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ON THE GROWTH OF ENTIRE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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The growth and the l-index boundedness of entire solutions of linear differential equations are investigated.

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Исследуются рост и ограниченность l-индекса целых решений линейных дифференциальных уравнений.

1. Let l be a positive continuous function on $[0, +\infty)$. An entire function f is said to be of bounded l-index [1] if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$

$$\frac{|f^{(n)}(z)|}{n! l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k! l^k(|z|)} : 0 \le k \le N\right\}.$$

The least of such N is called the l-index of f and is denoted by $N(l; f)$.

Let K be the class of positive analytic on $[0, +\infty)$ functions l such that $l'(x) = o(l^2(x))$ $(x \to +\infty)$, and Q be the class of positive continuous on $[0, +\infty)$ functions l such that $l(x + O(1/l(x))) = O(l(x))$ $(x \to +\infty)$. It is known [1] that if $l \in K$ and f is of bounded l-index then

$$\overline{\lim_{r\to+\infty}} \frac{\ln M(r, f)}{L(r)} \le N(l; f) + 1, \quad L(r) = \int_0^r l(x)dx, \tag{1}$$

where $M(r, f) = \max\{|f(z)| : |z| = r\}$.

For an entire function g with zeros a_k denote $G_g(r) = \bigcup_k \{z : |z - a_k| \le r/l(|a_k|)\}$, $r > 0$. It is proved [2] that if $l \in Q$, $g_0, g_1, \dots, g_n, h$ are entire functions of bounded l-index and for every $r > 0$ there exists $M = M(r) > 0$ such that $|g_j(z)| \le M l^j(|z|)|g_0(z)|$ for all $z \in \mathbb{C} \setminus G_{g_0}(r)$ then an entire solution f of the equation

$$g_0(z)w^{(n)} + g_1(z)w^{(n-1)} + \dots + g_n(z)w = h(z), \tag{2}$$

is of bounded l-index. Since there is no estimate from above of $N(l; f)$ in this statement we cannot use inequality (1) to estimate the growth of f. But if we impose some additional

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conditions on the coefficients of equation (2) and on function l we can indicate the estimate of growth of f without finding a quantity of its l-index.

2. The following lemma plays an important role in our investigations.

Lemma 1 [3]**.** *Let $l \in Q$. An entire function f is of bounded l-index if and only if there exist numbers $p \in \mathbb{Z}_+$, $C > 0$ and $R > 0$ such that*

$$\frac{|f^{(p+1)}(z)|}{l^{p+1}(|z|)} \leq C \max\left\{\frac{|f^{(k)}(z)|}{l^k(|z|)} : 0 \leq k \leq p\right\} \tag{3}$$

for all $z \in \mathbb{C}$, $|z| \geq R$.

We need also the following two lemmas.

Lemma 2. *Let $l \in K$. If there exist numbers $p \in \mathbb{Z}_+$ and $C > 0$ such that for z, $|z| \geq R$, inequality (3) holds then*

$$\varlimsup_{r\to+\infty} \frac{\ln M(r,f)}{L(r)} \leq \max\{1, C\}.$$

Proof. Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$, $x \geq R$ and

$$\Omega(x) = \max\left\{\frac{|f^{(k)}(\alpha x)|}{l^k(x)} : 0 \leq k \leq p\right\}.$$

The function Ω is positive continuous differentiable and

$$\Omega'(x) \leq \max\left\{\frac{d}{dx}\frac{|f^{(k)}(\alpha x)|}{l^k(x)} : 0 \leq k \leq p\right\}$$

on $[R, +\infty)$, except an enumerable set of points. Therefore, using the inequality $\frac{d}{dx}|\varphi(x)| \leq$ $\leq \left|\frac{d}{dx}\varphi(x)\right|$ which is true for complex-valued function φ of real variable outside an enumerable set of points, in view of (3), we have

$$\begin{aligned}\Omega'(x) &\leq \max\left\{\frac{|\alpha\, f^{(k+1)}(\alpha x)|}{l^k(x)} + \frac{|f^{(k)}(\alpha x)\, kl'(x)|}{l^{k+1}(x)} : 0 \leq k \leq p\right\} \leq \\ &\leq \max\left\{\frac{|f^{(k+1)}(\alpha x)|}{l^{k+1}(x)} l(x) + \frac{|f^{(k)}(\alpha x)|}{l^k(x)} \frac{k|l'(x)|}{l(x)} : 0 \leq k \leq p\right\} \leq \\ &\leq \Omega(x)\left(\max\{1,\, C\} l(x) + \frac{p|l'(x)|}{l(x)}\right).\end{aligned}$$

Since $l'(x) = o(l^2(x))$ $(x \to +\infty)$, we obtain $\Omega'(x) \leq (1+\varepsilon)\max\{1,\, C\} l(x)\Omega(x)$ for every $\varepsilon > 0$ and for all $x \geq x_0(\varepsilon) \geq R$, except an enumerable set of points. Therefore, there exists $r_0 \geq x_0(\varepsilon)$ such that

$$\Omega(r) \leq \Omega(r_0)\exp\left\{(1+\varepsilon)\max\{1,\, C\}\int_{r_0}^{r} l(x)dx\right\}, \quad r > r_0,$$

whence it follows that $|f(\alpha r)| \leq \Omega(r_0)\exp\{(1+\varepsilon)\max\{1,\, C\}L(r)\}$, $(r > r_0)$, that is, in view of arbitrarity of ε, we obtain the required inequality. □

The proof of following lemma is similar.

Lemma 3. *Let $l \in K$. If there exist numbers $p \in \mathbb{Z}_+$ and $C > 0$ such that for all z, $|z| \geq R$,*

$$\frac{|f^{(p+1)}(z)|}{(p+1)!l^{p+1}(|z|)} \leq C \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \leq k \leq p\right\} \tag{4}$$

then

$$\overline{\lim_{r\to+\infty}} \frac{\ln M(r,f)}{L(r)} \leq (p+1)\max\{1, C\}.$$

3. Now we will study the entire solutions of differential equation (2).

Theorem 1. *Let $l \in K \cap Q$, and for all z, $|z| \geq R$, the entire functions $g_0, g_1, \dots, g_n$ and h satisfy the conditions: a) $|g_j(z)| \leq m_j l^j(|z|)|g_0(z)|$, $(1 \leq j \leq n)$, b) $|g_j'(z)| < < M_j l^{j+1}(|z|)|g_0(z)|$, $(0 \leq j \leq n)$ and c) $|h'(z)| \leq Ml(|z|)|h(z)|$, where m_j, M_j and M are nonnegative constants. If an entire function f is a solution of (2) then f is of bounded l-index and*

$$\overline{\lim_{r\to+\infty}} \frac{\ln M(r,f)}{L(r)} \leq \max\{1, C\}, \quad C = \sum_{j=0}^{n} M_j + (M+1)\sum_{j=1}^{n} m_j + M. \tag{5}$$

Proof. At first we remark that condition b) with $j = 0$ implies $g_0(z) \neq 0$ for $|z| \geq R$. Further, since f satisfies (2),

$$g_0(z)f^{(n+1)}(z) + \sum_{j=0}^{n} g_j'(z)f^{(n-j)}(z) + \sum_{j=1}^{n} g_j(z)f^{(n-j+1)}(z) = h'(z),$$

and, in view of c), we have

$$|h'(z)| \leq Ml(|z|)|h(z)| \leq Ml(|z|)\sum_{j=0}^{n} |g_j(z)||f^{(n-j)}(z)|.$$

Thus, setting $m_0 = 1$, we have

$$|f^{(n+1)}(z)| \leq$$

$$\leq \frac{1}{|g_0(z)|}\left(Ml(|z|)\sum_{j=0}^{n}|g_j(z)||f^{(n-j)}(z)| + \sum_{j=0}^{n}|g_j'(z)||f^{(n-j)}(z)| + \sum_{j=1}^{n}|g_j(z)||f^{(n-j+1)}(z)|\right) \leq$$

$$\leq Ml(|z|)\sum_{j=0}^{n} m_j l^j(|z|)|f^{(n-j)}(z)| + \sum_{j=0}^{n} M_j l^{j+1}(|z|)|f^{(n-j)}(z)| + \sum_{j=1}^{n} m_j l^j(|z|)|f^{(n-j+1)}(z)|,$$

whence

$$\frac{|f^{(n+1)}(z)|}{l^{n+1}(|z|)} \leq Ml(|z|)\sum_{j=0}^{n} m_j \frac{|f^{(n-j)}(z)|}{l^{n-j}(|z|)} + \sum_{j=0}^{n} M_j \frac{|f^{(n-j)}(z)|}{l^{n-j}(|z|)} +$$

$$+\sum_{j=1}^{n} m_j \frac{|f^{(n-j+1)}(z)|}{l^{n-j+1}(|z|)} \leq \left(\sum_{j=0}^{n} M_j + (M+1)\sum_{j=1}^{n} m_j + M\right)\max\left\{\frac{|f^{(k)}(z)|}{l^k(|z|)} : 0 \leq k \leq n\right\},$$

i. e. f is of bounded l-index by Lemma 1 and estimate (5) holds by Lemma 2. □

Corollary [4]. *If in differential equation (2) all coefficients g_j and h are polynomials with $\deg g_j \le \deg g_0 = d\,(1 \le j \le n)$ and an entire function f satisfies (2), then f is of bounded index (l-index with $l(x) \equiv 1$) and*

$$\overline{\lim_{r\to+\infty}} \frac{\ln M(r,f)}{r} \le \max\left\{1, \frac{1}{|a_0|}\sum_{j=1}^{n}|a_j|\right\},$$

where a_j is the coefficient at z^d in g_j.

Indeed, since g_j and h are polynomials with $\deg g_j \le \deg g_0 = d\,(1 \le j \le n)$, therefore $|g_j(z)| \le (1+o(1))|a_j/a_0||g_0(z)|$, $(1 \le j \le n)$, $|g_j'(z)| = o(|g_0(z)|)$, $(0 \le j \le n)$ and $|h'(z)| = o(|h(z)|)$ as $|z| \to +\infty$. Thus, from Theorem 1 we easily obtain the necessary statement.

We remark that estimate (5), in general, is unimprovable. Indeed, consider the differential equation $\exp\{z^2\}w'' + 2\exp\{z^2\}w = 4z^2$. The function $f(z) = \exp\{-z^2\}$ is its solution and of bounded l-index with $l(x) = x$ for $x \ge 1$. Using Theorem 1 it is easy to show that $\overline{\lim}_{r\to+\infty} \ln M(r,f)r^{-2} \le 2$. But $\ln M(r,f) = r^2$, $L(r) = \frac{1+o(1)}{2}r^2\,(r \to +\infty)$. Thus, $\overline{\lim}_{r\to+\infty} \ln M(r,f)r^{-2} = 2$.

In the case when differential equation is homogeneous, the following more simple statement is true.

Theorem 2. *Let $l \in K \cap Q$, and entire functions $g_0, g_1, \dots, g_n$ satisfy the condition $|g_j(z)| \le m_j l^j(|z|)|g_0(z)|$, $(1 \le j \le n)$ for all z, $|z| \ge R$. If an entire function f is a solution of (2) with $h \equiv 0$ then f is of bounded l-index and*

$$\overline{\lim_{r\to+\infty}} \frac{\ln M(r,f)}{L(r)} \le \max\left\{1, \sum_{j=1}^{n} m_j\right\}. \tag{6}$$

Proof. Directly from (2) we have $|g_0(z)||f^{(n)}(z)| \le \sum_{j=1}^{n}|g_j(z)||f^{(n-j)}(z)|$, whence

$$\frac{|f^{(n)}(z)|}{l^n(|z|)} \le \sum_{j=1}^{n} m_j \frac{|f^{(n-j)}(z)|}{l^{n-j}(|z|)} \le \sum_{j=1}^{n} m_j \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \le k \le n-1\right\},$$

and it remains to use Lemmas 1 and 2. □

In general, one cannot improve estimate (6). The differential equation $w' + 2zw = 0$ with entire solution $f(z) = \exp\{-z^2\}$ and $l(x) = x$ for $x \ge 1$ demonstrates this.

Finally, using Lemmas 1 and 3 we supplement Theorems 1 and 2 with two statements which give estimates of $M(r,f)$ that are sometimes better than (5) and (6).

Theorem 3. *Let $l \in K \cap Q$, and entire functions $g_0, g_1, \dots, g_n$ and h satisfy the conditions a), b) and c) of Theorem 1. If an entire function f is a solution of (2) then it is of bounded l-index and*

$$\overline{\lim_{r\to+\infty}} \frac{\ln M(r,f)}{L(r)} \le \max\{n+1, 2M^*(M+2)\}, \quad M^* = \max_j\{1, m_j, M_j\}.$$

Proof. As in the proof of Theorem 1 we have

$$|f^{(n+1)}(z)| \le \frac{1}{|g_0(z)|}\left(Ml(|z|)\sum_{j=0}^{n}|g_j(z)||f^{(n-j)}(z)|+\right.$$
$$\left.+\sum_{j=0}^{n}|g_j'(z)||f^{(n-j)}(z)|+\sum_{j=1}^{n}|g_j(z)||f^{(n-j+1)}(z)|\right)\le$$
$$\le M^*\left((M+1)\sum_{j=0}^{n}l^{j+1}(|z|)|f^{(n-j)}(z)|+\sum_{j=1}^{n}l^{j}(|z|)|f^{(n-j+1)}(z)|\right)\le$$
$$\le M^*(M+2)\sum_{j=0}^{n}l^{j+1}(|z|)|f^{(n-j)}(z)|,$$

whence

$$\frac{|f^{(n+1)}(z)|}{(n+1)!l^{n+1}(|z|)}\le M^*(M+2)\sum_{j=0}^{n}\frac{|f^{(n-j)}(z)|}{(n-j)!l^{n-j}(|z|)}\frac{(n-j)!}{(n+1)!}\le$$
$$\le\frac{2M^*(M+2)}{n+1}\max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)}:0\le k\le n\right\},$$

because $\sum_{j=0}^{n}\frac{(n-j)!}{(n+1)!}\le\frac{2}{n+1}$. Therefore, by Lemma 1 the function f is of bounded l-index and by Lemma 3 the required estimate holds. □

Theorem 4. *Let $l\in K\cap Q$, and entire functions $g_0, g_1,\dots,g_n$ satisfy for all z, $|z|\ge R$, the conditions of Theorem 2. If an entire function f is a solution of (2) with $h\equiv 0$ then it is of bounded l-index and*

$$\overline{\lim_{r\to+\infty}}\frac{\ln M(r,f)}{L(r)}\le\max\{n,2M^*\},\quad M^*=\max_j\{1,\,m_j\}.$$

The proof of Theorem 4 is analogous to that of Theorems 2 and 3.

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