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M. M. Sheremeta

ON BEHAVIOUR OF THE MAXIMAL TERM OF DIRICHLET SERIES DERIVATIVE

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Let $\mu(\sigma, F)$ be the maximal term of Dirichlet series $F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}$, $ks = \sigma + it$, $0 \le \lambda_n \uparrow +\infty$, with the abscissa of absolute convergence $A \in (-\infty, +\infty]$.

Conditions on positive convex function Φ on $(-\infty, A)$ are studied in order that the relation $\ln \mu(\sigma, F) = (1 + o(1))\Phi(\sigma)$, $\sigma \uparrow A$, imply the relation $\mu(\sigma, F')/\mu(\sigma, F) = (1 + o(1))\Phi'(\sigma)$, $\sigma \uparrow A$.

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Пусть $\mu(\sigma, F)$ — максимальный член ряда Дирихле $F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}$, $s = \sigma + it$, $0 \le \lambda_n \uparrow +\infty$, с абсциссой абсолютной сходимости $A \in (-\infty, +\infty]$.

Изучаются условия на положительную выпуклую на $(-\infty, A)$ функцию Φ для того чтобы из соотношения $\ln \mu(\sigma, F) = (1 + o(1))\Phi(\sigma)$, $\sigma \uparrow A$, следовало соотношение $\mu(\sigma, F')/\mu(\sigma, F) = (1 + o(1))\Phi'(\sigma)$, $\sigma \uparrow A$.

1°. Let $0 = \lambda_0 < \lambda_n \uparrow +\infty$, $n \to \infty$, and the Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it,$$

has the abscissa of absolute convergence $A \in (-\infty, +\infty]$. For $\sigma < A$ let $\mu(\sigma) = \mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \ge 0\}$ and $\nu(\sigma) = \nu(\sigma, F) = \max\{n : |a_n| \exp(\sigma\lambda_n) = \mu(\sigma)\}$ be the maximal term and the central index, respectively.

By $\Omega(A)$ we denote the class of positive on $(-\infty, A)$ functions Φ such that their derivatives Φ' are continuous, positive and increasing to $+\infty$ on $(-\infty, A)$ functions. For $\Phi \in \Omega(A)$ let φ be the inverse function to Φ' and $\Psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(\sigma)$ be the function associated with Φ in the sense of Newton. Then [1] Ψ is continuous and increasing to A function on $(-\infty, A)$, and φ is continuous on $(0, +\infty)$ and increasing to A function. In [2] it is proved that if $\Phi \in \Omega(A)$ and

$$\varlimsup_{\sigma \to A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)} = 1 \tag{1}$$

then

$$\varlimsup_{\sigma \to A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\Phi'(\sigma)} \ge 1, \tag{2}$$

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and if, moreover,

$$\ln \Phi'(\sigma) = o(\Phi(\sigma)), \quad \sigma \to A, \tag{3}$$

then

$$\varlimsup_{\sigma\to A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\Phi'(\Psi^{-1}(\sigma))} \le 1. \tag{4}$$

The precision of estimate (4) is shown in [2] provided that there exists a number $\alpha \in [0, 1)$ such that the function $(\Phi'(\sigma))^{\alpha}/\Phi(\sigma)$ is nonincreasing on $[\sigma_0, A)$. This condition, generally speaking, cannot be removed. It is possible in (4) to replace $\Phi'(\Psi^{-1}(\sigma))$ by $\Phi'(\sigma)$ provided

$$\Phi'(\Psi(\sigma)) = (1 + o(1))\Phi'(\sigma), \quad \sigma \to A. \tag{5}$$

Then, in view of (2),

$$\varlimsup_{\sigma\to A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\Phi'(\sigma)} = 1. \tag{6}$$

In the case $A = +\infty$ condition (5) holds provided $\Phi \in \Omega(+\infty)$ and $\Phi(\sigma) = \sigma\alpha(\sigma)$ for $\sigma \ge \sigma_0$, where α is a slowly increasing function such that $\alpha\left(\frac{x^2\alpha'(x)}{\alpha(x)}\right) = (1 + o(1))\alpha(x)$ $(x \to +\infty)$. If $\Phi \in \Omega(+\infty)$ increases not slowlier than a power function or $\Phi \in \Omega(0)$, then condition (5) is false and equality (6), generally speaking, is not true. The question arises whether this equality is true, whenever

$$\ln \mu(\sigma, F) = (1 + o(1))\Phi(\sigma), \quad \sigma \to A. \tag{7}$$

We consider somewhat generalized case, when together with (1) the condition

$$\lim_{\sigma\to A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)} = \beta \in (0, 1]. \tag{8}$$

holds.

For $q \in (0, 1)$ and $\Phi \in \Omega(A)$ we put

$$\alpha_{\Phi}(q) = \varlimsup_{x\to+\infty} \frac{\Phi(\varphi(qx)) + qx(\varphi(x) - \varphi(qx))}{\Phi(\varphi(x))}.$$

Theorem 1. *Let the function $\Phi \in \Omega(A)$, $A \in (-\infty, +\infty]$, be such that $\alpha_{\Phi}(q) < \beta$ for every $q < \beta$ and*

$$\ln \Phi'(\sigma) = o(\Phi(\Psi(\sigma))), \quad \sigma \to A. \tag{9}$$

Then equalities (1) and (8) imply the following equality

$$\lim_{\sigma\to A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\Phi'(\sigma)} = \beta. \tag{10}$$

From Theorem 1 we obtain the following result.

Theorem 2. *If the function $\Phi \in \Omega(A)$, $A \in (-\infty, +\infty]$, be such that $\alpha_{\Phi}(q) < 1$ for every $q < 1$ and condition (9) holds, then (7) implies*

$$\frac{\mu(\sigma, F')}{\mu(\sigma, F)} = (1 + o(1))\Phi'(\sigma), \quad \sigma \to A. \tag{11}$$

2°. We need some lemmas.

Lemma 1 [2]. *For every Dirichlet series*

$$\lambda_{\nu(\sigma,F)} \leq \frac{\mu(\sigma,F')}{\mu(\sigma,F)} \leq \lambda_{\nu(\sigma,F')}.$$

Lemma 2 [2]. *If $\Phi \in \Omega(A)$, $A \in (-\infty,+\infty]$, and conditions (1) and (3) hold, then*

$$\lambda_{\nu(\sigma,F')} \leq (1+o(1))\Phi'(\Psi^{-1}(\sigma)), \quad \sigma \to A.$$

For investigation of the asymptotic behaviour of positive functions L'Hospital's rules are often used

$$\varliminf_{\sigma\to A} \frac{G'(\sigma)}{\Phi'(\sigma)} \leq \varliminf_{\sigma\to A} \frac{G(\sigma)}{\Phi(\sigma)} \leq \varlimsup_{\sigma\to A} \frac{G(\sigma)}{\Phi(\sigma)} \leq \varlimsup_{\sigma\to A} \frac{G'(\sigma)}{\Phi'(\sigma)}, \tag{12}$$

which are true under known conditions on smoothness of functions G and Φ. For convex functions G and Φ O. V. Bratishchev [3] showed the necessary and sufficient conditions in order that in (12) this or another inequality turns into quality. We adduce here two of his results necessary for us. We consider, that G is a convex function, G' is its right-handed derivative, and Φ is a convex and continuously differentiable function such that $\Phi'(x) \to +\infty$ $(x \to A)$.

Lemma 3 [3]. *In order that for every function G*

$$\left(\varlimsup_{\sigma\to A} \frac{G(\sigma)}{\Phi(\sigma)} = 1\right) \Longleftrightarrow \left(\varlimsup_{\sigma\to A} \frac{G'(\sigma)}{\Phi'(\sigma)} = 1\right), \tag{13}$$

it is necessary and sufficient that

$$\varlimsup_{\sigma\to A} \frac{1}{\Phi'(\sigma)} \inf\left\{\frac{\Phi(t)}{t-\sigma} : t > \sigma\right\} \leq 1. \tag{14}$$

We remark that if $\Phi \in \Omega(A)$ then Φ' is continuous and increasing to $+\infty$ on $(-\infty, A)$ and, therefore,

$$\inf\left\{\frac{\Phi(t)}{t-\sigma} : t > \sigma\right\} = \frac{\Phi(t(\sigma))}{t(\sigma)-\sigma},$$

where $t(\sigma)$ is the solution of the equation $\Phi'(t)(t-\sigma) - \Phi(t) = 0$ that is $t(\sigma) = \Psi^{-1}(\sigma)$ and

$$\frac{\Phi(t(\sigma))}{t(\sigma)-\sigma} = \Phi'(t(\sigma)) = \Phi'(\Psi^{-1}(\sigma)).$$

Therefore, in this case condition (14) is equivalent to condition (5).

Lemma 4 [3]. *In order that for every function G such that $G(\sigma) \leq (1+o(1))\Phi(\sigma)$, $\sigma \to A$, the equivalence*

$$\left(\varliminf_{\sigma\to A} \frac{G(\sigma)}{\Phi(\sigma)} = \beta\right) \Longleftrightarrow \left(\varliminf_{\sigma\to A} \frac{G'(\sigma)}{\Phi'(\sigma)} = \beta\right), \quad \beta \in (0,1], \tag{15}$$

holds, it is necessary and sufficient that for every $\varepsilon \in (0, \beta)$

$$\varlimsup_{\sigma\to A} \frac{\Phi(\sigma^*) + \Phi'(\sigma^*)(\sigma - \sigma^*)}{\Phi(\sigma)} < \beta, \tag{16}$$

where $\sigma^* = \max\{t < \sigma : \Phi'(t) = (\beta - \varepsilon)\Phi'(\sigma)\}$.

If $\Phi \in \Omega(A)$ then $\sigma^* = \varphi((\beta - \varepsilon)\Phi'(\sigma))$ and

$$\varlimsup_{\sigma\to A} \frac{\Phi(\sigma^*) + \Phi'(\sigma^*)(\sigma - \sigma^*)}{\Phi(\sigma)} =$$
$$= \varlimsup_{\sigma\to A} \frac{\Phi(\varphi((\beta-\varepsilon)\Phi'(\sigma))) + (\beta-\varepsilon)\Phi'(\sigma)(\varphi(\Phi'(\sigma)) - \varphi((\beta-\varepsilon)\Phi'(\sigma)))}{\Phi(\sigma)} =$$
$$= \varlimsup_{x\to+\infty} \frac{\Phi(\varphi((\beta-\varepsilon)x)) + (\beta-\varepsilon)x(\varphi(x) - \varphi((\beta-\varepsilon)x))}{\Phi(\varphi(x))} = \alpha_\Phi(\beta - \varepsilon).$$

Therefore, condition (16) is equivalent to the condition $\alpha_\Phi(q) < \beta$ for every $q < \beta$.

3°. We prove the theorems. Since Theorem 2 follows from Theorem 1 as $\beta = 1$, we need to prove Theorem 1.

From (1) it follows that $\ln\mu(\sigma, F) \le (1 + o(1))\Phi(\sigma)$, $\sigma \to A$. Therefore, using Lemma 4 with $G(\sigma) = \ln\mu(\sigma, F)$ and the equality

$$\ln\mu(\sigma, F) = \ln\mu(\sigma_0, F) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(t,F)} dt, \quad -\infty < \sigma_0 < \sigma < A, \tag{17}$$

in view of (8), we have

$$\lim_{\sigma\to A} \frac{1}{\Phi'(\sigma)} \lambda_{\nu(\sigma,F)} = \beta. \tag{18}$$

Thus, in view of Lemmas 1 and 2,

$$\frac{\ln\beta + \ln\Phi'(\sigma) + o(1)}{\Phi(\sigma)} \le \frac{\ln\lambda_{\nu(\sigma,F)}}{\Phi(\sigma)} \le \frac{\ln\mu(\sigma, F')}{\Phi(\sigma)} - \frac{\ln\mu(\sigma, F)}{\Phi(\sigma)} \le$$
$$\le \frac{\ln\lambda_{\nu(\sigma,F')}}{\Phi(\sigma)} \le \frac{\ln\Phi'(\Psi^{-1}(\sigma)) + o(1)}{\Phi(\sigma)}, \quad \sigma \to A,$$

that is, in view of (9),

$$\frac{\ln\mu(\sigma, F')}{\Phi(\sigma)} - \frac{\ln\mu(\sigma, F)}{\Phi(\sigma)} \to 0, \quad \sigma \to A.$$

Hence, in view of (1) and (8), we have

$$\beta = \varliminf_{\sigma\to A} \frac{\ln\mu(\sigma, F')}{\Phi(\sigma)} \le \varlimsup_{\sigma\to A} \frac{\ln\mu(\sigma, F')}{\Phi(\sigma)} = 1.$$

Therefore, using again Lemma 4 with $G(\sigma) = \ln\mu(\sigma, F')$ and equality (17), we obtain

$$\lim_{\sigma\to A} \frac{1}{\Phi'(\sigma)} \lambda_{\nu(\sigma,F')} = \beta. \tag{19}$$

Equality (10) follows from Lemma 1 and equalities (18) and (19). Theorem 1 is proved.

4°. Now, we study out for which functions Φ the condition $q < \beta \in (0, 1]$ implies the inequality $\alpha_\Phi(q) < \beta$.

First, let $\beta < 1$. If $\Phi \in \Omega(+\infty)$ and $\Phi(\sigma) = \sigma\alpha(\sigma)$ for $\sigma \geq \sigma_0$, where α is a slowly increasing function, then $\Phi'(\sigma) = \alpha(\sigma) + \sigma\alpha'(\sigma) = (1 + o(1))\alpha(\sigma)$, $\sigma \to +\infty$, that is Φ' is a slowly increasing function. Therefore, $\varphi(qt) = o(\varphi(t))$, $t \to +\infty$, for each $q \in (0, 1)$, whence, in view of convexity of Φ,

$$\frac{\Phi(\varphi(qx))}{\Phi(\varphi(x))} \leq \frac{\varphi(qx)}{\varphi(x)} \to 0, \quad x \to +\infty, \quad 0 < q < 1,$$

and, since

$$\lim_{x\to+\infty} \frac{x\varphi(x)}{\Phi(\varphi(x))} = \lim_{\sigma\to A} \frac{\Phi'(\sigma)\sigma}{\Phi(\sigma)} = 1,$$

then

$$\frac{qx(\varphi(x) - \varphi(qx))}{\Phi(\varphi(x))} = (1 + o(1))\frac{qx\varphi(x)}{\Phi(\varphi(x))} \to q, \quad x \to +\infty.$$

Thus, in this case $\alpha_\Phi(q) = q < \beta$ provided $q < \beta$ and the statement of Theorem 1 is true.

If $\Phi \in \Omega(+\infty)$ and $\Phi(\sigma) = \sigma^p \quad (p > 1)$ for $\sigma \geq \sigma_0$ then $\alpha_\Phi(q) = qp - (p-1)q^{p/(p-1)} > q$ and if $\Phi(\sigma) = e^{\varrho\sigma} \quad (\varrho > 0)$, then $\alpha_\Phi(q) = q - q\ln q > q$ that is in this cases the statement of Theorem 1, generally speaking, is not true. In the case $A = 0$, generally, it cannot yield a subclass of functions from $\Omega(0)$ such that $\alpha_\Phi(q) < \beta$ for $q < \beta < 1$.

If $\beta = 1$ then a situation is different. For example, if $\Phi \in \Omega(+\infty)$ and $\Phi(\sigma) = \sigma^p\,(p > 1)$ for $\sigma \geq \sigma_0$ then $\alpha_\Phi(q) = qp - (p-1)q^{p/(p-1)} < 1$ and if $\Phi(\sigma) = e^{\varrho\sigma}\,(\varrho > 0)$ then $\alpha_\Phi(q) = q - q\ln q < 1$ for each $q \in (0, 1)$. Thus, it can yield large subclasses of functions from $\Omega(+\infty)$ such that the statement of Theorem 2 holds. We can say the same about $\Omega(0)$. We mention only the case when $\Phi(\sigma) = |\sigma|^{-\varrho}$, $\varrho > 0$. Here $\varphi(t) = -(\varrho/t)^{1/(\varrho+1)}$ and $\Phi(\varphi(t)) = (t/\varrho)^{\varrho/(\varrho+1)}$, that is $\alpha_\Phi(q) = (\varrho + 1)q^{\varrho/(\varrho+1)} - q\varrho < 1$ for each $q \in (0, 1)$.

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Lviv National University, Faculty of Mechanics and Mathematics

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