By Remark 1, $F \in S_\psi(\Lambda)$.

Further, from (17) and (19) we deduce

$$
\begin{aligned}
\varkappa_k &= \frac{l_2(n_{k+1})\psi(\lambda_{n_{k+1}})}{\lambda_{n_k}} + m_k\psi(\lambda_{n_k}) = \\
&= \frac{\lambda_{n_{k+1}}\psi(\lambda_{n_{k+1}})}{\lambda_{n_{k+1}}-\lambda_{n_k}}\frac{l_2(n_{k+1})(\lambda_{n_{k+1}}-\lambda_{n_k})}{\lambda_{n_{k+1}}\lambda_{n_k}} + m_k\psi(\lambda_{n_k}) = \\
&= \frac{\lambda_{n_{k+1}}\psi(\lambda_{n_{k+1}})}{\lambda_{n_{k+1}}-\lambda_{n_k}}\left(m_{k+1} - m_k\frac{\psi(\lambda_{n_k})}{\psi(\lambda_{n_{k+1}})}\right) + m_k\psi(\lambda_{n_k}) = \\
&= \frac{m_{k+1}\lambda_{n_{k+1}}\psi(\lambda_{n_{k+1}})}{\lambda_{n_{k+1}}-\lambda_{n_k}} - m_k\frac{\lambda_{n_{k+1}}\psi(\lambda_{n_k})}{\lambda_{n_{k+1}}-\lambda_{n_k}} + m_k\psi(\lambda_{n_k}) = \\
&= -\frac{\ln a_{n_{k+1}}}{\lambda_{n_{k+1}}-\lambda_{n_k}} + m_k\frac{\psi(\lambda_{n_k})(\lambda_{n_{k+1}}-\lambda_{n_k}) - \lambda_{n_{k+1}}\psi(\lambda_{n_k})}{\lambda_{n_{k+1}}-\lambda_{n_k}} = \\
&= -\frac{m_k\lambda_{n_k}\psi(\lambda_{n_k}) + \ln a_{n_{k+1}}}{\lambda_{n_{k+1}}-\lambda_{n_k}} = \frac{\ln a_{n_k} - \ln a_{n_{k+1}}}{\lambda_{n_{k+1}}-\lambda_{n_k}}, \qquad k \geq 0.
\end{aligned}
$$

It is well known that if

$$
\varkappa_k = \frac{\ln|a_{n_k}| - \ln|a_{n_{k+1}}|}{\lambda_{n_{k+1}}-\lambda_{n_k}} \uparrow +\infty, \qquad k \uparrow +\infty,
$$

then $\mu(\varkappa_k, F) = a_{n_k}e^{\varkappa_k\lambda_{n_k}}$, $k \geq 0$. Therefore, by (15) and (19),

$$
\begin{aligned}
M(\varkappa_k, F) &\geq \sum_{p=[l_1(n_{k+1})]-n_k}^{n_{k+1}-n_k-1} a_{n_k+p}e^{\varkappa_k\lambda_{n_k+p}} = \mu(\varkappa_k, F)(n_{k+1} - [l_1(n_{k+1})]) \geq \\
&\geq \mu(\varkappa_k, F)h(l_2(n_{k+1})\psi(\lambda_{n_{k+1}})) = \mu(\varkappa_k, F)h(\varkappa_k\lambda_{n_{k+1}} - m_k\lambda_{n_{k+1}}\psi(\lambda_{n_{k+1}})) = \\
&= \mu(\varkappa_k, F)h(\varkappa_k\lambda_{n_{k+1}} + \ln a_{n_k}) = \mu(\varkappa_k, F)h(\ln\mu(\varkappa_k, F)), \qquad k \geq 0,
\end{aligned}
$$

and Theorem 2 is completely proved.

Proof of Theorem 1

Let $h_\varepsilon(x)=\exp\{\varphi^{-1}((1+\varepsilon)\varphi(x))-x\}$, $x\ge x_0$, for $\varepsilon>0$. Since $\varphi(\ln x)\in L_{SG}$, we obtain $h_\varepsilon(x)\to+\infty$ $(x\to+\infty)$ for any $\varepsilon>0$.

We assume that condition (7) is satisfied. Suppose that condition (4) is not fulfilled, i.e., there exist $F\in S_\psi(\Lambda)$, a number $\varepsilon>0$, and a sequence $\{x_p\}_{p=0}^\infty$, $x_p\to+\infty$ as $p\to\infty$, such that

$$\varphi(\ln M(x_p,F))\ge(1+\varepsilon)\varphi(\ln\mu(x_p,F)),\qquad p\ge 0. \tag{20}$$

Hence,

$$M(x_p,F)\ge\mu(x_p,F)h_\varepsilon(\ln\mu(x_p,F)),\qquad p\ge 0. \tag{21}$$

Then, by Theorem 2, there exist $l_1,l_2\in L$, and a sequence $\{n_k\}_{k=0}^\infty$, $n_k\to\infty$ as $k\to\infty$, such that

$$n_k-l_1(n_k)\ge h_\varepsilon(l_2(\psi(\lambda_{n_k}))\psi(\lambda_{n_k})),\qquad k\ge 0. \tag{22}$$

We set $l(x)=l_2(x)$, $x\ge 0$. It follows from (22) that

$$n_k\ge n_k-l_1(n_k)\ge\exp\{\varphi^{-1}((1+\varepsilon)\varphi(l(\psi(\lambda_{n_k}))\psi(\lambda_{n_k})))-l(\psi(\lambda_{n_k}))\psi(\lambda_{n_k})\},\qquad k\ge 0,$$

which contradicts (7). The sufficiency of the condition is proved.

Let us prove the necessity of condition (7). Suppose that condition (7) is not fulfilled, i.e., there exist a function $l\in L$, and a number $\varepsilon>0$ such that

$$\overline{\lim_{n\to\infty}}\frac{\varphi(\ln n+\psi(\lambda_n)l(\psi(\lambda_n)))}{\varphi(\psi(\lambda_n)l(\psi(\lambda_n)))}>1+\varepsilon.$$

We remark that $\varphi(\ln(x/2))\sim\varphi(\ln x)$ $(x\to+\infty)$, if $\varphi(\ln x)\in L_{SG}$. Therefore, there exists a sequence $\{n_k\}_{k=0}^\infty$, $n_k\to\infty$ as $k\to\infty$, such that

$$\varphi(\ln(n_k/2)+\psi(\lambda_{n_k})l(\psi(\lambda_{n_k})))>(1+\varepsilon)\varphi(\psi(\lambda_{n_k})l(\psi(\lambda_{n_k}))),\qquad k\ge 0. \tag{23}$$

We put $l_1(x)=x/2$, $l_2(x)=l(x)$, $x\ge 0$. Then from (23) we obtain (22). Hence by virtue of Theorem 2, there exist a function $F\in S_\psi(\Lambda)$, and a sequence $\{x_p\}_{p=0}^\infty$, $x_p\to+\infty$ as $p\to\infty$, such that relations (21) and (20) hold. Thus, if condition (7) is not fulfilled then there exists a function $F\in S_\psi(\Lambda)$, for which the relation $\varphi(\ln M(\sigma,F))\sim\varphi(\ln\mu(\sigma,F))$ $(\sigma\to+\infty)$ is not valid. Hence, Theorem 1 is proved.

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