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SOME PROBLEMS ON GROUPS OF FINITELY AUTOMATIC PERMUTATIONS

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Every automaton over a finite alphabet X determines a transformation of both the free monoid generated by X and the countable power X^{ω} (automatic permutation). Some problems concerning the group of all automatic permutations are formulated.

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Каждый автомат над конечным алфавитом X определяет преобразование свободного моноида над X и счетной степени X^{ω} (автоматная подстановка). Формулируются некоторые проблемы о группах автоматных подстановок множества.

1. *An initial Mealy automaton* over a finite alphabet X ($|X| > 1$) is a quadruple $A = \langle Q, q_0, \phi, \lambda\rangle$, where Q is a set of *internal states* of the automaton, $q_0 \in Q$ is its *initial state*, $\phi : Q \times X \to Q$ and $\lambda : Q \times X \to X$ are *the transition* and *output* functions respectively. The automaton A is finite if the set Q is finite and it is called *permutational* if for every $q \in Q$ the map $x \to \lambda(q, x)$ is a permutation of the alphabet X.

For two automata $A_1 = \langle Q_1, q_1, \phi_1, \lambda_1\rangle, A_2 = \langle Q_2, q_2, \phi_2, \lambda_2\rangle$ we define their *composition* A_1A_2 as the automaton $B = \langle Q, q, \phi, \lambda\rangle$, where

$$Q = Q_1 \times Q_2, q = (q_1, q_2),$$
$$\phi\left((s_1, s_2), x\right) = \left(\phi_1\left(s_1, x\right), \phi_2\left(s_2, \lambda_1\left(s_1, x\right)\right)\right),$$
$$\lambda\left((s_1, s_2), x\right) = \lambda_2(s_2, \lambda_1\left(s_1, x\right)).$$

If the automaton A_1 is permutational then its *inverse* automaton is the automaton $A_1^{-1} = \langle Q_1, q_1, \phi_1, \lambda_1^{-1}\rangle$, where $\lambda_1^{-1}\left(s, \lambda_1(s, x)\right) = x$ for every $x \in X$ and $s \in Q_1$.

We extend (see [1]) the functions ϕ and λ onto the set $Q \times X^*$, where X^* is the free monoid generated by X (the set of finite words over the alphabet X) by the following recursive rules:

$$\phi(q, 1) = q, \qquad \phi(q, wx) = \phi\left(\phi\left(q, w\right), x\right),$$
$$\lambda(q, 1) = 1, \qquad \lambda(q, wx) = \lambda\left(\phi\left(q, w\right), x\right),$$

where $x \in X$, $w \in X^*$ and $q \in Q$ are arbitrary elements.

An automaton A is *accessible* if for any state $q \in Q$ there exists a word $w \in X^*$ such that $\phi(q_0, w) = q$.

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Thus every automaton A defines a transformation π_A of the set X^* or of the set X^ω of all infinite sequences over the alphabet X (ω-words) by the equalities:

$$\pi_A(x_1x_2\ldots) = y_1y_2\ldots, \quad y_k = \lambda(q_0, x_1x_2\ldots x_k),$$

where $x_1x_2\ldots$ is a finite or infinite word. Every transformation defined by a (finite) automaton can be defined by a (finite) accessible automaton. A transformation π_A defined by an accessible automaton A, is a permutation if and only if the automaton A is permutational. Permutations defined by automata are called automatic.

The sets of all automatic permutations of the set X^* and of the set X^ω are groups. These groups are isomorphic as abstract groups but have different permutational properties. For instance, the group of automatic permutations of X^ω is transitive, while the group of automatic permutations of X^* is not. The group of all automatic permutations over the alphabet X is isomorphic to the automorphism group of a rooted n-adic tree, where $n = |X|$. It is also isomorphic to the full isometry group of a special ultrametric space [1, 2]. This makes possible to use combinatorial, geometrical and topological methods to study this group (see [2, 4]). In particular, one can introduce a natural complete ultrametric on the group of all automatic permutations (see [2]).

An automatic permutation is said to be *finitely automatic* if it can be defined by a finite automaton. Product of two automatic permutations is defined by their composition and since composition and inverses of two finite automata are again finite automata, the set of all finitely automatic permutations over the alphabet X is a group. This group will be denoted $FGA(X)$. The group $FGA(X)$ was introduced for the first time in [3].

2. The properties of the group $FGA(X)$, in the contrast to the group of all automatic permutations, are known very little. The main known facts about them are the following.

The group $FGA(X)$ is residually finite as a subgroup of the profinite group of all automatic permutations over the alphabet X.

Let π be an automatic permutation. If there exists a number n such that for every word $w = x_1x_2\ldots$ and its image $\pi(w) = y_1y_2\ldots$ we have $y_k = x_k$ for all $k > n$, then π is finitely automatic. Such permutations are called *finitary* and they form a dense subgroup of the group of all automatic permutations (and thus it is also dense in $FGA(X)$). The subgroup of all finitary automatic permutations was introduced in [5, 6]. In fact, this is the most simple subgroup of $FGA(X)$ and it can be studied by group theoretic methods. It was extensively studied in [7], where it is called *the base group*.

The group $FGA(X)$ can be constructed of symmetric groups $\mathrm{Symm}(X)$ using the general construction of *up*-wreath power of a permutation group, introduced in [8].

An infinite word $w \in X^\omega$ is said to be ultimately periodic if there exist finite words $u, v \in X^*$ such that $w = uvvv\ldots$. The set of all ultimately periodic words from X^ω is denoted by X^{up}. The set X^{up} is an orbit of the group $FGA(X)$ under its action on X^ω. The action of $FGA(X)$ on X^{up} is faithful [8].

The permutation group $(FGA(X), X^{up})$ is imprimitive. The imprimitivity blocks are intersections of the imprimitivity blocks of the whole group of automatic permutations with X^{up} [8].

A finite group G can be embedded into $FGA(X)$ if and only if it has a subnormal series $G = G_0 > G_1 > \ldots > G_s = \{1\}$ with all quotients embeddable into $\mathrm{Symm}(X)$ [9].

Any finitely generated subgroup of $FGA(X)$ has a solvable word problem.

The group $FGA(X)$ contains free subgroups of any finite rank for all $X, |X| > 1$ (see [10, 11]). For $|X| \geq 2^n$ the group $\mathrm{GL}(n, \mathbb{Z})$ is a subgroup of $FGA(X)$ [11].

The group $FGA(X)$ contains subgroups of Burnside type, i.e., infinite finitely generated torsion groups. Examples of such subgroups constructed using different languages (tree automorphisms, permutations of a segment, wreath products) can be found in [12]–[15].

For more special properties of the group of finitely automatic permutations see [16, 17].

3. Besides the properties, formulated in section **2**, the group $FGA(X)$ can be used in construction of different extremal examples of groups, in symbolic dynamics, ergodic theory etc. These investigations proved that more detailed study of the structure and properties of $FGA(X)$ is necessary. In this direction we emphasize the following problems.

3.1. Systems of generators of $FGA(X)$. We still do not know any not very excessive system of generators of $FGA(X)$. In particular, the following problem stated in [6] is still open.

Does there exist an irreducible generating system of $FGA(X)$?

3.2. Conjugacy classes in the group of all automatic permutations over an alphabet X (in terms of rooted trees) is described in [18, 19]. But similar description is not valid for $FGA(X)$ and no suitable modification is known. In connection with this the following questions arise.

(i) *Is the conjugacy problem algorithmically solvable in $FGA(X)$?*

(ii) *Describe the conjugacy classes of $FGA(X)$.*

(iii) *Investigate conjugacy problem for finitely generated subgroups of $FGA(X)$. Is it always solvable?*

3.3. Automorphism groups of different groups of automatic permutations where studied by S. Sidki [20], A. Brunner, S. Sidki [11] and Y. Lavreniuk [21]. Accoding to results of Y. Lavreniuk, the automorphism group of $FGA(X)$ coincides with the normalizer of $FGA(X)$ in the group of all automatic permutations over the alphabet X. *We conjecture that $FGA(X)$ is complete.*

3.4. Verbal subgroups of the group of finitary automatic permutations over an alphabet with two letters where studied by N.Smetaniuk and the second author [22]. They proved that every such a verbal subgroup is a member of the lower central series. The members of the lower central series in this case are clearly described by the polynomial representation introduced by L.Kaloujnine [23, 24]. But this description can not be applied directly for the group $FGA(X)$. Therefore the following questions arise.

(i) *Describe the members of the lower central series of the group $FGA(X)$, $|X| = 2$. In particular, how can the derived subgroup and abelianization be described.*

(ii) *Describe all verbal subgroups of $FGA(X)$. Are they all members of the lower central series?*

3.5. Embeddings of linear and free groups into $FGA(X)$ were studied by several authors (see [10, 11, 12]), but still for many groups it is not known if they are isomorphic to subgroups of $FGA(X)$. *In particular, this concerns the automorphism group of a free group, the braid groups, the Nottingham group and the Golod-Shafarevich torsion groups.*

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