ANATOLIY ASIROVYCH GOL'DBERG

The famous mathematician-analyst Anatoliy Asirovych Gol'dberg is 70. Half of those years were the years of active, strenuous and fruitful research and teaching work at Lviv Ivan Franko National University.

Anatoliy Asirovych was born on 2nd April 1930 in Kyiv into a family of intellectuals (his father was a doctor, and his mother was a teacher). Prior to the war, the Gol'dbergs lived in Zaporizhya for some years. After the war, they moved to Lviv, where Anatoliy finished secondary school in 1947 and entered the University. Anatoliy's father wanted his son to become a doctor but the latter took a great interest in mathematics and made up his mind to go in for it with his characteristic assiduity and persistence.

During the twenties and thirties the famous Lviv School of Mathematics was formed. Its leading representatives were University Professors H. Steinhaus and S. Banach. The well-known mathematical journal "Studia Matematica" was also published, many mathematical problems were raised and solved in "Kawiarnia szkocka", and new mathematical theories were created.

During the first years after the Second World War, the operation "Wisla" was carried out by Stalin's regime, which was the resettlement of ethnic groups. Many famous Polish mathematicians had to leave Lviv for Poland. At the same time a number of well-known mathematicians arrived in Lviv from different towns of the former Soviet Union among them L. Volkovysky, B. Gnedenko, O. Kovanko, Y. Lopatynsky, and I. Sokolov.

Anatoliy Gol'dberg showed an aptitude for mathematics as a schoolboy, but the final decision to take up mathematics was made after conversations the boy had with Professor O. Kovanko, Head of the Organizing Committee of the Lviv Mathematical Olympiad in 1947. Anatoliy went to win the Lviv mathematical Olympiad.

Gol'dberg's first steps in science were made under the guidance of professor M. Zarytsky when Anatoliy was a second-year university student. At that time, he was interested in the theory of Boolean algebras. Later he become a participant in Volkovysky and Sokolov's seminar where the problems and theorems of analysis from the well-known book by Pólya and Szegő were the subject of discussions. As Anatoliy Asirovych recollects, Professors L. Volkovysky and I. Sokolov as scientists, mathematicians, and individuals had the greatest influence on him. Later his research under the guidance of L. Volkovysky, was closely connected with the monograph by R. Nevanlinna "Eindeutige analytische Funktionen". It is L. Volkovysky, who is considered the founder of the Lviv School of complex analysis. In spite of the short period of his activities in Lviv, Volkovysky developed many disciples, such as the prominent mathematicians P. Belinsky, A. Gol'dberg, I. Danylyuk, V. Mykhalchuk, I. Pesin, Y. Rodin and Yu. Trokhymchuk.

The diploma paper of A. Gol'dberg, written under the guidance of Professor L. Volkovysky, refers to value distribution theory of meromorphic functions. This theory formed the main branch of the mathematical interests of the future mathematician. The foundation of this theory were laid down in the twenties by R. Nevanlinna. To formulate the main result of the diploma paper and the principal scientific achievements of A. Gol'dberg we shall use generally accepted notions and designations of value distribution theory of meromorphic functions [1–4].

Let f be a meromorphic function in the whole complex plane. The Nevanlinna deficiency $\delta(a, f)$ characterizes exceptional behaviour of a-points of f. If $\delta(a, f) > 0$, then a is a deficient value of f. The first fundamental theorem of value distribution theory yields $0 \le \delta(a, f) + \varepsilon(a, f) \le 1$, where $\varepsilon(a, f)$ is the ramification index of f at a. A general formulation of inverse problem of value distribution theory is the following.

Given a finite or infinite sequence of points $\{a_k\} \subset \overline{\mathbb{C}}$ and the corresponding nonnegative numbers $\delta(a_k)$, $\varepsilon(a_k)$ such that

$$0 < \delta(a_k) + \varepsilon(a_k) \le 1, \quad \sum_k \{\delta(a_k) + \varepsilon(a_k)\} \le 2. \tag{1}$$

Find a meromorphic function with assigned deficiencies $\delta(a_k)$ and indices $\varepsilon(a_k)$ at every a_k and without other deficiencies and indices. In his diploma paper, A. Gol'dberg solved the inverse problem under the assumptions:

(i) the number of deficient values is finite,

(ii) $\sum_k \delta(a_k) < 2$.

Only 20 years after the publication D. Drasin completely solved the inverse problem using Goldberg's method.

Gol'dberg's student years at the University were far from cloudless. Twice he was expelled from Comsomol*) and from the University for the far-fetched "bourgeois liberalism" and "the loss of Comsomol vigilance". These were not the only reasons for his not becoming a post-graduate student. At that time, there were some more serious obstacles on the road to the post-graduate course. The communist regime established "quotas" for people of certain nationalities and those considered politically unreliable who wished to enter institutions of higher education and post-graduate courses. The KGB "first department" painstakingly kept their eyes on this. Even after the death of the "leader of the nations", A. Gol'dberg's application for admission to post-graduate courses was rejected. Having brilliantly defended his diploma paper, the young university graduate went to work as a teacher of Physics and Mathematics at a secondary school in the village of Zabolotzi, near Brody. A. Gol'dberg's teaching load was approximately 30 hours a week instead of the 18 required by the curriculum. Nonetheless, he successfully conducted his research work, solving a number of principal problems of value distribution theory of meromorphic functions.

*) The Communist young league.

In 1955, Anatoliy Asirovych defended his candidate degree (Ph.D.) thesis. The dissertation was highly appreciated by his refferee, the well-known mathematician and professor of Kharkiv University B. Levin, with whom A. Gol'dberg maintained friendly and scientific contacts and worked in close collaboration thereafter. In his candidate dissertation, A. Gol'dberg constructed examples of meromorphic functions with infinite set of deficient values for the first time. Having developed the idea of constructing such examples, he went on to solve one more famous problem of value distribution theory. He proved the following theorem. Given ρ, $0 < \rho \le \infty$, and given an arbitrary, at most countable, subset $M \subset \overline{\mathbb{C}}$, there exists a meromorphic function of the order ρ whose set of deficient values coincides with M. Together with the result of Valiron, this theorem gives the full description of the structure of the deficient values of meromorphic functions of the arbitrary order ρ: it may be an arbitrary given, at most countable set if $0 < \rho \le +\infty$, and it contains at most one arbitrary point if $\rho = 0$.

The idea and methods of this proof appeared to be extremely fruitful and were widely used by many mathematicians. In particular, we should like to mention the monograph by the well-known English mathematician W. Hayman, published in 1964 [2].

From 1955 to 1963, A. Gol'dberg worked in Uzhorod and from 1963 to 1997 at Lviv University. In 1997, he left to join his relatives in Israel.

In 1965, Anatoliy Asirovych defended his dissertation for the doctor's degree on subject "Value distribution and asymptotic properties of entire and meromorphic functions". His doctor's dissertation is a fundamental work comprising over 600 pages. The principal directions in which the investigation was conducted were the following: whether the basic conclusions of the Nevanlinna theory have final character and whether they give in to the improvements; what properties of Picard exceptional values are kept for deficient values; what kind of connection is between Nevanlinna characteristics and other quantities that describe the asymptotic properties of meromorphic functions; what connections are between deficient values and distribution of arguments of a-points; asymptotic properties and value distribution of concrete classes of entire and meromorphic functions.

The solution of some of these problems required the creation of the new mathematical theories. Let us dwell in particular on the problem of the connection between the growth of the entire function on the rays and the distribution of its zeros. The growth characteristics of the entire function of order ρ on the rays are the Phragmen-Lindelöf indicator h_f and the lower indicator $\underline{h}_f$,

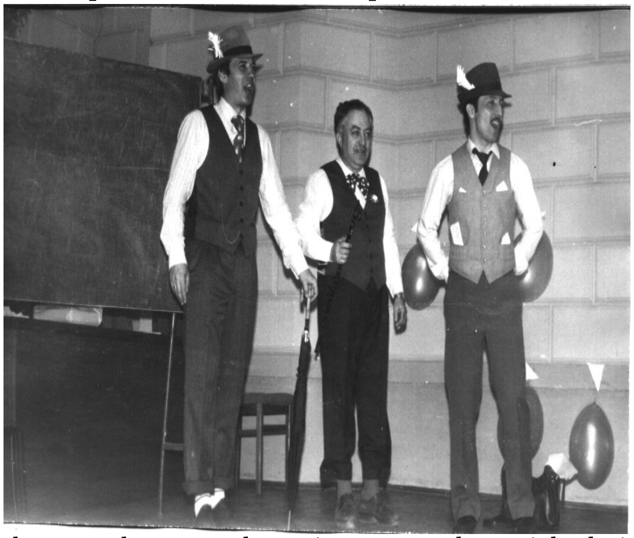
$$h_f(\xi) = \limsup_{r\to\infty} \frac{\log|f(r\xi)|}{r^\rho L(r)}, \quad \underline{h}_f(\xi) = \liminf_{r\to\infty} \frac{\log|f(r\xi)|}{r^\rho L(r)}, \quad \xi \in \mathbb{T},$$

where L is the slowly varying function such that h_f is bounded on the unit circle $\mathbb{T}$. The characteristics of zeros distribution of f are two set functions on $\mathbb{T}$, the upper and the lower angular densities of zeros, $\overline{\Delta}$ and $\underline{\Delta}$. The above-mentioned problem consists in the following: finding the connections between h_f, $\underline{h}_f$ and $\overline{\Delta}$, $\underline{\Delta}$. If $\overline{\Delta} = \underline{\Delta}$, then $h_f = \underline{h}_f$. The link can be written as an integral equality which expresses the indicator h_f through the angular density $\triangle = \overline{\Delta} = \underline{\Delta}$. It was established in the years 1938-39 by B. Levin and A. Pfluger. In this case $\triangle$ is a Borel measure on $\mathbb{T}$. If $\overline{\Delta} = \underline{\Delta}$, then $h_f \neq \underline{h}_f$ and both indicators are not expressed through $\overline{\Delta}$ and $\underline{\Delta}$ unique. In this case $\overline{\Delta}$ is a semi-additive measure on $\mathbb{T}$. The problem of finding of estimates for h_f, $\underline{h}_f$ leads to an integral with respect to this semi-additive measure. A. Gol'dberg introduced such a new kind of integral which is interesting on its own. With the help of the theory of this integral A. Gol'dberg obtained the sharp estimates of h_f and $\underline{h}_f$.

The scientific work of Anatoliy Asirovych is so great and varied that even we, his disciples and colleagues for many years at Lviv University, can hardly describe his main results completely and objectively. It is sufficient to mention that the range of his research also comprises differential equations, Riemann surfaces, functions of several complex variables, subharmonic functions, polyanalytic functions, conformal and quasiconformal mappings, algebraic and algebroid functions. One can get acquainted with the review and the list of A. Gol'dberg's works in [5] (the list does not contain article [9] in which A. Gol'dberg proved twenty years old Shah's conjecture concerning the index boundedness of Mittag-Leffler function, and the article [10]). He has over 150 scientific works to his name. Together with I. Ostrovsky, he wrote a fundamental monograph [3]. He also translated books from German [4] and English [2], surveyed the development of the theory of entire and meromorphic functions in [6–8]. A. Gol'dberg is a member of many editorial boards. He is the first editor-in-chief of "Matematychni Studii".

Creative collaboration unites A. Gol'dberg with many mathematicians from Kharkiv, Kyiv, Ufa, Krasnoyarsk, Novosybirsk, Rostov-on-Don, Vilnius, England, Germany, the USA, Finland, Israel, China, Romania and other parts of the world. In the nineties A. Gol'dberg guided the work of a group of Lviv mathematicians, supported by ISF and under the general guidance of Professor I. Laine supported by INTAS. The chances of Anatoliy Asirovych to meet his former colleagues are rare, but his cordial letters besides the discussion of concrete scientific problems, very often contain some fine humor, some everyday problems, and interesting stories. They testify to his gift of a real writer.

A. Gol'dberg's research work is closely connected with his teaching. In 1964, he initiated the seminar in analytical functions at Lviv University, which has become the foundation of the contemporary Lviv Scientific School of Complex Analysis. This seminar continues to function. The lectures of Prof. A. Gol'dberg in complex analysis, which he delivered for 35 years, at Lviv University can serve as an example for every teacher. He was always ready to provide consultation to the students working on their diploma and course papers. These weekly consultations were individual and lasted much longer than was required by the curriculum. The cheerful disposition, the fine and at times sharp humor of Professor A. Gol'dberg made him the beloved Professor of the students, a constant participant of the Mathematician Days at the department and the captain of the Wits'Club team.



Anatoliy Asirovych Gol'dberg has brought up a multitude of disciples. Among them are Doctors of Phisics and Mathematics M. Sheremeta, A. Eremenko, A. Kondratyuk, A. Mokhon'ko, M. Zabolotsky and Candidates of Science V. Tairova, F. Geche, T. Strochyk, H. Prokopovych, M. Girnyk, V. Mokhon'ko, M. Korenkov, O. Fridman, E. Gleiser, O. Sokolovska, M. Strochyk, and V. Piana. One should also mention M. Andrashko, S. Tushkanov, V. Grinstein, Y. Haida, V. Boychuk, O. Hyzha, L. Plakhta, I. Sheikhet, and I. Tkach who worked under A. Gol'dberg's guidance and published either some independent works or works written together with their teacher. From Lviv School of Complex Analysis there are doctors of Physics and Mathematics B. Vynnytsky and O. Skaskiv. Even now, living far from Ukraine, A. Gol'dberg continues to assist his disciples and exchanges letters and e-mail with them.

We wish our dear Anatoliy Asirovych good health, new scientific achievements and firmness in all his life's adversities.

Gol'dberg's Ph. D. Students

1. Tairova, V. G. (1964). On some problems of the theory of Riemann surfaces.
2. Geche, F.I. (1965). Some theorems on the growth of entire functions and vector-valued functions of several complex variables and their applications.
3. Strochik, T. V. (1969). Conformal representation of half-strips.
4. Kondratyuk, A.A. (1969). Bounds for indicators of entire functions.
5. Sheremeta, M. N. (1969). Connection between the asymptotic behavior of analytic functions and coefficients of their power series expansions.
6. Prokopovich, G. S. (1974). Distribution of values and fix-points of superpositions of meromorphie functions.
7. Girnyk, M. A. (1977). Asymptotic properties of some classes of functions analytic or subharmonic in a disc.
8. Mokhon'ko, A. Z. (1978). Some applications of the theory of analytic functions to differential equations and planar curves.
9. Mokhon'ko, V. D. (1978). Generalizations and applications of the lemma on the logarithmic derivative.
10. Eremenko, N. E. (1979). Asymptotic properties of meromorphic and subharmonic functions.
11. Korenkov, N. E. (1979). Asymptotic properties of logarithmic derivatives of entire functions.
12. Fridman, A. N. (1981). Lower bounds of analytic and subharmonic functions.

13. Zabolotskii, N.V. (1982). Asymptotic properties of meromorphic and δ-subharmonic functions of the slow growth.
14. Gleizer, E. V. (1988). Asymptotic behavior of meromorphic and δ-subharmonic functions with a special distribution of values masses.
15. Sokolovskaya, O. P. (1989). Asymptotic relations for δ-subharmonic functions of order or lower order less than one.
16. Strochik, N. N. (1991). The growth of several quantities characterizing asymptotic properties of meromorphic functions.
17. Pyana, V.A. (1993). Uniqueness theorems for algebraic and algebroid functions.

D. Sci. Theses defended by students of A. A. Gol'dberg

1. Sheremeta, M. N. (1987). Asymptotic properties of entire functions given by power and Dirichlet series.
2. Eremenko, A. E. (1987). Asymptotic properties and value distribution of meromopphic functions.
3. Kondratyuk, A. A. (1988). Method of Fourier and Fourier-Laplace series for meromorphic and subharmonic functions of completely regular growth.
4. Mokhon'ko, A. Z. (1994). New relations between the Nevanlinna characteristics and their applications.
5. Zabolotskii, M. V. (1999). Asymptotic properties of analytic and subharmonic functions of non-rapid growth.

REFERENCES

1. Nevanlinna R. *Eindeutige analytisce Funktionen,* Berlin, Springer Verlag, 1936.
2. Hayman W. K. *Meromorphic functions,* Oxford and Clarendon Press, 1964.
3. Gol'dberg A. A., Ostrovskii I. V. *The distribution of values of meromorphic functions,* Nauka, Moscow, 1970, 592 pp. (in Russian)
4. Wittich H., *Neue Untersuchungen über eindeutige analytische Funktionen,* Springer, Verlag, Berlin, 1955.
5. Eremenko A., Ostrovskii I., Sodin M., *Anatolii Asirovich Gol'dberg,* Complex variables theory and application **37** (1998), no. 1–4, 1–51.
6. Gol'dberg A. A., Levin B. Ya., Ostrovskii I. V. *General theory of entire and meromorphic functions,* In: History of the native mathematics, vol. 4, part 1. Naukova Dumka. Kyiv 1970 pp.9–48 (in Russian)
7. Gol'dberg A. A. *Meromorphic functions,* In: Result of Science and Technique. Mathematical Analysis, vol.10, VINITI, Moscow, 1973, pp. 5-97. (Russian); Engl. transl.: J. Soviet Math. 4 (1975), pp.157-216.
8. Gol'dberg A. A., Levin B. Ya., Ostrovskii I. V. *Entire and Meromorphic functions,* In: Complex Analysis. One variable. Results of Science and Technique. Contemporary Problems of Mathematics. Fundamental Directions, vol.85. VINITI. Moscow. 1991. Pp.5-186.(Russian); English transl.: Complex Analysis. One variable 1, Encyclopedia of Math. Sci., Springer-Verlag, Berlin.
9. Gol'dberg A. A. *An estimate of logarithmic derivative modulus of Mittag-Leffler function and its application,* Matem. Studii **5** (1996), 21–30. (in Ukrainian)
10. Girnyk M. O., Gol'dberg A. A. *Approximation of subharmonic function by logarithms of moduli of entire functions in integral metrics,* Dopovidi NAN Ukrainy, (2000), №2, 37–39.

A. A. Kondratyuk, . M. Sheremeta