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O. B. Skaskiv, I. E. Chyzhykov

THE NEVANLINNA CHARACTERISITICS AND MAXIMUM MODULUS OF GAP POWER SERIES

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Let f be an analytic in $\{z : |z| < 1\}$ function represented by the lacunary power series $f(z) = \sum_{k=0}^{+\infty} a_k z^{n_k}$. We obtain general conditions on the exponents and the growth of f which provide the asymptotic equality

$$T(r, f) = (1 + o(1)) \ln M_f(r)$$

as $r \to 1 - 0$ outside an exceptional set, where $T(r, f)$ is the Nevanlinna characteristics of f, $M_f(r)$ is the maximum modulus of f on the circle $\{z : |z| = r\}$.

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Для аналитической в $\{z : |z| < 1\}$ функции f, представленной лакунарным степенным рядом $f(z) = \sum_{k=0}^{+\infty} a_k z^{n_k}$, получены общие условия на показатели и рост f, обеспечивающие асимптотическое соотношение

$$T(r, f) = (1 + o(1)) \ln M_f(r)$$

при $r \to 1 - 0$ вне исключительного множества, где $T(r, f)$ — характеристика Неванлинны f, $M_f(r)$ — максимум модуля f на окружности $\{z : |z| = r\}$.

1. Introduction

We use the standard notations of the Nevanlinna theory [1] . Let f be an analytic in $D_R = \{z : |z| < R\}$, $0 < R \le +\infty$, function represented by the power series

$$f(z) = \sum_{k=0}^{+\infty} a_k z^{n_k}. \tag{1.1}$$

We denote $M_f(r) = \max\{|f(z)| : |z| = r\}$, $\mu(r, f) = \max\{|a_k| r^{n_k} : k \ge 0|\}$. In the case $R = +\infty$ T. Murai [2] established that if the condition

$$\sum_{k=1}^{+\infty} \frac{1}{n_k} < +\infty \tag{1.2}$$

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is fulfilled then the function f of form (1.1) has no finite deficient values (i.e. $(\forall a \in \mathbb{C})$ $\delta(a, f) = 0$).

Actually condition (1.2) provides [2] for $r \to +\infty$ outside a set of finite logarithmic measure the relation

$$T(r, f) = (1 + o(1)) \ln M_f(r), \tag{1.3}$$

that is stronger than the assertion $(\forall a \in \mathbb{C})$: $\delta(a, f) = 0$.

The question on the absence of Picard's exceptional values was well studied for an analytic in D_1 function of the form (1.1) with the Hadamard gaps (i.e. $\frac{n_{k+1}}{n_k} \geq q > 1$, $k \geq 1$) [4–8].

It was proved in [2] (see also [6]) that in this case an unbounded function f takes every value $a \in \mathbb{C}$ infinitely often in D_1.

On the other hand, in [9, Th. 3] it was shown that if f is an analytic in D_1 function of form (1.1) and the conditions $k = O(n_k^{1-\beta})$ as $k \to +\infty$, and $\rho[f] > \frac{1-\beta}{\beta}$ where $\rho[f] = \overline{\lim}_{r\to 1-0} \frac{\ln\ln M_f(r)}{-\ln(1-r)}$, $0 < \beta < 1$, are satisfied then

$$(\forall a \in \mathbb{C}): \overline{\lim_{r\to 1-0}} \frac{N(r, a, f)}{\ln M_f(r)} = 1.$$

The same conclusion is obtained [9, Th. 4] if f has Hadamard gaps and

$$\overline{\lim_{r\to 1-0}} \frac{\ln M_f(r)}{-\ln(1-r)} > 0.$$

In our paper we establish general sufficient condition that provide (1.3) for an analytic in D_1 function f of form (1.1) and indicate conditions under which f has no deficient values.

2. Main results

Let us introduce necessary notations. We define the counting function of the exponents of series (1.1) by $n_f(t) = \sum_{0<n_k\leq t} 1$, and put $N_f(t) = \int_0^t \frac{n_f(s)}{s} ds$. It is obvious that for fixed $n_f(t)$ there exists a continuous increasing function $\tilde{n}_f(t)$ such that $\tilde{n}_f(t) \sim n_f(t)$ as $t \to +\infty$. We put $\hat{n}_f(t) = \max\{\tilde{n}_f(t), \ln^+ t\}$.

For $p \in \mathbb{R}$ and a measurable set $E \subset \mathbb{R}$ we define the relative left-side lower density at the point $p \in \mathbb{R}$ by the equality $d_p E = \underline{\lim}_{x\to p-0} \operatorname{mes}(E \cap [x, p))/(p - x)$.

We are going to prove the following theorems.

Theorem 1. *Let f be an analytic in D_1 function of the form (1.1), $\Phi_1(t)$ a positive continuous function such that $\Phi_1(t)/t \uparrow +\infty$ as $t \uparrow +\infty$. If*

$$\overline{\lim_{r\to 1-0}} \frac{\ln \mu(r, f)}{(1-r)\Phi_1\left(\frac{1}{1-r}\right)} > 0 \tag{2.1}$$

and

$$\lim_{t\to+\infty} t \int\limits_{\Phi_1(t)}^{+\infty} \frac{\hat{n}_f(s)}{s^2} ds = 0, \tag{2.2}$$

then (1.3) holds as $r \to 1 - 0$ ($r \notin E_1$, $d_1 E_1 = 0$).

Theorem 2. *Let f and Φ_1 be the functions from Theorem 1. If*

$$\varliminf_{r\to 1-0} \frac{\ln \mu(r,f)}{(1-r)\Phi_1\left(\frac{1}{1-r}\right)} > 0 \tag{2.3}$$

and

$$\varliminf_{t\to+\infty} t \int\limits_{\Phi_1(t)}^{+\infty} \frac{\hat{n}_f(s)}{s^2} ds = 0, \tag{2.4}$$

then (1.3) holds as $r \to 1-0$ ($r \notin E_2$, $d_1 E_2 = 0$).

We note that using the approach of [2] one can deduce from Theorems 1 and 2 the following

Theorem 3. *Suppose that conditions (2.1) and (2.2) hold with $N_f(s)$ instead of $\hat{n}_f(s)$ for an analytic in D_1 function f of the form (1.1). Then ($\forall a \in \mathbb{C}$): $\delta(a,f) = 0$.*

Theorem 4. *Suppose that conditions (2.3) and (2.4) hold with $N_f(s)$ instead of $\hat{n}_f(s)$ for an analytic in D_1 function f of the form (1.1). Then ($\forall a \in \mathbb{C}$): $\delta(a,f) = 0$.*

It is interesting to compare the following corollaries with the results of P. Nichols and L. Sons [9].

Corollary 1. *If the function f satisfies*

$$\varlimsup_{r\to 1-0} \frac{\ln M_f(r)}{(1-r)^{\frac{\beta-1}{\beta}}} = +\infty \quad \text{and} \quad \varlimsup_{t\to+\infty} \frac{n_f(t)}{t^{1-\beta}} < +\infty,$$

then $\forall a \in \mathbb{C}$ we have $\delta(a,f) = 0$.

Corollary 2. *If the function f satisfies*

$$\varlimsup_{r\to 1-0} \frac{\ln M_f(r)}{-\ln(1-r)} = +\infty \quad \text{and} \quad \varlimsup_{t\to+\infty} \frac{n_f(t)}{\ln t} < +\infty,$$

then $\ln M_f(r) \sim T(r,f)$ as $r \to 1-0$ outside a set of zero left-side lower density at 1.

3. Preliminaries

To prove Theorems 1 and 2 we need the following lemmas.

Lemma 1. *Let $f(z)$ be an analityc in D_1 function of form(1.1), and $-\ln(1-r) = o\big(\ln \mu(r,f)\big)$ as $r \to 1-0$. Then*

$$\ln \mu(r,f) \sim \ln M_f(r)$$

as $r \to 1-0$ except possible a set $E \subset [0,1)$ of r such that $\int_E \frac{dr}{1-r} < +\infty$.

Lemma 1 follows from Theorem 1 [10], where the same assertion is proved for Dirichlet series with exponents having a positive step. It is sufficient to change the variable.

Throughout the paper we suggest that $\gamma(\sigma)$ and $\Gamma(\sigma)$ are positive increasing to $+\infty$ continuous function on $[-1,0)$ and $[1,+\infty)$, respectively, $\Phi_1(t)$ such as in Theorem 1.

Lemma 2. *If the conditions*

$$\varlimsup_{\sigma\to -0}\frac{\gamma(\sigma)}{|\sigma|\Phi_1\left(\frac{1}{|\sigma|}\right)}>0, \tag{3.1}$$

$$\lim_{t\to+\infty} t\int\limits_{\Phi_1(t)}^{+\infty}\frac{\Gamma(v)}{v^2}dv=0, \tag{3.2}$$

hold, then

$$\frac{\Gamma(v_0(\sigma))}{v_0(\sigma)|\sigma|}=o(1) \tag{3.3}$$

as $\sigma\to -0$ ($\sigma\in[-1;0)\setminus E_3$), $d_0E_3=0$, where $v_0(\sigma)$ is the unique solution of the equation $\Gamma(v_0)=\gamma(\sigma)$.

Lemma 3. *If the conditions of Lemma 2 are satisfied, then $\forall\sigma\in[-1;0)\setminus E_4$, $d_0E_4=0$, we have*

$$\gamma\Big(\sigma+\frac{\Gamma(v_0(\sigma))}{v_0(\sigma)}\Big)\le\gamma(\sigma)+\frac{1}{\beta(v_0(\sigma))}, \tag{3.4}$$

where $\beta(v)$ is a positive continuous increasing to $+\infty$ function such that (3.2) holds with the function $\Gamma_1(v)=\Gamma(v)\beta(v)$ instead of $\Gamma(v)$. Moreover, $E_3\subset E_4$.

The proofs of Lemmas 2 and 3 are similar to those of Lemma 3 [11] and Lemmas 9 and 10 [12]. The difference between them is that in [12] in condition (3.2) the author uses $\Phi(t)=\Phi_1(t)/t$ instead $\Phi_1(t)$. We note that in the case $\varphi(t)\asymp\varphi_1(t)$ ($t\to+\infty$) Lemmas 2 and 3 are equivalent to the lemmas proved in [12], here φ and φ_1 are the function inverse to Φ and Φ_1, respectively.

Proof of Lemma 2. Let $\sigma_j\uparrow 0$ ($j\to+\infty$) be a sequence such that for $\Phi(t)=\frac{\Phi_1(t)}{t}$ we have $\Phi\Big(\frac{1}{|\sigma_j|}\Big)=O\big(\gamma(\sigma_j)\big)$. It follows from (3.2) that

$$\Gamma\big(\Phi_1(y)\big)=o\big(\Phi(y)\big),\quad y\to+\infty. \tag{3.5}$$

Therefore we obtain

$$\frac{1}{|\sigma_j|}\le\varphi\big(K_1\gamma(\sigma_j)\big)=\varphi\big(K_1\Gamma\big(v_0(\sigma_j)\big)\big)=\varphi\Big(o\Big(\Phi\big(\varphi_1\big(v_0(\sigma_j)\big)\big)\Big)\Big)\le\varphi_1\big(v_0(\sigma_j)\big). \tag{3.6}$$

for some $K_1>0$ as $j\to+\infty$

Since (3.5) implies

$$\varepsilon(x)\overset{\text{def}}{=}\frac{\Gamma(x)}{x}\varphi_1(x)\to 0,\quad x\to+\infty, \tag{3.7}$$

$\alpha_j=\big(\max\{\varepsilon(v):v\ge v_0(\sigma_{j-1})\}\big)^{\frac12}$ is a nondecreasing sequence, and $\alpha_j\to 0$ as $j\to+\infty$. Denote $\sigma_j^*=\sigma_j-\alpha_j(\sigma_j-\sigma_{j-1})$ and $E_3=\bigcup_{j=1}^{+\infty}[\sigma_j^*;\sigma_j]$. For the measure of the set E_3 we have the following estimate:

$$\operatorname{mes}\big(E_3\cap[\sigma_{j-1};0)\big)\le\sum_{k=j}^{+\infty}(\sigma_k-\sigma_k^*)=$$

$$=\sum_{k=j}^{+\infty}\alpha_k(\sigma_k-\sigma_{k-1})\le\alpha_j\sum_{k=j}^{+\infty}(\sigma_k-\sigma_{k-1})=\alpha_j|\sigma_{j-1}|.$$

This implies $d_0 E_3 = 0$.

Now we assume that $\sigma \in [\sigma_1^*; 0) \setminus E_3$. Then $\sigma \in (\sigma_j; \sigma_{j+1}^*)$ for some $j \geq 1$. Thus

$$|\sigma| \geq |\sigma_{j+1}^*| = \alpha_{j+1}|\sigma_j| + (1-\alpha_{j+1})|\sigma_{j+1}| \geq \alpha_{j+1}|\sigma_j|.$$

Applying (3.6) we obtain

$$\frac{\Gamma\big(v_0(\sigma)\big)}{v_0(\sigma)|\sigma|} \leq \frac{\Gamma\big(v_0(\sigma)\big)}{v_0(\sigma)\alpha_{j+1}|\sigma_j|} \leq \frac{\varepsilon\big(v_0(\sigma)\big)}{\alpha_{j+1}} \frac{\varphi_1\big(v_0(\sigma_j)\big)}{\varphi_1\big(v_0(\sigma)\big)} \leq \sqrt{\varepsilon\big(v_0(\sigma)\big)}.$$

The last inequality together with (3.7) finish the proof of Lemma 2. □

Proof of Lemma 3. We use the notation from the proof of Lemma 2. Let $\gamma_0(\sigma) = \ln\gamma(\sigma) + \big(\beta\big(v_0(\sigma)\big)\big)^{-1}$,

$$E_5 = \{\sigma \in [-1; 0) \setminus E_3 : \ln\gamma(\sigma + \delta_0(\sigma)) \geq \gamma_0(\sigma)\},$$

where $\delta_0(\sigma) = \frac{\Gamma(v_0(\sigma))}{v_0(\sigma)}$. We denote $E_5(\sigma) = E_5 \cap [\sigma; 0)$. If $E_5(\sigma) = \varnothing$ for some $\sigma < 0$, then the lemma is proved. Thus, assume that $E_5(\sigma) \neq \varnothing$ for all $\sigma < 0$. We define $\tau_1 = \inf\{\sigma : \sigma \in E_5\}$, $T_1 = \inf\{\sigma : \ln\gamma(\sigma) = \gamma_0(\tau_1)\}$. Then

$$\ln\gamma(\tau_1 + \delta_0(\tau_1)) \geq \gamma_0(\tau_1) = \ln\gamma(T_1),$$

i.e. $0 < T_1 - \tau_1 \leq \delta_0(\tau_1)$. For $n \geq 2$ we put also

$$\tau_n = \inf\{\sigma : \sigma \in E_5(T_{n-1})\}, \quad T_n = \inf\{\sigma : \ln\gamma(\sigma) = \gamma_0(\tau_n)\}.$$

Then

$$\ln\gamma(\tau_n + \delta_0(\tau_n)) \geq \gamma_0(\tau_n) = \ln\gamma(T_n),$$

i.e.

$$0 < T_n - t_n < \delta_0(\tau_n), \quad n \geq 1. \tag{3.8}$$

Moreover,

$$\ln\gamma(\tau_{n+1}) - \ln\gamma(\tau_n) \geq \ln\gamma(T_n) - \ln\gamma(\tau_n) = \frac{1}{\beta(v_0(\tau_n))}, \quad n \geq 1. \tag{3.9}$$

We note that $E_5 \subset \bigcup_{n=1}^{+\infty}[\tau_n; T_n]$. Taking into account (3.8), we get

$$\operatorname{mes} E_5(\sigma) \leq \begin{cases} \sum_{k=n}^{+\infty} \delta_0(\tau_k), & \sigma \in (\tau_n; T_n], \\ \sum_{k=n+1}^{+\infty} \delta_0(\tau_k), & \sigma \in (T_n; \tau_{n+1}]. \end{cases} \tag{3.10}$$

Remark that in accordance with inequality (3.9) and definition of $v_0(\sigma)$

$$\delta_0(\tau_k) = \frac{\Gamma(v_0(\tau_k))}{v_0(\tau_k)} \leq \frac{\Gamma_1(v_0(\tau_k))}{v_0(\tau_k)}\big(\ln\gamma(\tau_{k+1}) - \ln\gamma(\tau_k)\big) =$$

$$= \frac{\Gamma_1(v_0(\tau_k))}{v_0(\tau_k)}\Big(\ln\Gamma_1(v_0(\tau_{k+1})) - \ln\Gamma_1(v_0(\tau_k)) - \ln\beta(v(\tau_{k+1})) + \ln\beta(v_0(\tau_k))\Big) \leq$$

$$\leq \frac{\Gamma_1(v_0(\tau_k))}{v_0(\tau_k)}\Big(\ln\Gamma_1(v_0(\tau_{k+1})) - \ln\Gamma_1(v_0(\tau_k))\Big) \leq \frac{\Gamma_1(v_0(\tau_{k+1})) - \Gamma_1(v_0(\tau_k))}{v_0(\tau_k)}.$$

If $v_0(\tau_{k+1})\le 2v_0(\tau_k)$, then

$$\delta_0(\tau_k)\le 2\int\limits_{v_0(\tau_k)}^{v_0(\tau_{k+1})}\frac{d\Gamma_1(t)}{t}=2\Big(\frac{\Gamma_1(v_0(\tau_{k+1}))}{v_0(\tau_{k+1})}-\frac{\Gamma_1(v_0(\tau_k))}{v_0(\tau_k)}+\int\limits_{v_0(\tau_k)}^{v_0(\tau_{k+1})}\frac{\Gamma_1(t)}{t^2}dt\Big). \tag{3.11}$$

Else we have $v_0(\tau_{k+1})>2v_0(\tau_k)$, and

$$\delta_0(\tau_k)=\Gamma(v_0(\tau_k))\Big(\frac{1}{v_0(\tau_k)}-\frac{1}{v_0(\tau_{k+1})}\Big)\frac{1}{1-\frac{v_0(\tau_k)}{v_0(\tau_{k+1})}}<$$
$$<2\int\limits_{v_0(\tau_k)}^{v_0(\tau_{k+1})}\frac{\Gamma(t)}{t^2}dt\le\int\limits_{v_0(\tau_k)}^{v_0(\tau_{k+1})}\frac{\Gamma_1(t)}{t^2}dt. \tag{3.12}$$

Let n such that $\sigma_j\in(T_n;\tau_{n+1}]$, (σ_j) is the sequence from Lemma 2. Then $\gamma(\sigma_j)\le\gamma(\tau_{n+1})$, and in the case $\sigma_j\in(\tau_n;T_n]$ we have

$$\gamma(\sigma_j)\le\gamma(T_n)=\gamma(\tau_n)\exp\Big\{\frac{1}{\beta(v_0(\tau_n))}\Big\}=(1+o(1))\gamma(\tau_n),\quad j\to+\infty.$$

From the last inequality we obtain similarly to (3.6) for $\sigma_j\in(\tau_n;T_n]$,

$$\frac{1}{|\sigma_j|}\le\varphi\big(K_1\gamma(\sigma_j)\big)\le\varphi\big(2K_1\gamma(\tau_n)\big)=\varphi\big(K_1\Gamma\big(v_0(\tau_n)\big)\big)\le$$
$$\le\varphi\Big(o\Big(\Phi\big(\varphi_1\big(v_0(\tau_n)\big)\big)\Big)\Big)\le\varphi_1\big(v_0(\tau_n)\big),\quad j\to+\infty. \tag{3.13}$$

Remark now that for $\sigma_j\in(\tau_n;T_n]$ (3.10)–(3.13) implies

$$\frac{1}{|\sigma_j|}\operatorname{mes}E_5(\sigma_j)\le\frac{1}{|\sigma_j|}\sum_{k=n}^{+\infty}\delta_0(\tau_k)\le 2\varphi_1(v_0(\tau_n))\int\limits_{v_0(\tau_n)}^{+\infty}\frac{\Gamma_1(t)}{t^2}dt, \tag{3.14}$$

and for $\sigma_j\in(T_n;\tau_{n+1}]$ it follows from (3.6), (3.10)–(3.13) that

$$\frac{1}{|\sigma_j|}\operatorname{mes}E_5(\sigma_j)\le\varphi_1(v_0(\tau_{n+1}))\sum_{k=n+1}^{+\infty}\delta_0(\tau_k)\le$$
$$\le 2\varphi_1(v_0(\tau_{n+1}))\int\limits_{v_0(\tau_{n+1})}^{+\infty}\frac{\Gamma_1(t)}{t^2}dt\le 2\varphi_1(v_0(\tau_{n+1}))\int\limits_{v_0(\tau_{n+1})}^{+\infty}\frac{\Gamma_1(t)}{t^2}dt. \tag{3.15}$$

Applying condition (3.2) with the function $\Gamma_1(t)=\Gamma(t)\beta(t)$ instead of $\Gamma(t)$, to (3.14) and (3.15) we obtain the assertion of the lemma. Moreover,

$$\operatorname{mes}\Big(\big(E_3\cup E_5\big)\cap[\sigma_j;0)\Big)\le\operatorname{mes}E_3(\sigma_j)+\operatorname{mes}E_5(\sigma_j)=o\big(|\sigma_j|\big),\quad j\to+\infty,$$

i.e. $d_0\big(E_3\cup E_5\big)=0$. Therefore, it is enough to define $E=E_3\cup E_5$. Lemma 3 is proved. □

Lemma 4. *If the conditions*

$$\varliminf_{\sigma\to-0}\frac{\gamma(\sigma)}{|\sigma|\Phi_1\left(\frac{1}{|\sigma|}\right)}>0, \tag{3.16}$$

$$\varliminf_{t\to+\infty} t\int\limits_{\Phi_1(t)}^{+\infty}\frac{\Gamma(v)}{v^2}dv=0, \tag{3.17}$$

hold, then

$$\frac{\Gamma(v_0(\sigma))}{v_0(\sigma)|\sigma|}=o(1)$$

as $\sigma\to-0$ *(*$\sigma\in[-1;0)\setminus\tilde{E}_3$*),* $d_0\tilde{E}_3=0$.

Lemma 5. *If under the condition of Lemma 4*

$$\ln t=O(\Gamma(t))\quad\text{as}\quad t\to+\infty,$$

then there is a continuous positive function $\beta(v)$ *increasing to* $+\infty$ *such that (3.2) holds with the function* $\Gamma_1(v)=\Gamma(v)\beta(v)$ *instead of* $\Gamma(v)$ *and* $\forall\sigma\in[-1;0)\setminus\tilde{E}_4$ *(*$d_0\tilde{E}_4=0$*)*

$$\gamma\Big(\sigma+\frac{\Gamma(v_0(\sigma))}{v_0(\sigma)}\Big)\le\gamma(\sigma)+\frac{1}{\beta(v_0(\sigma))},$$

Moreover, $\tilde{E}_3\subset\tilde{E}_4$.

The proofs of Lemma 4 and 5 are similar to those of Lemmas 2 and 3.

Lemma 6. *Let conditions (2.1) of Theorem 1 and (3.2) of Lemma 2 hold with* $\Gamma(t)=n_f^*(t)\stackrel{\text{def}}{=}\hat{n}_f(t)\beta(t)$*, where* $\beta(t)$ *as above. Then*

$$\Lambda(r)\stackrel{\text{def}}{=}\sum_{n_k>v(r)}|a_k|r^{n_k}=o(1),\quad r\to1-0, r\notin E_6,$$

where $d_1E_6=0$*, and* $v(r)$ *is the unique solution of the equation* $n_f^*(v)=3\ln\mu(r,f)$.

Proof of Lemma 6. We put $\sigma=r-1$, $\gamma(\sigma)=3\ln\mu(1+\sigma,f)$. Thus the conditions of Lemmas 2 and 3 hold. They yield that outside a set E_6 such that $d_1E_6=0$ we have

$$3\ln\mu\Big(r+\frac{n_f^*(v(r))}{v(r)},f\Big)\le3\ln\mu(r,f)+\frac{1}{\beta(v(r))}, \tag{3.18}$$

where $v(r)$ is the unique solution of the equation $n_f^*(v)=3\ln\mu(r,f)$. Inequality (3.18) implies for $r\to1-0$, $r\notin E_6$,

$$\mu(r+\delta^*(r),f)\le\mu(r,f)\exp\Big\{\frac{1}{3\beta(v(r))}\Big\}=(1+o(1))\mu(r,f), \tag{3.19}$$

where $\delta^*(r) = n_f^*(v(r))/v(r)$. Using (3.19), we obtain as $r \to 1-0$, $r \notin E_6$,

$$\Lambda(r) = \sum_{n_k > v(r)} |a_k| r^{n_k} \le \mu(r+\delta^*(r), f) \sum_{n_k > v(r)} \Big(\frac{r}{r+\delta^*(r)}\Big)^{n_k} \le$$
$$\le (1+o(1))\mu(r,f)\Big(\frac{r}{r+\delta^*(r)}\Big)^{v(r)} \sum_{n=0}^{+\infty}\Big(1+\frac{\delta^*(r)}{r}\Big)^{-n} =$$
$$= (1+o(1))\mu(r,f)\exp\Big\{-v(r)\ln\Big(1+\frac{\delta^*(r)}{r}\Big)\Big\}\frac{1}{\delta^*(r)} =$$
$$= (1+o(1))\mu(r,f)\exp\big\{-(1+o(1))v(r)\delta^*(r) - \ln\delta^*(r)\big\} =$$
$$= (1+o(1))\mu(r,f)\exp\big\{-(1+o(1))n_f^*(v(r)) + \ln v(r)\big\}.$$

Take into account that $\ln t \le \frac{1}{2}\hat{n}_f(t) = o(n_f^*(t))$ as $t \to +\infty$ and recall that $n_f^*(v) = 3\ln\mu(r,f)$. Thus,

$$\Lambda(r) \le (1+o(1))\mu(r,f)\exp\big\{-(1+o(1))n_f^*(v(r))\big\} =$$
$$= \exp\big\{-(2+o(1))\ln\mu(r,f)\big\} = o(1), \quad r \to 1-0, r \notin E_6.$$

Lemma 6 is proved. □

Due to T. Murai [2] we have the following lemma.

Lemma 7. *For arbitrary trigonometric polynomial $P(t) = \sum_k \hat{P}(k)e^{ikt}$ with n non-zero coefficients we have*

$$\hat{m}(P) \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \ln^+|P(t)|\,dt \ge \ln^+\max|\hat{P}(k)| - K_4 n,$$

where K_4 is a constant independent of n.

4. Proofs of the theorems

Proof of Theorem 1. Let $P_r(\theta) = \sum_{n_k \le v(r)} a_k r^{n_k} e^{i\theta n_k}$. Because of Lemma 6 for $r \in [r_1, 1) \setminus E_6$, we have

$$|f(re^{i\theta}) - P_r(\theta)| \le \Lambda(r) \le 1.$$

Therefore, for such r the equality $\mu(r,f) = \mu(r,P_r)$ holds. Now we apply Lemma 7. This yields

$$\hat{m}(P_r) \ge \ln^+\max\{|a_k|r^{n_k} : n_k \le v(r)\} - K_4 n(v(r)) \ge$$
$$\ge \ln\mu(r,f) - o(n_f^*(v(r))) = (1+o(1))\ln\mu(r,f).$$

According to Lemma 1 the relation $\ln\mu(r,f) = (1+o(1))\ln M_f(r)$ holds outside a set E_7 of r such that $\int_{E_7}\frac{d\tau}{1-\tau} < +\infty$. Then $\forall\eta > 0$ for $1 > r > r_\eta$

$$\frac{1}{1-r}\operatorname{mes}\big(E_7 \cap [r;1)\big) = \frac{1}{1-r}\int\limits_{E_7\cap[r;1)} d\tau \le \int\limits_{E_7\cap[r;1)} \frac{d\tau}{1-\tau} \le \eta,$$

i.e.

$$D_1E_7 \stackrel{\text{def}}{=} \overline{\lim_{r\to 1-0}} \frac{1}{1-r}\operatorname{mes}\big(E_7\cap[r;1)\big)=0.$$

Thus, outside a set $E_8 = E_6 \cup E_7$ we have

$$T(r,f)=m(r,f)\geq m(r,P_r)-m(r,f-P_r)-\ln 2=\hat{m}(P_r)-\ln 2\geq \\ \geq (1+o(1))\ln\mu(r,f)=(1+o(1))\ln M_f(r),$$

and $d_1E_8 \leq d_1E_6 + D_1E_7 = 0$. Theorem 1 is proved. □

Applying Lemmas 2 and 3 we deduced Lemma 6, and after that using Lemmas 7 and 1 we deduced Theorem 1. Similarly, one can obtain Theorem 2 from Lemmas 4 i 5 instead of Lemmas 2 and 3, and from Lemmas 7 and 1.

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Lviv National University, Faculty of Mechanics and Mathematics

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