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PROPERTIES OF INTEGRALS WHICH HAVE THE TYPE OF DERIVATIVES OF VOLUME POTENTIALS FOR PARABOLIC SYSTEMS WITH DEGENERATION ON THE INITIAL HYPERPLANE

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Lemmas on the properties of integrals which have type of derivatives of volume potentials generated by fundamental matrix of solutions of the Cauchy problem for parabolic systems with degeneration on the initial hyperplane are proved.

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Доказаны леммы о свойствах интегралов типа производных от объемных потенциалов, порожденных фундаментальной матрицей решений задачи Коши для параболических систем с вырождением на начальной гиперплоскости.

1. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, $\mathbb{R} \equiv \mathbb{R}^1$, $\mathbb{R}_+ \equiv (0, \infty)$, $\overline{\mathbb{R}}_+ \equiv [0, \infty)$, $\Pi_H \equiv \{(t, x) \,|\, t \in H,\ x \in \mathbb{R}^n\}$, $H \subset \overline{\mathbb{R}}_+$, and let n, b, N be given positive integer numbers, T is a positive number, continuous functions $\alpha\colon [0, T] \to \overline{\mathbb{R}}_+$ and $\beta\colon [0, T] \to \overline{\mathbb{R}}_+$ such that $\alpha(0)\beta(0) = 0$ and $\alpha(t) > 0$, $\beta(t) > 0$ for $t > 0$.

The paper is concerned with the study of properties of integrals of the type

$$u(t, x) \equiv \int_0^t \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} M(t, \tau, x - \xi) f(\tau, \xi) d\xi, \quad (t, x) \in \Pi_{(0,T]}. \tag{1}$$

The kernel M has properties of the derivatives of the fundamental matrix for solutions Z of the Cauchy problem for parabolic by Petrovsky systems of N equations with degeneration on the initial hyperplane of the form

$$\alpha(t)\partial_t u(t, x) = \beta(t) \sum_{1 \le |k| \le 2b} a_k(t)\partial_x^k u(t, x) + a_0(t)u(t, x) + f(t, x), \quad (t, x) \in \Pi_{(0,T]}. \tag{2}$$

The structure of the matrix Z, estimates of derivatives of Z and its other properties are given in [1, 2].

Properties of integral (1) are described by that the function u belongs to the appropriate functional spaces, which depend on the spaces to which the function f belongs.

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It should be noted that integral (1) may be singular, the convergence of which is non-absolute and essentially it is based on the fact that the average of the kernel M is equal to zero (see condition A_1).

The results of the paper can be considered as a generalization of the corresponding lemmas from [3] for the case of parabolic systems with degeneration on the initial hyperplane. They will be applied to study properties of volume potentials, to obtain a priori estimates of the solutions and to establish the well-posedness of the Cauchy problem and of problems with weight initial conditions or without initial conditions depending on a type of the degeneration [4] for systems (2), in which the coefficients $a_k, |k| \le 2b$, depend on all independent variables. Applications of the results obtained here will be given in the forthcoming publications.

2. Besides the functions α and β, also we shall use the continuous function $\delta\colon [0,T] \to \overline{\mathbb{R}}_+$ such that

$$\forall\, t \in (0,T]:\; 0 < \delta(t) \le \beta(t), \qquad \int_0^T \frac{\delta(\theta)}{\alpha(\theta)} d\theta < \infty. \tag{3}$$

We assume that the functions β and δ are monotone and nondecreasing.

We denote by $\mathbb{C}_{NN}$ and $\mathbb{C}_N$ respectively the sets of all square N-dimensional matrices $M = (M_{ij})_{i,j=1}^N$ and all columns $f = (f_j)_{j=1}^N$ of height N, elements of which are complex numbers; $|M| \equiv \max_{1\le i\le N} \sum_{j=1}^N |M_{ij}|$ and $|f| \equiv \max_{1\le j\le N} |f_j|$, if $M \in \mathbb{C}_{NN}$ and $f \in \mathbb{C}_N$; $A(t,\tau) \equiv \int_\tau^t \frac{d\theta}{\alpha(\theta)}$, $B(t,\tau) \equiv \int_\tau^t \frac{\beta(\theta)}{\alpha(\theta)} d\theta$, $\Delta(t,\tau) \equiv \displaystyle\int_\tau^t \frac{\delta(\theta)}{\alpha(\theta)} d\theta$, $E^d(t,\tau) \equiv \exp\{dA(t,\tau)\}$, $E_c(t,\tau,x) \equiv \exp\{-c(B(t,\tau))^{1-q}|x|^q\}$, $0 < \tau < t \le T$, $x \in \mathbb{R}^n$, $d \in \mathbb{R}$, $c \in \mathbb{R}_+$, $q \equiv 2b(2b-1)^{-1}$; $\Delta_t^{t'} f(t,x) \equiv f(t,x) - f(t',x)$, $\Delta_x^{x'} f(t,x) \equiv f(t,x) - f(t,x')$, $\Delta_{t,x}^{t',x'} f \equiv f(t,x) - f(t',x')$; $t \le t'$; η is the Heaviside function.

We define the parabolic distance between points (t,x) and (t',x') by the formula

$$p(t,x;t',x') \equiv ((A(t',t))^{1/b} + |x - x'|^2)^{1/2}$$

and the Hölder conditions with respect to t, x we shall assume to be concerned with this distance.

3. Let us describe properties of the kernel M of integral (1). As the kernel of this integral, let us take the functions $M(t,\tau,x), 0 < \tau < t \le T$, $x \in \mathbb{R}^n$, with values in $\mathbb{C}_{NN}$, which can be represented in the form

$$M(t,\tau,x) \equiv (B(t,\tau))^{-\nu-n/(2b)} \Omega(t,\tau,(B(t,\tau))^{-1/(2b)}x), \quad 0 < \tau < t \le T, \quad x \in \mathbb{R}^n, \tag{4}$$

where $\nu \in (0,1]$, and the function $\Omega(t,\tau,x)$, $0 < \tau < t \le T$, $x \in \mathbb{R}^n$, with the values in $\mathbb{C}_{NN}$, is continuous and it satisfies the conditions below with some numbers $c > 0$, $d \in \mathbb{R}$ and $\gamma \in (0,1]$:

A_1.

$$\int_{\mathbb{R}^n} \Omega(t,\tau,x) dx = 0, \quad 0 < \tau < t \le T; \tag{5}$$

A_2. $\exists\, C > 0 \quad \forall \{t,\tau\} \subset (0,T]$, $\tau < t$, $\forall\, x \in \mathbb{R}^n$:

$$|\Omega(t,\tau,x)| \le C E^d(t,\tau) \exp\{-c|x|^q\}; \tag{6}$$

A_3. $\exists C > 0 \quad \forall \{t, \tau\} \subset (0, T]$, $\tau < t$, $\forall \{x, x'\} \subset \mathbb{R}^n$:

$$|\Delta_x^{x'} \Omega(t, \tau, x)| \leq C E^d(t, \tau)|x - x'|^\gamma (\exp\{-c|x|^q\} + \exp\{-c|x'|^q\}); \tag{7}$$

A_4. $\exists C > 0 \quad \forall \{t, t', \tau\} \subset (0, T]$, $\tau < t < t'$, $\forall x \in \mathbb{R}^n$:

$$|\Delta_t^{t'} \Omega(t, \tau, x)| \leq C(A(t', t))^{\gamma/(2b)} (B(t, \tau))^{-\gamma/(2b)} E^d(\tilde{t}, \tau) \exp\{-c|x|^q\}, \tag{8}$$

where $\tilde{t} \equiv t + (t' - t)\eta(d)$.

So, the definition of the function M contains the number ν, d, c and γ, which assume are considered to be given. By $\mathfrak{M}(\nu, d, c, \gamma)$ we denote a set of all functions M determined by formula (4), in which the function Ω satisfies conditions $A_1 - A_4$ with given $\{\nu, \gamma\} \subset (0, 1]$, $d \in \mathbb{R}$ and $c \in \mathbb{R}_+$.

It should be noted that for $\nu = 1$ integral (1) with the function $M \in \mathfrak{M}(\nu, d, c, \gamma)$ is treated as the limit

$$\lim_{h \to 0} \int_0^{t-h} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}_n} M(t, \tau, x - \xi) f(\tau, \xi) d\xi,$$

which exists for suitable f, because of condition A_1.

4. Let us define spaces to which the functions f and u belong. They are the spaces of functions which are continuous or satisfy Hölder condition and which have certain restrictions as $t \to 0$ and $|x| \to \infty$. Their behaviour as $t \to 0$ will be described by the function

$$(\delta(t))^\mu (\Delta(t, 0))^r E^{-d}(T, t), \quad t \in (0, T],$$

and as $|x| \to \infty$ by the function

$$\Psi(t, x) \equiv \exp\{k(t)|x|^q\}, \quad t \in [0, T], \ x \in \mathbb{R}^n,$$

where $\mu = 0, 1$; $r \in \mathbb{R}$; $k(t) \equiv c_0 a (c_0^{2b-1} - (T - B(T, t)) a^{2b-1})^{1-q}$, $t \in [0, T]$, (and $k(0) \equiv 0$, if $B(T, 0) = \infty$), c_0 is a fixed number from the interval $(0, c)$, c is the constant from conditions A_2–A_4, and the number a such that $0 \leq a < c_0 T^{1-q}$.

The function k monotonically increases from $k(0)$ to $k(T)$ and it has the following property [2]:

$$\forall \{x, \xi\} \subset \mathbb{R}^n \quad \forall \{t, \tau\} \subset (0, T], \quad \tau < t:$$
$$-c_0(B(t, \tau))^{1-q}|x - \xi|^q + k(\tau)|\xi|^q \leq k(t)|x|^q,$$

with the help of which the inequality

$$E_{c_0}(t, \tau, x - \xi)\Psi(\tau, \xi) \leq \Psi(t, x), \quad 0 < \tau < t \leq T, \quad \{x, \xi\} \subset \mathbb{R}^n, \tag{9}$$

is proved.

For given numbers $\lambda \in (0, 1]$, $\mu \in \{0, 1\}$ and $r \in \mathbb{R}$ we denote by $C_{\mu,r}^{\lambda,\lambda/(2b)}$, $C_{\mu,r}^{\lambda,0}$ and $C_{\mu,r}^{0,0}$ the spaces of continuous functions $u \colon \Pi_{(0,T]} \to \mathbb{C}_N$, for which the corresponding norms

$$\|u\|_{\mu,r}^{\lambda,\lambda/(2b)} \equiv \|u\|_{\mu,r}^{0,0} + [u]_{\mu,r}^{\lambda,\lambda/(2b)}, \qquad \|u\|_{\mu,r}^{\lambda,0} \equiv \|u\|_{\mu,r}^{0,0} + [u]_{\mu,r}^{\lambda,0}$$

and $\|u\|_{\mu,r}^{0,0}$, where

$$\|u\|_{\mu,r}^{0,0} \equiv \sup_{(t,x) \in \Pi_{(0,T]}} \frac{|u(t, x)| E^d(T, t)}{\Psi(t, x)(\delta(t))^\mu (\Delta(t, 0))^r},$$

$$[u]^{\lambda,\lambda/(2b)}_{\mu,r} \equiv \sup_{\substack{\{(t,x),(t',x')\}\subset\Pi_{(0,T]}\\ (t,x)\neq(t',x')}} \left(\frac{|\Delta^{t',x'}_{t,x} u|(p(t,x;t',x'))^{-\lambda}}{(\delta(t))^{\mu}(\Delta(\bar{t},0))^{r-\lambda/(2b)}E^{-d}(T,\tilde{t})}(\Psi(t,x)+\Psi(t',x'))^{-1}\right),$$

$$[u]^{\lambda,0}_{\mu,r} \equiv \sup_{\substack{\{x,x'\}\subset\mathbb{R}^n\\ x\neq x'}} \frac{|\Delta^{x'}_{x} u(t,x)|(\Psi(t,x)+\Psi(t,x'))^{-1}}{|x-x'|^{\lambda}(\delta(t))^{\mu}(\Delta(t,0))^{r-\lambda/(2b)}E^{-d}(T,t)},$$

are finite. Here $\bar{t} \equiv t + (t'-t)\eta(r-\lambda/(2b))$ and $\tilde{t} \equiv t + (t'-t)\eta(d)$.

For given numbers $\lambda \in (0,1]$, $\mu \in \{0,1\}$, $r \geq 0$ and $p \geq 0$, introduce the spaces $C^0_{\mu,r,p}$ and $C^\lambda_{\mu,r,p}$ of continuous function $u\colon \Pi_{(0,T]} \to \mathbb{C}_N$, for which the following values are finite: for $C^0_{\mu,r,p}$ $W_0[u;t]$ for all $t \in (0,T]$ and $\|u\|^0_{\mu,r,p}$, and for $C^\lambda_{\mu,r,p}$ $W_0[u;t]$ and $W_\lambda[u;t]$ for all $t \in (0,T]$ and $\|u\|^\lambda_{\mu,r,p} \equiv \|u\|^0_{\mu,r,p} + [u]^\lambda_{\mu,r,p}$, where

$$W_0[u;t] \equiv \sup_{x\in\mathbb{R}^n} \frac{|u(t,x)|}{\Psi(t,x)}, \quad W_\lambda[u;t] \equiv \sup_{\substack{\{x,x'\}\subset\mathbb{R}^n\\ x\neq x'}} \frac{|\Delta^{x'}_x u(t,x)||x-x'|^{-\lambda}}{\Psi(t,x)+\Psi(t,x')},$$

$$\|u\|^0_{\mu,r,p} \equiv \int_0^T W_0[u;t](\delta(t))^{-\mu}(\Delta(t,0))^{-r}(B(T,t))^{p}E^{d}(T,t)\frac{dt}{\alpha(t)},$$

$$[u]^\lambda_{\mu,r,p} \equiv \int_0^T W_\lambda[u;t](\delta(t))^{-\mu}(\Delta(t,0))^{-r}(B(T,t))^{p+\lambda/(2b)}E^{d}(T,t)\frac{dt}{\alpha(t)}.$$

5. Let us formulate the main lemma of this paper.

Lemma 1. *Let $M \in \mathfrak{M}(\nu,d,c,\gamma)$ and function u is determined by formula (1). Then the following statements are valid:*

a) if $\nu+\gamma/(2b) < 1$ and $f \in C^{0,0}_{1,r}$ (when $r \leq \nu+\gamma/(2b)-1$ we assume, in addition, that $F_0 \equiv \int_0^T W_0[f;\tau](\Delta(\tau,0))^{-\nu-\gamma/(2b)}E^d(T,\tau)\frac{d\tau}{\alpha(\tau)} < \infty$), than $u \in C^{\gamma,\gamma/(2b)}_{0,r-\nu+1}$ and the estimate

$$\|u\|^{\gamma,\gamma/(2b)}_{0,r-\nu+1} \leq C(\|f\|^{0,0}_{1,r} + \chi_r F_0), \tag{10}$$

is valid, where $\chi_r \equiv \eta(\nu+\gamma/(2b)-1-r)$;

b) if $f \in C^{\lambda,0}_{1,r}$, $\lambda \in (0,1]$, and the additional assumption from statement a) is fulfilled, then with $1-\nu-(\gamma-\lambda)/(2b) > 0$ we have $u \in C^{\gamma,\gamma/(2b)}_{0,r-\nu+1}$ and

$$\|u\|^{\gamma,\gamma/(2b)}_{0,r-\nu+1} \leq C(\|f\|^{\lambda,0}_{1,r} + \chi_r F_0), \tag{11}$$

and with $1-\nu-(\gamma-\lambda)/(2b) < 0$ we have $u \in C^{\lambda,\lambda/(2b)}_{0,r-\nu+1}$ and

$$\|u\|^{\lambda,\lambda/(2b)}_{0,r-\nu+1} \leq C(\|f\|^{\lambda,0}_{1,r} + \chi_r F_0). \tag{12}$$

The constants C in inequalities (10)–(12) depend only on the constants C from conditions A_2, A_3 and A_4, and also they depend on the numbers $n, b, r, \nu, d, c, \gamma, \lambda, \beta(T)$ and $\delta(T)$.

Proof. a) Using the notation

$$I^r_\nu(t_1,t_2,t_3) \equiv \int_{t_1}^{t_2} (B(t_3,\tau))^{-\nu}(\Delta(\tau,0))^{r}\frac{\delta(\tau)}{\alpha(\tau)}d\tau, \quad t_1 \in [0,T), \quad \{t_2,t_3\} \subset (0,T], \tag{13}$$

and the equality

$$\int_{\mathbb{R}^n} (B(t,\tau))^{-n/(2b)} E_{c-c_0}(t,\tau,x-\xi)d\xi = C, \quad 0<\tau<t\le T, \quad x\in\mathbb{R}^n, \tag{14}$$

with the help of (4), (6), (9) and of the definition of the norm $\|f\|^{0,0}_{1,r}$ with $r>-1$ we have

$$\begin{aligned} |u(t,x)| &\le C\int_0^t (B(t,\tau))^{-\nu-n/(2b)} E^d(t,\tau)\frac{d\tau}{\alpha(\tau)}\int_{\mathbb{R}^n} E_c(t,\tau,x-\xi)|f(\tau,\xi)|d\xi = \\ &= C\int_0^t (B(t,\tau))^{-\nu-n/(2b)} E^d(t,\tau)E^{-d}(T,\tau)(\Delta(\tau,0))^r\frac{\delta(\tau)}{\alpha(\tau)}d\tau\int_{\mathbb{R}^n} E_{c-c_0}(t,\tau,x-\xi)\times \\ &\quad\times(E_{c_0}(t,\tau,x-\xi)\Psi(\tau,\xi))\frac{|f(\tau,\xi)|E^d(T,\tau)}{\Psi(\tau,\xi)\delta(\tau)(\Delta(\tau,0))^r}d\xi \le \\ &\le CE^{-d}(T,t)\Psi(t,x)\|f\|^{0,0}_{1,r}I^r_\nu(0,t,t), \quad (t,x)\in\Pi_{(0,T]}. \end{aligned} \tag{15}$$

It should be noticed that for any $t\in(0,T]$ there exists a unique point $t_1\in(0,t)$ such that: $\Delta(t_1,0)=B(t,t_1)$ with $\tau<t_1$ $\Delta(\tau,0)<B(t,\tau)$ and with $\tau>t_1$ $\Delta(\tau,0)>B(t,\tau)$. Indeed, the function $\Delta(\cdot,0)$ monotonically increases, and $B(t,\cdot)$ monotonically decreases, besides, $\Delta(0,0)=B(t,t)=0$. We notice also that the inequalities

$$\Delta(t,0)/2\le\Delta(t_1,0)\le\Delta(t,0)\le\Delta(T,0), \quad t\in(0,T], \tag{16}$$

are valid. The two last inequalities are obvious, and the first one follows from the inequality

$$\Delta(t_1,0)=B(t,t_1)\ge\Delta(t,t_1)=\Delta(t,0)-\Delta(t_1,0),$$

which is true as a consequence of (3).

Let us estimate the integral $I^r_\nu(0,t,t)$ with $r>\nu-1$. We write it in a form of sum of the integrals $I^r_\nu(0,t_1,t)$ and $I^r_\nu(t_1,t,t)$. With the help of (3) and (16) we have

$$\begin{aligned} I^r_\nu(0,t_1,t) &\le \int_0^{t_1}(\Delta(\tau,0))^{r-\nu}d\Delta(\tau,0) = \frac{1}{r-\nu+1}(\Delta(t_1,0))^{r-\nu+1}\le \\ &\le \frac{1}{r-\nu+1}(\Delta(t,0))^{r-\nu+1}; \end{aligned} \tag{17}$$

$$\begin{aligned} I^r_\nu(t_1,t,t) &\le -\int_{t_1}^t (B(t,\tau))^{-\nu}dB(t,\tau)(\Delta(t,0))^r = \frac{1}{1-\nu}(B(t,t_1))^{1-\nu}(\Delta(t,0))^r\le \\ &\le \frac{1}{1-\nu}(\Delta(t,0))^{r-\nu+1}, \quad \text{if} \quad r\ge 0; \end{aligned} \tag{18}$$

$$\begin{aligned} I^r_\nu(t_1,t,t) &\le -\int_{t_1}^t (B(t,\tau))^{-\nu+r}dB(t,\tau) = \frac{1}{r-\nu+1}(B(t,t_1))^{r-\nu+1}\le \\ &\le \frac{1}{r-\nu+1}(\Delta(t,0))^{r-\nu+1}, \quad \text{if} \quad r<0. \end{aligned} \tag{19}$$

So,

$$I^r_\nu(0,t,t)\le C(\Delta(t,0))^{r-\nu+1}, \quad r>\nu-1, \tag{20}$$

and by (15) we obtain the estimate

$$\|u\|^{0,0}_{0,r-\nu+1} \le C\|f\|^{0,0}_{1,r}. \tag{21}$$

Similarly, if $r \le \nu - 1$, then

$$|u(t,x)| \le CE^{-d}(T,t)\Psi(t,x)\Big((\Delta(t,0))^{r-\nu+1}(\Delta(T,0))^{-r+\nu+\gamma/(2b)-1}\times$$
$$\times\int_0^{t_1} W_0[f;\tau](\Delta(\tau,0))^{-\nu-\gamma/(2b)}E^d(T,\tau)\frac{d\tau}{\alpha(\tau)} + \int_{t_1}^{t}(B(t,\tau))^{-\nu}\frac{\delta(\tau)}{\alpha(\tau)}d\tau(\Delta(t_1,0))^r\|f\|^{0,0}_{1,r}\Big) \le$$
$$\le CE^{-d}(T,t)\Psi(t,x)(\Delta(t,0))^{r-\nu+1}(F_0 + \|f\|^{0,0}_{1,r}), \quad (t,x)\in\Pi_{(0,T]},$$

and hence, with $r \le \nu - 1$ the estimate

$$\|u\|^{0,0}_{0,r-\nu+1} \le C(F_0 + \|f\|^{0,0}_{1,r}) \tag{22}$$

holds.

Estimates (21) and (22) join in the following estimate:

$$\|u\|^{0,0}_{0,r-\nu+1} \le C(\|f\|^{0,0}_{1,r} + \chi'_r F_0), \tag{23}$$

where $\chi'_r \equiv \eta(\nu - 1 - r)$.

Let $(t,x),(t',x')$ be arbitrary fixed points of the layer $\Pi_{(0,T]}$, $t \le t'$, and let $p \equiv p(t,x;t',x')$. Let us estimate the difference $\Delta^{t',x'}_{t,x}u$.

When $p^{2b} > \Delta(t,0)$, with the help of estimate (23) we obtain

$$|\Delta^{t',x'}_{t,x}u| \le |u(t,x)| + |u(t',x')| \le C(\|f\|^{0,0}_{1,r} + \chi'_r F_0)((\Delta(t,0))^{r-\nu+1}E^{-d}(T,t)\Psi(t,x)+$$
$$+(\Delta(t',0))^{r-\nu+1}E^{-d}(T,t)\Psi(t',x')) \le$$
$$\le C(\|f\|^{0,0}_{1,r} + \chi'_r F_0)p^{\gamma}(\Delta(\bar t,0))^{r-\nu+1-\gamma/(2b)}E^{-d}(T,\tilde t)(\Psi(t,x)+\Psi(t',x')), \tag{24}$$

where $\bar t \equiv t + (t'-t)\eta(r-\nu+1-\gamma/(2b))$ and $\tilde t \equiv t + (t'-t)\eta(d)$.

Indeed, since because of monotonicity of δ and the inequality $A(t',t) \le p^{2b}$

$$\Delta(t',0) = \Delta(t,0) + \Delta(t',t) \le p^{2b} + \delta(T)A(t',t) \le (1+\delta(T))p^{2b},$$

then with $r - \nu + 1 - \gamma/(2b) \le 0$ we have

$$(\Delta(t,0))^{r-\nu+1} \le (\Delta(t,0))^{r-\nu+1-\gamma/(2b)}p^{\gamma},$$

$$(\Delta(t',0))^{r-\nu+1} \le (\Delta(t,0))^{r-\nu+1-\gamma/(2b)}(\Delta(t',0))^{\gamma/(2b)} \le C(\Delta(t,0))^{r-\nu+1-\gamma/(2b)}p^{\gamma},$$

and with $r - \nu + 1 - \gamma/(2b) > 0$

$$(\Delta(t,0))^{r-\nu+1} \le (\Delta(t,0))^{r-\nu+1-\gamma/(2b)}p^{\gamma} \le (\Delta(t',0))^{r-\nu+1-\gamma/(2b)}p^{\gamma},$$

$$(\Delta(t',0))^{r-\nu+1} = (\Delta(t',0))^{r-\nu+1-\gamma/(2b)}(\Delta(t',0))^{\gamma/(2b)} \le C(\Delta(t',0))^{r-\nu+1-\gamma/(2b)}p^{\gamma}.$$

Let us consider the case $p^{2b} < \Delta(t,0)$. We have

$$|\Delta^{t',x'}_{t,x}u| \le \int_0^t \frac{d\tau}{\alpha(\tau)}\int_{\mathbb{R}^n}|\Delta^{t',x'}_{t,x}M(t,\tau,x-\xi)||f(\tau,\xi)|d\xi+$$

$$+\int_t^{t'} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} |M(t',\tau,x'-\xi)||f(\tau,\xi)|d\xi \equiv J_1 + J_2. \tag{25}$$

Using (4), (6), (9), (13) and (14), we get

$$J_2 \le C \int_t^{t'} (B(t',\tau))^{-\nu-n/(2b)} E^d(t',\tau) E^{-d}(T,\tau)(\Delta(\tau,0))^r \frac{\delta(\tau)}{\alpha(\tau)} d\tau \times$$

$$\times \int_{\mathbb{R}^n} E_{c-c_0}(t',\tau,x'-\xi)(E_{c_0}(t',\tau,x'-\xi)\Psi(\tau,\xi)) \frac{|f(\tau,\xi)|E^d(T,\tau)d\xi}{\Psi(\tau,\xi)\delta(\tau)(\Delta(\tau,0))^r} \le$$

$$\le CE^{-d}(T,t')\Psi(t',x')\|f\|_{1,r}^{0,0} I_\nu^r(t,t',t'). \tag{26}$$

To estimate $I_\nu^r(t,t',t')$, we use the monotonicity of the functions β and δ and the inequalities $A(t',t) \le p^{2b} < \Delta(t,0)$. We have

$$I_\nu^r(t,t',t') \le \frac{1}{1-\nu}(\Delta(\hat{t},0))^r (B(t',t))^{1-\nu} \le C(\Delta(\hat{t},0))^r (A(t',t))^{1-\nu},$$

where $C \equiv (\beta(T))^{1-\nu}/(1-\nu)$, $\hat{t} \equiv t + (t'-t)\eta(r)$. With $r < 0$

$$I_\nu^r(t,t',t') \le C(\Delta(t,0))^r p^{2b(1-\nu-\gamma/(2b))} p^\gamma \le C(\Delta(t,0))^{r-\nu+1-\gamma/(2b)} p^\gamma, \tag{27}$$

and with $r \ge 0$

$$I_\nu^r(t,t',t') \le C((\Delta(t,0))^r + (\Delta(t',t))^r)(A(t',t))^{1-\nu} \le$$

$$\le C(\Delta(t,0))^{r-\nu+1-\gamma/(2b)} p^\gamma + C(\delta(T))^r (A(t',t))^{r-\nu+1} \le C(\Delta(t,0))^{r-\nu+1-\gamma/(2b)} p^\gamma. \tag{28}$$

From (26)–(28) it follows the estimate

$$J_2 \le C\|f\|_{1,r}^{0,0} p^\gamma (\Delta(t,0))^{r-\nu+1-\gamma/(2b)} E^{-d}(T,t')\Psi(t',x'). \tag{29}$$

In order to estimate J_1, at first, let us prove for the difference $\Delta M \equiv \Delta_{t,x}^{t',x'} M(t,\tau,x-\xi)$ the inequality

$$|\Delta M| \le Cp^\gamma (B(t,\tau))^{-\gamma/(2b)} (B(t,\tau))^{-\nu-n/(2b)} E_{c_1}(t,\tau,x-\xi) +$$

$$+(B(t',\tau))^{-\nu-n/(2b)} E_{c_1}(t',\tau,x'-\xi)) E^d(\tilde{t},\tau), \tag{30}$$

where $c_0 < c_1 < c$, $\tilde{t} \equiv t + (t'-t)\eta(d)$.

We shall distinguish the following cases: 1) $p^{2b} \ge B(t,\tau)$, 2) $p^{2b} < B(t,\tau)$. In the first case, we obtain estimate (30) immediately from (4), (6) and from the inequality $|\Delta M| \le |M(t,\tau,x-\xi)| + |M(t',\tau,x'-\xi)|$. In case 2) note that

$$\Delta M = \Delta_t^{t'} (B(t,\tau))^{-\nu-n/(2b)} \Omega(t,\tau,(B(t,\tau))^{-1/(2b)}(x-\xi)) +$$

$$+(B(t',\tau))^{-\nu-n/(2b)} \Delta_t^{t'} \Omega(t,\tau,y)|_{y=(B(t,\tau))^{-1/(2b)}(x-\xi)} + (B(t',\tau))^{-\nu-n/(2b)} \Delta'\Omega, \tag{31}$$

where

$$\Delta'\Omega \equiv \Omega(t',\tau,(B(t,\tau))^{-1/(2b)}(x-\xi)) - \Omega(t',\tau,(B(t',\tau))^{-1/(2b)}(x'-\xi)).$$

Consider the function

$$\varphi(\theta) \equiv (B(t,\tau))^{-\nu-n/(2b)} - (B(t,\tau) + \theta B(t',t))^{-\nu-n/(2b)}, \quad \theta \in [0,1].$$

Since

$$|\Delta_t^{t'}(B(t,\tau))^{-\nu-n/(2b)}| = |\varphi(1)| = \left|\int_0^1 \varphi'(\theta)d\theta\right| \le \max_{\theta\in[0,1]} |\varphi'(\theta)|,$$

we have

$$|\Delta_t^{t'}(B(t,\tau))^{-\nu-n/(2b)}| \le CB(t',t)(B(t,\tau))^{-\nu-1-n/(2b)}. \tag{32}$$

Let us estimate $\Delta'\Omega$. Because of (7) we have

$$|\Delta'\Omega|(E_c(t,\tau,x-\xi)+E_c(t',\tau,x'-\xi))^{-1}E^{-d}(t',\tau) \le$$
$$\le C|(B(t,\tau))^{-1/(2b)}(x-\xi)-(B(t',\tau))^{-1/(2b)}(x'-\xi)|^{\gamma} \le$$
$$\le C\Big((B(t,\tau))^{-1/(2b)}|x-x'|+((B(t,\tau))^{-1/(2b)}-(B(t',\tau))^{-1/(2b)})|x'-\xi|\Big)^{\gamma} \le$$
$$\le C\bigg((B(t,\tau))^{-\gamma/(2b)}p^{\gamma}+\Big((B(t',\tau))^{-1/(2b)}|x'-\xi|\Big)^{\gamma}(B(t,\tau))^{-\gamma/(2b)}|\Delta_t^{t'}(B(t,\tau))^{1/(2b)}|^{\gamma}\bigg). \tag{33}$$

Since

$$|\Delta_t^{t'}(B(t,\tau))^{1/(2b)}| = \frac{B(t',t)}{2b}\int_0^1 (B(t,\tau)+\theta B(t',t))^{1/(2b)-1}d\theta \le$$
$$\le \frac{1}{2b}(B(t',t))^{1/(2b)}\int_0^1 \theta^{1/(2b)-1}d\theta = (B(t',t))^{1/(2b)} \le (\beta(T)A(t',t))^{1/(2b)} \le (\beta(T))^{1/(2b)}p,$$

then from (33) it follows the estimate

$$|\Delta'\Omega| \le Cp^{\gamma}(B(t,\tau))^{-\gamma/(2b)}(1+(B(t',\tau))^{-\gamma/(2b)}|x'-\xi|^{\gamma})(E_c(t,\tau,x-\xi)+$$
$$+E_c(t',\tau,x'-\xi))E^d(t',\tau).$$

Having used the inequality

$$|z|^l\exp\{-c|z|^q\} \le C\exp\{-c_1|z|^q\}, \quad z\in\mathbb{R}^n,\ l>0,\ c_0<c_1<c, \tag{34}$$

we shall arrive at the estimate

$$|\Delta'\Omega| \le Cp^{\gamma}(B(t,\tau))^{-\gamma/(2b)}(E_{c_1}(t,\tau,x-\xi)+E_{c_1}(t',\tau,x'-\xi))E^d(t',\tau). \tag{35}$$

From (6), (8), (31)–(33) and (35) estimate (30) follows in case 2).

With the help of (9), (13), (14), (20) and (30) with $r>\nu+\gamma/(2b)-1$ we get

$$J_1 \le C\|f\|_{1,r}^{0,0}p^{\gamma}E^{-d}(T,\tilde t)(\Psi(t,x)+\Psi(t',x'))I^r_{\nu+\gamma/(2b)}(0,t,t) \le$$
$$\le C\|f\|_{1,r}^{0,0}p^{\gamma}E^{-d}(T,\tilde t)(\Psi(t,x)+\Psi(t',x'))(\Delta(t,0))^{r-\nu-\gamma/(2b)+1}. \tag{36}$$

Similarly, when $r\le\nu+\gamma/(2b)-1$, using (16) we have

$$J_1 \le Cp^{\gamma}E^{-d}(T,\tilde t)(\Psi(t,x)+\Psi(t',x'))\bigg((\Delta(t,0))^{r-\nu-\gamma/(2b)+1}\times$$
$$\times(\Delta(T,0))^{-r+\nu+\gamma/(2b)-1}\int_0^{t_1} W_0[f;\tau](\Delta(\tau,0))^{-\nu-\gamma/(2b)}E^d(T,\tau)\frac{d\tau}{\alpha(\tau)}+$$
$$+\|f\|_{1,r}^{0,0}\int_{t_1}^{t}(B(t,\tau))^{-\nu-\gamma/(2b)}\frac{\delta(\tau)}{\alpha(\tau)}d\tau(\Delta(t_1,0))^r\bigg) \le$$
$$\le C(F_0+\|f\|_{1,r}^{0,0})p^{\gamma}E^d(T,\tilde t)(\Psi(t,x)+\Psi(t',x')). \tag{37}$$

From (24), (25), (29), (36) and (37) the estimate

$$[u]^{\gamma,\gamma/(2b)}_{0,r-\nu+1} \le C(\|f\|^{0,0}_{1,r} + \chi_r F_0),$$

follows and by this result and (23) estimate (10) holds.

b) Because of condition (5) we represent integral (1) in the form

$$u(t,x) = \int_0^{t_1} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} M(t,\tau,x-\xi) f(\tau,\xi) d\xi + \\ + \int_{t_1}^{t} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} M(t,\tau,x-\xi) \Delta^x_\xi f(\tau,\xi) d\xi, \qquad (t,x) \in \Pi_{(0,T]},$$

where the point t_1 is the same as before.

With the help of (4), (6), (9), (13), (14), (17)–(19) and (34) with $r > \nu - 1$ we get

$$|u(t,x)| \le CE^{-d}(T,t)\Psi(t,x)(\|f\|^{0,0}_{1,r} I^r_\nu(0,t_1,t) + [f]^{\lambda,0}_{1,r} I^{r-\lambda/(2b)}_{\nu-\lambda/(2b)}(t_1,t,t)) \le \\ \le C\|f\|^{\lambda,0}_{1,r} E^{-d}(T,t)\Psi(t,x)(\Delta(t,0))^{r-\nu+1}, \quad (t,x) \in \Pi_{(0,T]}. \tag{38}$$

Similarly, when $r \le \nu - 1$ by (15) we have

$$|u(t,x)| \le CE^{-d}(T,t)\Psi(t,x)\Bigg((\Delta(t,0))^{r-\nu+1} \times \\ \times (\Delta(T,0))^{-r+\nu+\gamma/(2b)-1} \int_0^{t_1} W_0[f;\tau](\Delta(\tau,0))^{-\nu-\gamma/(2b)} E^d(T,\tau) \frac{d\tau}{\alpha(\tau)} + \\ + \int_{t_1}^{t} (B(t,\tau))^{-\nu+\lambda/(2b)} \frac{\delta(\tau)}{\alpha(\tau)} d\tau (\Delta(t_1,0))^{r-\lambda/(2b)} [f]^{\lambda,0}_{1,r}\Bigg) \le \\ \le CE^{-d}(T,t)\Psi(t,x)(\Delta(t,0))^{r-\nu+1}(F_0 + [f]^{\lambda,0}_{1,r}), \quad (t,x) \in \Pi_{(0,T]}. \tag{39}$$

From (38) and (39) it follows the estimate

$$\|u\|^{0,0}_{0,r-\nu+1} \le C(\|f\|^{\lambda,0}_{1,r} + \chi'_r F_0). \tag{40}$$

Let us estimate the difference $\Delta^{t',x'}_{t,x} u$. It is sufficient to consider the case, when $p^{2b} < \Delta(t,0)$. If $p^{2b} \ge \Delta(t,0)$, then under condition (40) we have the estimate

$$|\Delta^{t',x'}_{t,x} u| \le C(\|f\|^{\lambda,0}_{1,r} + \chi'_r F_0) p^\gamma (\Delta(\bar{t},0))^{r-\nu+1-\gamma/(2b)} E^{-d}(T,\tilde{t})(\Psi(t,x) + \Psi(t',x')), \tag{41}$$

where $\bar{t}$ and $\tilde{t}$ are the same as in (24).

Let the points t_1 and t_2 be such that $B(t,t_1) = \Delta(t_1,0)$ and $B(t,t_2) = p^{2b}$. If $t_1 < t_2$, than by condition (5) we write

$$\Delta^{t',x'}_{t,x} u = \int_0^{t_1} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} \Delta^{t',x'}_{t,x} M(t,\tau,x-\xi) f(\tau,\xi) d\xi + \\ + \int_{t_1}^{t_2} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} \Delta^{t',x'}_{t,x} M(t,\tau,x-\xi) \Delta^x_\xi f(\tau,\xi) d\xi + \int_{t_2}^{t} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} M(t,\tau,x-\xi) \Delta^x_\xi f(\tau,\xi) d\xi -$$

$$-\int_{t_2}^{t'} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} M(t',\tau,x'-\xi)\Delta_\xi^{x'} f(\tau,\xi)d\xi \equiv \sum_{j=1}^{4} K_j. \quad (42)$$

Using (9), (13), (14), (17) and (30), we get with $r > \nu + \gamma/(2b) - 1$

$$|K_1| \le C\|f\|_{1,r}^{0,0} p^\gamma E^{-d}(T,\tilde{t})(\Psi(t,x)+\Psi(t',x'))I^r_{\nu+\gamma/(2b)}(0,t_1,t) \le$$

$$\le C\|f\|_{1,r}^{0,0} p^\gamma E^{-d}(T,\tilde{t})(\Psi(t,x)+\Psi(t',x'))(\Delta(t,0))^{r-\nu-\gamma/(2b)+1},$$

and with $r \le \nu + \gamma/(2b) - 1$

$$|K_1| \le Cp^\gamma E^{-d}(T,\tilde{t})(\Psi(t,x)+\Psi(t',x'))(\Delta(t,0))^{r-\nu-\gamma/(2b)+1}\times$$

$$\times(\Delta(T,0))^{-r+\nu+\gamma/(2b)-1}\int_0^{t_1} W_0[f;\tau](\Delta(\tau,0))^{-\nu-\gamma/(2b)}E^d(T,\tau)\frac{d\tau}{\alpha(\tau)} \le$$

$$\le CF_0 p^\gamma E^{-d}(T,\tilde{t})(\Psi(t,x)+\Psi(t',x')(\Delta(t,0))^{r-\nu-\gamma/(2b)+1}).$$

Let us estimate K_2. With the help of (9), (13), (14), (30) and (34) we obtain

$$|K_2| \le C[f]_{1,r}^{\lambda,0} p^\gamma E^{-d}(T,\tilde{t})(\Psi(t,x)+\Psi(t',x'))\times$$

$$\times\int_{t_1}^{t_2}(B(t,\tau))^{-\gamma/(2b)}(\Delta(\tau,0))^{r-\lambda/(2b)}((B(t,\tau))^{-\nu+\lambda/(2b)}+$$

$$+(B(t',\tau))^{-\nu+\lambda/(2b)} + p^\lambda(B(t',\tau))^{-\nu})\frac{\delta(\tau)}{\alpha(\tau)}d\tau \le C[f]_{1,r}^{\lambda,0}p^\gamma\times$$

$$\times E^{-d}(T,\tilde{t})(\Psi(t,x)+\Psi(t',x'))(I^{r-\lambda/(2b)}_{\nu+(\gamma-\lambda)/(2b)}(t_1,t_2,t) + p^\lambda I^{r-\lambda/(2b)}_{\nu+\gamma/(2b)}(t_1,t_2,t)). \quad (43)$$

Since $p^\lambda = (B(t,t_2))^{\lambda/(2b)} \le (B(t,\tau))^{\lambda/(2b)}, \tau \le t_2$, we obtain

$$p^\lambda I^{r-\lambda/(2b)}_{\nu+\gamma/(2b)}(t_1,t_2,t) \le I^{r-\lambda/(2b)}_{\nu+(\gamma-\lambda)/(2b)}(t_1,t_2,t). \quad (44)$$

Let $1-\nu-(\gamma-\lambda)/(2b) > 0$. Then with the help of (3) and (16) we have

$$I^{r-\lambda/(2b)}_{\nu+(\gamma-\lambda)/(2b)}(t_1,t_2,t) \le (\Delta(t_3,0))^{r-\lambda/(2b)}\int_{t_1}^{t_2}(B(t,\tau))^{-\nu-(\gamma-\lambda)/(2b)}\frac{\beta(\tau)}{\alpha(\tau)}d\tau \le$$

$$\le C(\Delta(t,0))^{r-\lambda/(2b)}(\Delta(t_1,0))^{1-\nu-(\gamma-\lambda)/(2b)} \le C(\Delta(t,0))^{r-\nu-\gamma/(2b)+1}, \quad (45)$$

where $t_3 = t_2$ with $r \ge \lambda/(2b)$ and $t_3 = t_1$ with $r < \lambda/(2b)$. Similarly, if $1-\nu-(\gamma-\lambda)/(2b) < 0$, then

$$I^{r-\lambda/(2b)}_{\nu+(\gamma-\lambda)/(2b)}(t_1,t_2,t) \le (\Delta(t_3,0))^{r-\lambda/(2b)}\int_{t_1}^{t_2}(B(t,\tau))^{-\nu-(\gamma-\lambda)/(2b)}\frac{\beta(\tau)}{\alpha(\tau)}d\tau \le$$

$$\le C(\Delta(t,0))^{r-\lambda/(2b)}(B(t,t_2))^{1-\nu-(\gamma-\lambda)/(2b)} =$$

$$= C(\Delta(t,0))^{r-\lambda/(2b)}p^{2b(1-\nu)+\lambda-\gamma} \le C(\Delta(t,0))^{r-\nu-\lambda/(2b)+1}p^{\lambda-\gamma}. \quad (46)$$

From (43)–(46) it follows the estimate

$$|K_2| \le C[f]_{1,r}^{\lambda,0}E^{-d}(T,\tilde{t})(\Psi(t,x)+\Psi(t',x'))P_{\gamma\lambda},$$

where

$$P_{\gamma\lambda} \equiv \begin{cases} p^{\gamma}(\Delta(t,0))^{r-\nu-\gamma/(2b)+1}, & \text{if } \ 1-\nu-(\gamma-\lambda)/(2b) > 0, \\ p^{\lambda}(\Delta(t,0))^{r-\nu-\lambda/(2b)+1}, & \text{if } \ 1-\nu-(\gamma-\lambda)/(2b) < 0. \end{cases}$$

After using (3), (4), (6), (9), (14) and (34), we obtain

$$|K_3| \le C[f]^{\lambda,0}_{1,r} E^{-d}(T,t)\Psi(t,x)(\Delta(t_4,0))^{r-\lambda/(2b)} \int_{t_2}^{t} (B(t,\tau))^{-\nu+\lambda/(2b)} \frac{\beta(\tau)}{\alpha(\tau)} d\tau,$$

where $t_4 = t$ with $r \ge \lambda/(2b)$ and $t_4 = t_2$ with $r < \lambda/(2b)$. Since, due to (16) $\Delta(t_2,0) \ge \Delta(t_1,0) \ge \Delta(t,0)/2$ and

$$\int_{t_2}^{t} (B(t,\tau))^{-\nu+\lambda/(2b)} \frac{\beta(\tau)}{\alpha(\tau)} d\tau = \frac{1}{1-\nu+\lambda/(2b)} p^{2b(1-\nu)+\lambda},$$

we have

$$|K_3| \le C[f]^{\lambda,0}_{1,r} E^{-d}(T,t)\Psi(t,x)(\Delta(t,0))^{r-\lambda/(2b)} p^{2b(1-\nu)+\lambda} \le C[f]^{\lambda,0}_{1,r} E^{-d}(T,t)\Psi(t,x)P_{\gamma\lambda}.$$

The following estimate can be obtained in the similar way

$$|K_4| \le C[f]^{\lambda,0}_{1,r} E^{-d}(T,t')\Psi(t',x')P_{\gamma\lambda}.$$

For the case $t_1 \ge t_2$, instead of (42) we use the representation $\Delta^{t',x'}_{t,x} u = K_1 + K_3^{'} + K_4^{'}$, where K_1 is the same as in (42), and $K_3^{'}$ and $K_4^{'}$ differ from K_3 and K_4, respectively, by that t_2 is replaced by t_1. The estimates $K_3^{'}$ and $K_4^{'}$ are obtained similarly to the estimates K_3 and K_4.

From (40), (41) and estimates for $K_j, 1 \le j \le 4, K_3^{'}$ and $K_4^{'}$ estimates (11) and (12) follow. □

Lemma 2. *Let the kernel of integral (1) be the matrix*

$$M(t,\tau,x-\xi;t,x) \equiv (B(t,\tau))^{-\nu-n/(2b)} \Omega(t,\tau,(B(t,\tau))^{-1/(2b)}(x-\xi);t,x),$$

$$0 < \tau < t \le T, \quad \{x,\xi\} \subset \mathbb{R}^n,$$

where $\nu \in (0,1]$, *and the matrix* $\Omega(t,\tau,x;\theta,y)$ *as a function of* t,τ *and* x *satisfies conditions* A_1*–*A_4 *uniformly with respect to* $(\theta,y) \in \Pi_{(0,T]}$ *and the condition*

$$\exists\, C > 0 \quad \forall \{t,\tau\} \subset (0,T], \tau < t, \forall\, x \in \mathbb{R}^n \quad \forall \{(\theta,y),(\theta',y')\} \subset \Pi_{(0,T]} :$$

$$|\Delta^{\theta',y'}_{\theta,y} \Omega(t,\tau,x;\theta,y)| \le C(p(\theta,y;\theta',y'))^{\lambda} E^{d}(t,\tau)\exp\{-c|x|^{q}\},$$

$c > 0, d \in \mathbb{R}, 0 < \lambda < \gamma \le 1,\ 1-\nu-(\gamma-\lambda)/(2b) < 0.$ *Then if* f *satisfies the conditions of statement b) of Lemma 1, then* $u \in C^{\lambda,\lambda/(2b)}_{0,r-\nu+1}$ *and inequality (12) is valid.* The proof is carried out in the same way as in Lemma 1, but while estimating $\Delta^{t',x'}_{t,x} u$ the following additional summands appear:

$$\int_0^{t_1} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} (M(t,\tau,x-\xi;t,x) - M(t,\tau,x-\xi;t',x'))f(\tau,\xi)d\xi,$$

$$\int_{t_1}^{t} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} (M(t,\tau,x-\xi;t,x) - M(t,\tau,x-\xi;t',x'))\Delta_\xi^x f(\tau,\xi)d\xi.$$

The necessary estimates of these summands are obtained similarly to estimate (40).

Lemma 3. *Statement a) of Lemma 1 is true for the integral*

$$u(t,x) = \int_0^t \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} L(t,\tau,x,\xi)f(\tau,\xi)d\xi, \quad (t,x) \in \Pi_{(0,T]}, \tag{47}$$

if the matrix L satisfies the inequalities

$$|L(t,\tau,x,\xi)| \le C(B(t,\tau))^{-\nu-n/(2b)} E_c^d(t,\tau,x-\xi),$$
$$|\Delta_{t,x}^{t',x'} L(t,\tau,x,\xi)| \le C(p(t,x;t',x'))^\gamma ((B(t,\tau))^{-\nu-(n+\gamma)/(2b)} E_c^d(t,\tau,x-\xi)+$$
$$+(B(t',\tau))^{-\nu-(n+\gamma)/(2b)} E_c^d(t',\tau,x'-\xi)), \quad 0<\tau<t\le t'\le T, \quad \{x,x',\xi\}\subset\mathbb{R}^n, \tag{48}$$

where $E_c^d(t,\tau,x) \equiv E_c(t,\tau,x)E^d(t,\tau)$.

If L also satisfies the conditions

$$\left|\int_{\mathbb{R}^n} L(t,\tau,x,\xi)d\xi\right| \le C(B(t,\tau))^{-\nu+\lambda/(2b)} E^d(t,\tau),$$

$$|\Delta_{t,x}^{t',x'} \int_{\mathbb{R}^n} L(t,\tau,x,\xi)d\xi| \le C(p(t,x;t',x'))^\gamma (B(t,\tau))^{-\nu-(\gamma-\lambda)/(2b)} E^d(\tilde{t},\tau),$$
$$0<\tau<t\le t'\le T, \quad \{x,x',\xi\}\subset\mathbb{R}^n, \tag{49}$$

then for integral (47) statement b) of Lemma 1 holds.

The proof of Lemma 3 is based on the representation

$$u(t,x) = \int_0^{t_1} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} L(t,x;\tau,\xi)f(\tau,\xi)d\xi+$$

$$+\int_{t_1}^{t} \frac{d\tau}{\alpha(\tau)} \int_{\mathbb{R}^n} L(t,x;\tau,\xi)\Delta_\xi^x f(\tau,\xi)d\xi + \int_{t_1}^{t} \left(\int_{\mathbb{R}^n} L(t,x;\tau,\xi)d\xi\right) f(\tau,x)\frac{d\tau}{\alpha(\tau)}.$$

The first two summands are estimated in the same way as the corresponding integrals from Lemma 1. Inequalities (49) ensure the necessary estimates for the last summand.

Lemma 4. *Let u be a function that is represented by formula (1) with the kernel M of form (4), for which conditions A_1–A_3 hold. If $\nu+\gamma/(2b)<1, r\ge 0$ and $f\in C^0_{1,r,1-\nu}$, then $u\in C^\gamma_{0,r,0}$ and the estimate*

$$\|u\|^\gamma_{0,r,0} \le C\|f\|^0_{1,r,1-\nu} \tag{50}$$

is valid. If $f\in C^\lambda_{1,r,1-\nu}, r\ge 0$ and $1-\nu-(\gamma-\lambda)/(2b)>0$, then $u\in C^\gamma_{0,r,0}$ and the estimate

$$\|u\|^\gamma_{0,r,0} \le C[f]^\lambda_{1,r,1-\nu} \tag{51}$$

holds.

The similar statements are true for integral (47), the kernel of which satisfies conditions (48) and (49) with $t'=t$, but in estimate (51) the value $[f]^\lambda_{1,r,1-\nu}$ must be replaced by $\|f\|^\lambda_{1,r,1-\nu}$.

Proof. Let us dwell upon the proof of the lemma only for integral (1). With the help of (3), (4), (6), (7), (9) and (14) with $\nu+\gamma/(2b)<1$ we get

$$|u(t,x)| \le C(\Delta(t,0))^r E^{-d}(T,t)\Psi(t,x)\int_0^t W_0[f;\tau](\delta(\tau))^{-1}\times$$
$$\times(\Delta(\tau,0))^{-r}E^d(T,\tau)(B(t,\tau))^{-\nu}\frac{\beta(\tau)}{\alpha(\tau)}d\tau,$$
$$|\Delta_x^{x'}u(t,x)| \le C|x-x'|^\gamma(\Delta(t,0))^r E^{-d}(T,t)(\Psi(t,x)+$$
$$+\Psi(t,x'))\int_0^t W_0[f;\tau](\delta(\tau))^{-1}(\Delta(\tau,0))^{-r}E^d(T,\tau)\times$$
$$\times(B(t,\tau))^{-\nu-\gamma/(2b)}\frac{\beta(\tau)}{\alpha(\tau)}d\tau, \qquad (t,x)\in\Pi_{(0,T]},$$

hence

$$W_0[u;t](\Delta(t,0))^{-r}E^d(T,t) \le C\beta(t)\int_0^t W_0[f;\tau](\delta(\tau))^{-1}\times$$
$$\times(\Delta(\tau,0))^{-r}E^d(T,\tau)(B(t,\tau))^{-\nu}\frac{d\tau}{\alpha(\tau)}, \tag{52}$$

$$W_\lambda[u;t](\Delta(t,0))^{-r}E^d(T,t) \le C\beta(t)\int_0^t W_0[f;\tau](\delta(\tau))^{-1}\times$$
$$\times(\Delta(\tau,0))^{-r}E^d(T,\tau)(B(t,\tau))^{-\nu-\gamma/(2b)}\frac{d\tau}{\alpha(\tau)}. \tag{53}$$

After multiplying inequalities (52) and (53) by $(\alpha(t))^{-1}$ and $(B(T,t))^{\gamma/(2b)}(\alpha(t))^{-1}$, and after integrating the results with respect to $t\in(0,T]$ and having changed the order of integrating, we get respectively

$$\|u\|^0_{0,r,0} \le C\|f\|^0_{1,r,1-\nu}, \quad [u]^\gamma_{0,r,0} \le C\|f\|^0_{1,r,1-\nu}.$$

Estimate (50) follows from these inequalities.

Let $1-\nu-(\gamma-\lambda)/(2b)>0$. Writing because of (5) the representation

$$u(t,x)=\int_0^t\frac{d\tau}{\alpha(\tau)}\int_{\mathbb{R}^n}M(t,\tau,x-\xi)\Delta_\xi^x f(\tau,\xi)d\xi, \quad (t,x)\in\Pi_{(0,T]},$$

in the same way as above, first we obtain the inequality

$$|u(t,x)| \le C(\Delta(t,0))^r E^{-d}(T,t)\Psi(t,x)\int_0^t W_\lambda[f;\tau](\delta(\tau))^{-1}\times$$
$$\times(\Delta(\tau,0))^{-r}E^d(T,\tau)(B(t,\tau))^{-\nu+\lambda/(2b)}\frac{\beta(\tau)}{\alpha(\tau)}d\tau, \quad (t,x)\in\Pi_{(0,T]}, \tag{54}$$

and then the estimate

$$\|u\|^0_{0,r,0} \le C[f]^\lambda_{1,r,1-\nu}. \tag{55}$$

In order to obtain an estimate for $\Delta_x^{x'} u(t,x)$, we shall distinguish two cases: $|x-x'|^{2b} \geq B(T,t)$ and $|x-x'|^{2b} < B(T,t)$. In the first case, using (54), we have

$$|\Delta_x^{x'} u(t,x)| \leq C|x-x'|^{\gamma}(\Delta(t,0))^r E^{-d}(T,t)(\Psi(t,x)+$$
$$+\Psi(t,x'))(B(T,t))^{-\gamma/(2b)}\beta(t)\int_0^t W_\lambda[f;\tau](\delta(\tau))^{-1}\times$$
$$\times(\Delta(\tau,0))^{-r}E^d(T,\tau)(B(t,\tau))^{-\nu+\lambda/(2b)}\frac{d\tau}{\alpha(\tau)}. \tag{56}$$

In the second case we note that

$$\Delta_x^{x'} u(t,x) = \int_0^{t_1}\frac{d\tau}{\alpha(\tau)}\int_{\mathbb{R}^n}\Delta_x^{x'} M(t,\tau,x-\xi)\Delta_\xi^x f(\tau,\xi)d\xi+$$
$$+\int_{t_1}^t\frac{d\tau}{\alpha(\tau)}\int_{\mathbb{R}^n} M(t,\tau,x-\xi)\Delta_\xi^x f(\tau,\xi)d\xi - \int_{t_1}^t\frac{d\tau}{\alpha(\tau)}\int_{\mathbb{R}^n} M(t,\tau,x'-\xi)\Delta_\xi^{x'} f(\tau,\xi)d\xi, \tag{57}$$

where the point t_1 is such that $B(t,t_1) = |x-x'|^{2b}$. The estimates of the integrals from (57) are carried out in the same way as before, and we finally obtain the estimate

$$|\Delta_x^{x'} u(t,x)| \leq C|x-x'|^{\gamma}(\Delta(t,0))^r E^{-d}(T,t)(\Psi(t,x)+$$
$$+\Psi(t,x'))\beta(t)\int_0^t W_\lambda[f;\tau](\delta(\tau))^{-1}(\Delta(\tau,0))^{-r}E^d(T,\tau)(B(t,\tau))^{-\nu-(\gamma-\lambda)/(2b)}\frac{d\tau}{\alpha(\tau)}. \tag{58}$$

Estimate (51) follows from (55), (56) and (58). □

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