УДК 512.544

SOLVABLE GROUPS WITH MANY CONDITIONS ON NILPOTENT-BY-ČERNIKOV SUBGROUPS

O.D. Artemovych

O.D. Artemovych. *Solvable groups with many conditions on nilpotent-by-Černikov subgroups*, Matematychni Studii, **13** (2000) 23–32.

We prove that non-"nilpotent-by-Černikov" group in which no non-trivial section is perfect has an infinite ascending chain of non-"nilpotent-by-Černikov" subgroups and give a characterization of the solvable groups with the maximal condition on non-"nilpotent-by-Černikov" subgroups.

О.Д. Артемович. *Разрешимые группы, насыщенные подгруппами, являющимися расширениями нильпотентных групп при помощи черниковских групп* // Математичні Студії. – 2000. – Т.13, №1. – С.23–32.

Доказано, что группа без нетривиальных совершенных секций, не являющаяся расширением нильпотентной группы при помощи черниковской, имеет бесконечный убывающий ряд, состоящий из подгрупп, не являющихся расширением нильпотентной группы при помощи черниковской. Охарактеризовано разрешимые группы с условием максимальности для подгрупп, не являющихся расширением нильпотентной группы при помощи черниковской группы.

0. Let $\mathfrak{X}$ be a class of groups. In the last few years there has been increasing interest in the structure of groups with many $\mathfrak{X}$-subgroups and, in particular, groups with the maximal or minimal condition on non-$\mathfrak{X}$-subgroups. We say that G satisfies the minimal condition on non-$\mathfrak{X}$-subgroups (briefly Min-$\overline{\mathfrak{X}}$) if for every descending chain $\{G_n \mid n \in \mathbb{N}\}$ of subgroups of G there exists a number $n_0 \in \mathbb{N}$ such that the subgroups G_n are $\mathfrak{X}$-groups for all $n \geq n_0$. The maximal condition on non-$\mathfrak{X}$-subgroups of G (briefly Max-$\overline{\mathfrak{X}}$) is defined dually, namely, one says that G satisfies Max-$\overline{\mathfrak{X}}$ if there is no infinite ascending chain of non-$\mathfrak{X}$-subgroups of G. S.N. Černikov (see [1]) and V.P. Šunkov [2] sdudied groups with the minimal condition on non-abelian subgroups, while D.I. Zaĭtsev and L.A. Kurdachenko [3] characterized the locally almost solvable groups with the maximal condition on non-abelian subgroups. Every minimal non-$\mathfrak{X}$ group, i.e. a non-$\mathfrak{X}$ group in which every proper subgroup is a $\mathfrak{X}$-group, satisfies Max-$\overline{\mathfrak{X}}$ and Min-$\overline{\mathfrak{X}}$. Many authors investigated the minimal non-"nilpotent-by-finite" groups. M.F. Newman and J. Wiegold [4] described the structure of a minimal non-nilpotent group with a maximal subgroup. Examples constructed by H. Heineken and I.J. Mohamed [5] have evoked considerable interest to groups whose all proper subgroups are nilpotent [6-10], abelian-by-finite [11-14], nilpotent-by-finite [15-16] or hypercentral-by-finite

2000 *Mathematics Subject Classification:* 20E15, 20F16.

[17], respectively. It is natural to extend the above problems by replacing the terms "finite group" by "Černikov group" like in [18-20].

In this paper we prove that every non-"nilpotent-by-Černikov" group in which no non-trivial section is perfect with the minimal condition on non-"nilpotent-by-Černikov" subgroups Min-$\overline{N\check{C}}$ is a nilpotent-by-Černikov group. We also characterize the solvable groups with the maximal condition on non-"nilpotent-by-Černikov" subgroups Max-$\overline{N\check{C}}$.

Throughout this paper p and q always denote distinct primes and $\mathbb{C}_{p^\infty}$ the quasicyclic p-group. For any group G, $Z(G)$ means the centre of G, $G', G'', \ldots, G^{(n)}$ the terms of derived series of G, RG the group ring of G over a commutative ring R, τG the set of all torsion elements of G and $\gamma_c G$ the term of the lower central series of G. Let also $\mathbb{F}_p$ be the finite field with p elements, $\mathbb{Z}$ the group (or set) of rational integers and $\mathbb{Q}$ the rational number field.

Most of the standard notation can be found in [21–23].

1. In this part we study the groups in which no non-trivial section is perfect with the minimal condition on non-"nilpotent-by-Černikov" subgroups Min-$\overline{N\check{C}}$.

Lemma 1.1. *Let G be a group satisfying Min-$\overline{N\check{C}}$ and H be a subgroup of G. Then:*

(i) H satisfies Min-$\overline{N\check{C}}$;

(ii) if H is normal in G, then the quotient group G/H satisfies Min-$\overline{N\check{C}}$;

(iii) if H is a normal non-"nilpotent-by-Černikov" subgroup of G, then G/H is a Černikov group.

Proof is immediate.

Lemma 1.2. *Let G be a non-perfect (i.e. $G \neq G'$) locally graded group with the "nilpotent-by-Černikov" commutator subgroup G'. If G satisfies Min-$\overline{N\check{C}}$, then it is a "nilpotent-by-Černikov" group.*

Proof. Suppose that G has a proper non-"nilpotent-by-Černikov" subgroup. Since G satisfies Min-$\overline{N\check{C}}$, it contains a subgroup S which is a minimal non-"nilpotent-by-Černikov" group. Then by Theorem A of [20], $S = S'$ and consequently $S \leq G'$, a contradiction. Therefore all proper subgroups of G are nilpotent-by-Černikov. From Theorem A of [20] it follows that G is a nilpotent-by-Černikov group, as desired. □

Corollary 1.3. *Let G be a non-perfect group with all proper normal subgroups nilpotent-by-Černikov. If G satisfies Min-$\overline{N\check{C}}$, then G is a nilpotent-by-Černikov group.*

Remark 1.4. Let G be a locally graded group satisfying Min-$\overline{N\check{C}}$. Then one of the following holds:

(1) G is a nilpotent-by-Černikov group;

(2) G is a non-"nilpotent-by-Černikov" group, but every proper subgroup of G is nilpotent-by-Černikov;

(3) G contains a proper subgroup S of finite index which is a group of type (2).

Unfortunately it is not known if there exists a group possessing the property (2) from Remark 1.4.

Lemma 1.1, Corollary 1.3 and Remark 1.4 yield the following

Proposition 1.5. *Let G be a group in which no non-trivial section is perfect (in particular, G is a solvable group). Then G satisfies Min-$\overline{N\check{C}}$ if and only if it is a nilpotent-by-Černikov group.*

Consequently every solvable non-"nilpotent-by-Černikov" group has an infinite ascending chain of non-"nilpotent-by-Černikov" subgroups.

Remark 1.6. Let c be a non-negative integer. In the same manner as before, Theorem C of [20] and Theorem 1 of [18] imply that a locally graded non-"abelian-by-Černikov" (respectively non-"Černikov-by-nilpotent of class $\leq c$") group G has an infinite ascending chain of non-"abelian-by-Černikov" (respectively non-"Černikov-by-nilpotent of class $\leq c$") subgroups.

2. In this part we establish some properties of groups with the maximal condition on non-"nilpotent-by-Černikov" subgroups Max-$\overline{N\check{C}}$.

Lemma 2.1. *Let G be a group satisfying Max-$\overline{N\check{C}}$ and H be a subgroup of G. Then:*

(i) H satisfies Max-$\overline{N\check{C}}$;

(ii) if H is normal in G, then the quotient group G/H satisfies Max-$\overline{N\check{C}}$;

(iii) if H is a normal non-"nilpotent-by-Černikov" subgroup of G, then G/H satisfies Max.

Proof is immediate.

Lemma 2.2. *If $G = AB$ is the product of two normal nilpotent-by-Černikov subgroups A and B, then G is a nilpotent-by-Černikov group.*

Proof. Let N be a normal nilpotent subgroup of A with the Černikov quotient group A/N. By Lemma 4.7 of [24] the normal closure $R = N^G$ of N in G is a nilpotent subgroup with the Černikov quotient group A/R. Let S be a G-invariant subgroup of B such that the quotient B/S is a Černikov group. Since G/RS is a product of two normal Černikov subgroups, G is a nilpotent-by-Černikov group. □

Proposition 2.3. *Let G be a group satisfying Max-$\overline{N\check{C}}$. Then either G is a nilpotent-by-Černikov group or the quotient group G/G' is finitely generated.*

Proof. If the commutator subgroup G' is not a nilpotent-by-Černikov group, then G/G' is finitely generated by Lemma 2.1(*iii*). Therefore we assume that G' is a nilpotent-by-Černikov subgroup and, furthermore, the quotient G/G' is not finitely generated. It is well known (see e.g.[21, Theorem 21.3]) that $\overline{G} = G/G' = \overline{N} \times \overline{D}$ is a group direct product of the divisible part $\overline{D}$ of $\overline{G}$ and a reducible subgroup $\overline{N}$. Let N (respectively D) is an inverse image of $\overline{N}$ (respectively N) in G.

Assume that G is not a nilpotent-by-Černikov group.

(1) Let the divisible part $\overline{D}$ of $\overline{G}$ is a non-trivial subgroup. By our hypothesis and Lemma 2.1 $\overline{D}$ cannot be written as a product $\overline{D} = \overline{A_1} \cdot \overline{A_2}$ of two subgroups $\overline{A_1}$ and $\overline{A_2}$ with the infinite indices $|\overline{D} : \overline{A_i}|$ $(i = 1, 2)$. This yields $\overline{D} \cong \mathbb{C}_{p^\infty}$ for some prime p. From this it follows that $G/N \cong \mathbb{C}_{p^\infty}$ and so N is an extension of a nilpotent subgroup S by a Černikov subgroup. By Lemma 4.7 of [24] the normal closure $R = S^G$ is a nilpotent group with the Černikov quotient group N/R. But then the quotient group G/N is also Černikov, a contradiction.

(2) Now, suppose that $\overline{D}$ is trivial. Let B be an inverse image for a p-basic subgroup $\overline{B}$ of the quotient group $\overline{G}$ in G (see [21, §32]).

Let $\overline{B} \neq \overline{G}$. Then G/B is a p-divisible abelian group. If G/B is torsion-free then, as proved in [25] (see also [1, Chapter 2, §6]), it contains a subgroup $\overline{L}$ which is isomorphic to a p-divisible subgroup $\mathbb{Q}^{(p)} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\} \right\}$ of the additive group $\mathbb{Q}^+$ of the rational number field $\mathbb{Q}$. Since $\overline{L}$ contains a subgroup $\overline{H}$ with the p-quasicyclic quotient group $\overline{L}/\overline{H}$, $\overline{G}$ has a p-quasicyclic homomorphic image. As above this yields that G is a nilpotent-by-Černikov group, a contradiction. Thus $\overline{B} = \overline{G}$. Since in view of our hypothesis $\overline{G}$ is not a finitely generated group, $\overline{G}$ can be written as a group direct product $\overline{G} = \overline{G_1} \times \overline{G_2}$, where every factor is infinitely generated. But then G is a product of two proper normal nilpotent-by-Černikov subgroups. By Lemmas 2.1 and 2.2 G is a nilpotent-by-Černikov group. □

Corollary 2.4. *Let G be a torsion group in which no non-trivial section is perfect. Then G satisfies Max-$\overline{N\check{C}}$ if and only if G is a nilpotent-by-Černikov group.*

Example 2.5. There are the non-“nilpotent-by-Černikov” groups with Max-$\overline{N\check{C}}$.
Indeed, let $G = A \rtimes \langle x \rangle$, where $A \cong \mathbb{C}_{p^\infty}$, $\langle x \rangle$ is an infinite cyclic group and $[a, x] = a^p$ $(a \in A)$. It is easy to see that G satisfies Max-$\overline{N\check{C}}$.

Lemma 2.6. *Let G be a group satisfying Max-$\overline{N\check{C}}$. If G has a normal subgroup S with the nilpotent-by-Černikov quotient group G/S, then either G is a nilpotent-by-Černikov group or G/S is a finitely generated group.*

Proof. If G/S is a nilpotent-by-“infinite Černikov” group, then obviously that G is nilpotent-by-Černikov. Therefore we assume that G/S is a nilpotent-by-finite quotient group and G is a non-“nilpotent-by-Černikov” group. Let M be a normal subgroup of G of finite index such that the quotient group $\overline{M} = M/S$ is nilpotent. By Lemma 2.1 M satisfies Max-$\overline{N\check{C}}$ and in view of our hypothesis M is a non-“nilpotent-by-Černikov group. By Theorem 21.3 of [21] $M_1 = \overline{M}/\overline{M'} = D_1 \times F_1$ is a group direct product of the divisible part D_1 and a reducible subgroup F_1.

(1) Suppose that the divisible part D_1 is not trivial. Then M has a proper subgroup X such that $M/X \cong \mathbb{C}_{p^\infty}$ for some prime p. So M is a nilpotent-by-Černikov group, a contradiction.

(2) Let the divisible part D_1 is trivial. By B_1 we denote a p-basic subgroup of M_1. If $B_1 = M_1$, then in view of our hypothesis B_1 is a finitely generated subgroup and so is $\overline{M}$, as desired. Therefore we assume that $B_1 \neq M_1$. Then M_1/B_1 is a p-divisible group. If the Sylow p-subgroup of M_1/B_1 is nontrivial or M_1/B_1 is a non-torsion group, then M is a nilpotent-by-Černikov group. Assume that M_1/B_1 is a torsion p'-group. By Proposition 18.3 of [21] and our hypothesis the quotient group M_1/B_1 is infinite. Since M is non-“nilpotent-by-Černikov” group, M_1/B_1 has an infinite Sylow q-subgroup for some prime q. Without restricting of generality, we can assume that M_1/B_1 is an infinite q-group. It is obvious that M_1/B_1 is not equal to its basic subgroup and thus M_1/B_1 has a q-quasicyclic quotient group. This yields that M is a nilpotent-by-Černikov group, a contradiction. □

The next lemma is an extension of Lemma 3.1 from [26].

Lemma 2.7. *Let G be a nilpotent p-group and B be a normal subgroup of finite exponent of G such that G/B is a divisible group. If D is the maximal divisible abelian subgroup of G, then $G = BD$.*

Proof. We use induction on the nilpotency class c of G. If $c = 1$ then the assertion follows from Proposition 21.3 of [21].

Suppose that $c > 1$ and the assertion holds for $c - 1$. Let $R = \gamma_c G$ and $\overline{G} = G/R$. If $\overline{U} = U/R$ is the maximal divisible abelian subgroup of $\overline{G}$, then by induction hypothesis $\overline{G} = \overline{U}\ \overline{B}$. Since $R \leq Z(G)$, the quotient group

$$UZ(G)/Z(G) \cong U/(U \cap Z(G)) \cong (U/R)/((U \cap Z(G))/R)$$

is divisible abelian. By Corollary 4.13 of [27] $\gamma_2(UZ(G)) = \gamma_2 U$ is a divisible group. Let $U_1 = U/U'$, $R_1 = RU'/U'$. Since the quotient U_1/R_1 is divisible, we conclude that $U_1 = V_1 R_1$, where V_1 is a divisible part of U_1. Then $U = VR$, where V is a divisible group. Thus $G = UB = VRB = VR$. □

3. In this part we characterize the solvable groups satisfying Max-$\overline{N\check{C}}$.

Let D be a commutative Dedekind domain and A a R-right module. By Spec(D) we denote the set $\{P \mid P \text{ is a non-zero prime ideal of } D\}$. Moreover the set $A_P = \{a \in A \mid aP^n = \{0\} \text{ for some } n = n(a) \in \mathbb{N}\}$ is called the P-component of A for some $P \in \text{Spec}(D)$. It is well known (see e.g. [28, Theorem 9.4]) that $A = A_P$ implies that A has a basic submodule B, i.e. B satisfies the following conditions:

(1) B is a direct sum of cyclic submodules;
(2) B is a pure submodule of A;
(3) A/B is a divisible D-module.

Lemma 3.1. *Let $G = A \rtimes \langle x \rangle$ be the semidirect product of a normal abelian subgroup A of prime exponent p and an infinite cyclic subgroup $\langle x \rangle$. Then G satisfies Max-$\overline{N\check{C}}$ if and only if it is either a polycyclic group or a nilpotent-by-Černikov group.*

Proof. ($\Leftarrow$) Obviously.

($\Rightarrow$) It is clear that A is a right $\mathbb{F}_p\langle x \rangle$-module over a Dedekind ring $\mathbb{F}_p\langle x \rangle$ with x acting on A by the conjugation and A is a D-torsion module. Suppose that A is not a finitely generated $\mathbb{F}_p\langle x \rangle$-module and G is not a nilpotent-by-Černikov group. By Corollary 3.6 of [28] (see also Proposition 2.4 of [29, Chapter 8, §8.2])

$$A = \sum\nolimits^{\oplus}_{P \in \text{Spec}(\mathbb{F}_p\langle x \rangle)} A_P \qquad (*)$$

is a module direct sum of its P-components A_P. Then in the decomposition (*) there are only finite number of non-trivial summands all of them except for one (say A_P) being finite. Let B be a basic submodule of A_P. In view of our hypothesis $B = A_P$ is a module direct sum of finitely many cyclic submodules and consequently $A_P \rtimes \langle x \rangle$ is a nilpotent-by-Černikov subgroup of finite index in G. □

Lemma 3.2. *Let $G = D \rtimes \langle u \rangle$ be the semidirect product of a normal divisible abelian torsion-free subgroup D and an infinite cyclic subgroup $\langle u \rangle$. If D is an injective right $\mathbb{Q}\langle u \rangle$-module (with u acting on D by the conjugation) and G satisfies Max-$\overline{N\check{C}}$, then G is a nilpotent-by-Černikov group.*

Proof. Suppose that G is a non-"nilpotent-by-Černikov" group. Then D is an indecomposable $\mathbb{Q}\langle u\rangle$-module. By Theorem 2.4 of [30] $D \cong E(\mathbb{Q}\langle u\rangle/I)$, where $E(\mathbb{Q}\langle u\rangle/I)$ is an injective envelope of $\mathbb{Q}\langle u\rangle/I$ and I is an irreducible ideal of $\mathbb{Q}\langle u\rangle$. By Lemma 7.12 of [31] I is a P-primary ideal for some $P \in \mathrm{Spec}(\mathbb{Q}\langle u\rangle)$ and by Proposition 3.1 of [30] $D \cong E(\mathbb{Q}\langle u\rangle/P)$. Let $A_1 = \{x \in D \mid xP = \{0\}\}$. By Theorem 3.4 of [30] A_1 and the field $\mathbb{Q}\langle u\rangle/P$ are isomorphic as linear $(\mathbb{Q}\langle u\rangle/P)$-spaces. Since $\mathrm{Ann}_{\mathbb{Q}\langle u\rangle}(A_1) \neq \{0\}$, by Corollary 7.3 of [28] A_1 is a module direct sum of cyclic $\mathbb{Q}\langle u\rangle$-submodules. In view of Proposition 2.2 of [30] A_1 is a cyclic $\mathbb{Q}\langle u\rangle$-submodule. Hence $A_1 \rtimes \langle u^m\rangle$ is a nilpotent subgroup for some integer $m \geq 2$. By Z we denote the centre $Z(A_1 \rtimes \langle u^m\rangle)$. Without restricting of generality we can assume that $Z(A_1 \rtimes \langle u\rangle) = 1$. Since the quotient group $A_1/(A_1 \cap Z)$ is torsion-free by Mal'cev Theorem (see [22, Proposition 5.2.19]), the subgroup $A_1 \cap Z$ is pure in A_1 (see [21, §26]). Consequently $A_1 \cap Z$ is a divisible subgroup of A_1. Moreover Z is a $\langle u\rangle$-invariant subgroup. This means that $A_1 \cap Z$ is an injective right $\mathbb{Q}\langle u\rangle$-submodule of A_1 and hence $A_1 \cap Z = A_1$. By U we denote the quotient group $\langle u\rangle/\langle u^m\rangle$. Let π be a finite set of distinct primes $p_1, \ldots, p_l$ ($l \geq 2$). By Lemma 2.3 of [16] the nontrivial right $\mathbb{Z}U$-module A_1 has a submodule N such that A_1/N is a torsion π-group. Clearly that N is a normal in G. Let $C = D/N$. Then C is a right $\mathbb{Q}\langle u\rangle$-module with the action induced by the conjugation of u on A_1. Moreover C is a P-module (see e.g.[28, §3]) and by Theorem 9.4 of [28] C has a basic submodule B. If $B = C$, then G/N is a nilpotent-by-Černikov group, a contradiction. Hence $B \neq C$ and by Theorem 5.28 of [28] the quotient module C/B is injective. In view of Proposition 2.2 of [30] C/B is a nilpotent-by-Černikov group. By Lemma 2.6 G is a nilpotent-by-Černikov group. □

Lemma 3.3. *Let $G = A \rtimes \langle u\rangle$ be the semidirect product of a normal abelian torsion-free subgroup A and an infinite cyclic subgroup $\langle u\rangle$. If G satisfies Max-$\overline{N\check{C}}$ then it is either a polycyclic group or a nilpotent-by-Černikov group.*

Proof. By Theorem 21.3 of [21] $A = D \times F$ is a group direct product of the divisible part D and a reducible subgroup F. We also assume that G is neither a polycyclic group nor a nilpotent-by-Černikov group. Then G is a locally "nilpotent-by-finite" group.

(1) First, suppose that the divisible part D is trivial. Let B be a p-basic subgroup of F for some prime p. If $B = F$ is a finitely generated subgroup, then G is polycyclic. Assume that $B = F$ is not finitely generated. Then by Lemma 3.1 the quotient group G/B^p is nilpotent-by-Černikov and by Lemma 2.6 so is G, a contradiction.

Let $B \neq F$. If B is not finitely generated, then by Lemma 26.1 of [21] $\overline{B} = B/B^p$ is a p-pure subgroup of $\overline{F} = F/B^p$ and then by Proposition 27.1 of [21] $\overline{F} = \overline{B} \times \overline{C}$ is a group direct product of an infinite abelian subgroup $\overline{B}$ of prime exponent p and a p-divisible subgroup $\overline{C}$. Thus F/F^p is an infinite group and by Lemma 3.1 G/F^p is a nilpotent-by-Černikov group. In view of Lemma 2.6 G is the ones, a contradiction. Hence B is a finitely generated subgroup.

(a) Suppose that the quotient group F/B is torsion and has a non-trivial p-subgroup. Then we can assume that $F/B \cong \mathbb{C}_{p^\infty}$. Therefore $\langle B, u\rangle$ is a nilpotent-by-Černikov subgroup and by Lemma 3.7 of [1, Chapter 3, §5] $\langle B, u\rangle = N \rtimes \langle u\rangle$ for some G-invariant subgroup N of F. Since $\langle B, u\rangle \neq G$, we can assume that $B = N$. Moreover $\langle B, u^m\rangle$ is a nilpotent subgroup for some integer m and therefore B has a $\langle B, u^m\rangle$-invariant subgroup B_1 with the infinite cyclic quotient group B/B_1. Let $\hat{G}_1 = (F \rtimes \langle u^m\rangle)/B_1 = \hat{F} \rtimes \langle \hat{u}^m\rangle$. It is clear that $\hat{F} \cong \mathbb{Q}^{(p)}$. Since $F \rtimes \langle u^m\rangle$ is a non-"nilpotent-by-Černikov" subgroup, $[F, \langle u^m\rangle] \nsubseteq B_1$ in view of Proposition 2.3 and consequently $\hat{u}^m$ induces an automorphism of $\hat{F}$ of infinite order. But as stated in Exercise 5 of [32, §113]

$$\operatorname{Aut}\mathbb{Q}^{(p)} \cong \mathbb{Z}_2 \times \mathbb{Z}$$

and therefore $\hat{G}_1$ is not a locally "nilpotent-by-finite" group, a contradiction.

(b) If F/B is a p'-group, then it has an infinite Sylow q-subgroup of finite index for some prime q. Therefore we can assume that F/B is an infinite q-group. If S/B is a basic subgroup of F/B then, in the same manner as before we can prove that S/B is finitely generated and $F/S \cong \mathbb{C}_{p^\infty}$, a contradiction.

(c) Now suppose that the quotient group F/B is mixed. Without restricting of generality we may assume that F is a p-divisible group. Then $H = \zeta_\alpha G \cdot \langle u\rangle$ is a hypercentral group. Assume that $F/((\zeta_\alpha G)\cap F)$ is a finitely generated group. In view of Proposition 2.3 the quotient group H/H' is finitely generated. Then $H = H'E$ for some finitely generated subgroup E. If $\overline{H} = H/\gamma_{c+2}H$, where c is the nilpotent class of E, then $\overline{H} = \overline{H'}\ \overline{E} = \overline{E}$. This yields that $L = [L, E]$, where $L = \gamma_{c+2}H$, and so

$$H = LE = [L, \langle u\rangle]E. \qquad (**)$$

Since the quotient F/L is a finitely generated abelian p-divisible group, it is finite and consequently $L \rtimes \langle u\rangle$ is a subgroup of finite index in G. In view of our hypothesis and Lemma 2.6 L/L^p is a finite quotient group for all primes p. By (**) L is a divisible group. Lemma 3.2 yields a contradiction.

If the quotient group $F/((\zeta_\alpha G)\cap F)$ is not finitely generated, then without restricting of generelity we can assume that the centre $Z(G)$ is trivial. Let K be any finitely generated subgroup of F. Then $\langle K, u^s\rangle$ is a nilpotent subgroup for some integer s. By X we denote the subgroup $Z(\langle K, u^s\rangle)\cap F$. Then X is a finitely generated $\langle u\rangle$-invariant subgroup and moreover X is a non-trivial right $\mathbb{Z}U$-module, where $U = \langle u\rangle/\langle u^s\rangle$ and the action of U on X is induced by the conjugation of u on X. By Lemma 2.3 of [21] X has a proper $\mathbb{Z}U$-submodule Y such that X/Y is a p-group. It is clear that Y is a finitely generated normal subgroup in G. Then by Theorem 21.3 of [13] $F/Y = T/Y \times M/Y$, where $T/Y \cong \mathbb{C}_{p^\infty}$ and M/Y is some subgroup. If $S = (M/Y)/\tau(M/Y)$ is not finitely generated then G is a nilpotent-by-Černikov group. Therefore S is a finitely generated p-divisible torsion-free group. This means that M/Y is a finite group. Hence $F/Y_1 \cong \mathbb{C}_{p^\infty}$ for some finitely generated G-invariant subgroup Y_1, a contradiction.

(2) Now suppose that the divisible part D is non-trivial. From the part (1) of this proof it follows that G/D is a polycyclic group or G is a nilpotent-by-Černikov group. Assume that G/D is a polycyclic quotient group. Then $\langle F, u\rangle$ is a finitely generated nilpotent-by-finite subgroup and moreover $\langle F, u\rangle = N \rtimes \langle u\rangle$ for some abelian subgroup N of A. Lemma 3.2 yields that $D \rtimes \langle u^s\rangle$ is a nilpotent subgroup for some integer s. Furthermore, the subgroup $N \rtimes \langle u^m\rangle$ is nilpotent for some integer m. Put $k = \min\{m, s\}$. Then $A \rtimes \langle u^k\rangle$ is a nilpotent subgroup of finite index in G, a contradiction. □

Proposition 3.4. *Let $G = NU$ be the product of a normal nilpotent subgroup N and a polycyclic abelian subgroup U. If G satisfies Max-$\overline{N\check{C}}$, then it is of one of the following types:*

(i) G is a polycyclic group;

(ii) G is a nilpotent-by-Černikov group;

(iii) $G = DV$ *is a product of a normal divisible abelian p-subgroup* D *and a polycyclic subgroup* V.

Proof. Suppose that G satisfies Max-$\overline{N\check{C}}$ and G is neither a polycyclic group nor a nilpotent-by-Černikov group. By Lemmas 3.1, 3.3 and 2.2 we can assume that the subgroup N is torsion and consequently N is a π-group for some finite set π of primes. Thus $G = QU_1$, where Q is the infinite Sylow q-subgroup of N for some $q \in \pi$ and U_1 is a polycyclic subgroup. Put $\overline{G} = G/Q'$. Then by Theorem 21.3 of [21] $\overline{Q} = \overline{D} \times \overline{X}$ is a group direct product of the divisible part $\overline{D}$ and a reducible subgroup $\overline{X}$.

(a) First, assume that $\overline{D}$ is trivial. By $\overline{B}$ we denote a basic subgroup of $\overline{X}$. If $\overline{X} = \overline{B}$ and $\overline{B}$ is a finitely generated subgroup, then G is a polycyclic group, a contradiction. Let $\overline{X} = \overline{B}$ and $\overline{B}$ is not a finitely generated subgroup. Then the quotient group $(\overline{X} \rtimes \langle \overline{u} \rangle)/\overline{X}^q$ is nilpotent-by-Černikov for every element $\overline{u}$ of $\overline{U}$ by Lemma 3.1. As consequence of Lemma 2.6 the quotient group $G/Q'X^q$ is polycyclic, a contradiction. Thus $\overline{X} \neq \overline{B}$.

If $\overline{B}$ is not finitely generated then by Lemma 2.7 and Theorem 21.3 of [12] $\overline{X}/\overline{B}^q = \overline{\overline{S}} \times \overline{\overline{B_1}}$ is a group direct product of its divisible part $\overline{\overline{S}}$ and an infinite abelian subgroup $\overline{\overline{B_1}}$ of prime exponent q. This yields that $|\overline{X} : \overline{X}^q| = \infty$.

Then by Lemmas 3.1 and 2.6 the quotient group $\overline{G}/\overline{X}^q$ is polycyclic, a contradiction. Therefore $\overline{B}$ is a finitely generated subgroup and $\overline{X}$ is a finite-by-divisible group, a contradiction with Theorem 1.16 of [1] and Proposition 2.4.

(b) Let $\overline{D}$ be a non-trivial subgroup. By D we denote an inverse image of $\overline{D}$ in G. Lemma 2.6 implies that G/D is a polycyclic quotient group. Put $G_1 = G/Q''D'$ and $Q_1 = Q/Q''D'$. Assume that the quotient group $Y = {Q_1}'/({Q_1}')^q$ is an infinite group. By Lemma 2.7 $Y = A{\cdot}C$ is a product of its non-trivial divisible part A and an infinite abelian $G_1/({Q_1}')^q$-invariant subgroup C of prime exponent q. Applying Lemmas 3.1 and 2.6 we conclude that G is a nilpotent-by-Černikov group, a contradiction. Hence $|{Q_1}' : ({Q_1}')^q| < \infty$. So Y is an abelian group by Theorem 1.16 of [1] and consequently ${Q_1}' = ({Q_1}')^q$ is a divisible abelian q-group. This yields that $D/D'Q''$ is a divisible abelian subgroup of finite index in Q_1. Since $D/D'Q'' \cong (D/Q'')/(D'Q''/Q'')$ and D is a nilpotent subgroup, the quotient group D/Q'' is a divisible abelian group. Moreover $|Q/Q'' : D/Q''| < \infty$ and hence Q/Q'' is a central-by-finite group. This means that the commutator subgroup Q' is finite and therefore D is a divisible abelian q-subgroup, as desired. □

Lemma 3.5. *Let* $G = DU$ *be the product of a normal divisible abelian subgroup* D *and an abelian polycyclic subgroup* U. *If* G *satisfies Max-*$\overline{N\check{C}}$*, then one of the following holds:*

(i) G *is a nilpotent-by-Černikov group;*

(ii) if $u \in U$ *and* $D\langle u \rangle$ *is a non-"nilpotent-by-Černikov" subgroup, then* u *has infinite order,* D *is a p-group for some prime* p *and* $[D, \langle u \rangle] = D$.

Proof. Let u be an element of U such that $D\langle u \rangle$ is not a nilpotent-by-Černikov subgroup. Then by Proposition 3.4 D is a p-group for some prime p. If $[D, \langle u \rangle] \neq D$, then $[D, \langle u \rangle]\langle u \rangle$ is a normal nilpotent-by-Černikov subgroup of G. Thus G is a nilpotent-by-Černikov group in view of Lemma 2.2, a contradiction. □

If G is a non-"nilpotent-by-Černikov" group, by $X_0(G)$ we denote the set

$$\bigcap \{H \mid H \text{ is a non-“nilpotent-by-Černikov” subgroup of } G\}.$$

Theorem 3.6. *Let G be a solvable group. Then G satisfies Max-$\overline{N\check{C}}$ if and only if it is of one of the following types:*

(i) G is a polycyclic group;

(ii) G is a nilpotent-by-Černikov group;

(iii) $G = DW$ is a product of a normal divisible abelian p-subgroup D and a non-torsion polycyclic subgroup W and moreover $X_0(G) = D$.

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Let G satisfies Max-$\overline{N\check{C}}$ and G is neither a polycyclic group nor a nilpotent-by-Černikov. By n we denote the derived length of G. Then there exists an integer k ($0 \leq k \leq n$ and $G^{(0)} = G$) such that $G^{(k)}$ is not a nilpotent-by-Černikov subgroup, but $G^{(k+1)}$ is a nilpotent-by-Černikov. By Proposition 2.3 the quotient group $G/G^{(k+1)}$ is polycyclic. From Proposition 3.4 it follows that $G^{(k+1)}$ has a G-invariant divisible abelian p-subgroup D such that $G^{(k+1)}/D$ is a polycyclic group. If K is any non-"nilpotent-by-Černikov" subgroup of G, then from $DK/(D \cap K) = \overline{D} \rtimes \overline{K}$ it follows that $\overline{D}$ is a trivial subgroup. Hence $X_0(G) \geq D$ and by Mal'cev Theorem (see [22, Proposition 5.4.16]) $X_0(G) = D$. □

REFERENCES

1. Черников С.Н. *Группы с заданными свойствами систем подгрупп*, М.: Наука, 1980, 384 с.
2. Шунков В.П. *Об абстрактных характеризациях некоторых линейных групп,* В кн.: Алгебра. Матрицы и матричные группы, Красноярск, 1970, 3–54.
3. Зайцев Д.И., Курдаченко Л.А. *Группы с условием максимальности для неабелевых подгрупп,* Укр. мат. ж. **43** (1991), no. 7-8, 926–930.
4. Newman M.F., Wiegold J. *Groups with many nilpotent subgroups,* Archiv Math. **15** (1964), 241–250.
5. Heineken H., Mohamed I.J. *A group with trivial centre satisfying the normalizer condition*, J. Algebra **10** (1968), 368–376.
6. Meldrum J.D.P. *On the Heineken-Mohamed groups*, J. Algebra **27** (1973), 437–444.
7. Hartley B. *A note on the normalizer condition,* Proc. Cambr. Phil. Soc. (1973), no. 74, 11–15.
8. Bruno B., Phillips R.E. *On multipliers of Heineken-Mohamed type groups,* Rend. Sem. Mat. Padova **85** (1991), 133–146.
9. Menegazzo F. *Groups of Heineken-Mohamed,* J. Algebra **171** (1995), 807–825.
10. Smith H. *Groups with few non-nilpotent subgroups,* Glasgow Math. J. **39** (1997), 141–151.
11. Беляев В.В. *Локально конечные группы с почти абелевыми собственными подгруппами,* Сиб. мат. ж. **24** (1983), 11–17.
12. Bruno B. *On groups with abelian-by-finite proper subgroups,* Boll. Un. Mat. Ital. **6** (1984), 3-B, 797–807.
13. Bruno B. *Special q-groups and $\mathbb{C}_{p^\infty}$-groups of automorphisms*, Archiv Math. **48** (1987), 15–24.
14. Артемович О.Д. *Неразложимые метабелевы группы*, Укр. мат. ж. **42** (1990), 1252–1254.
15. Bruno B. *On p-groups with "nilpotent-by-finite"proper subgroups*, Boll. Un. Mat. Ital. **7** (1989), 3-A, 45–51.
16. Bruno B., Phillips R.E. *A note on groups with nilpotent-by-finite proper subgroups*, Archiv Math. **65** (1995), 369–374.
17. Artemovych O.D. *On indecomposable groups and groups with hypercentral-by-finite proper subgroups* Publ. Mathematicae (Debrecen) **53** (1998), 163–175.

18. Otal J., Peña J.M. *Groups in which every proper subgroup is Černikov-by-nilpotent or nilpotent-by-Černikov*, Archiv Math. **51** (1988), 193–197.

19. Otal J., Peña J.M. *Nilpotent-by-Černikov CC-groups*, J. Austral Math. Soc. **A53** (1992), 120–130.

20. Napolitani F., Pegoraro E. *On groups with nilpotent-by-Černikov proper subgroups*, Archiv Math. **69** (1997), 89–94.

21. Фукс Л. *Бесконечные абелевы группы. Т. 1*, М.: Мир, 1974, 336 с.

22. Robinson D.J.S. *A course in the theory of groups*, New York e.a.: Springer, 1980, 481 p.

23. Фейс К. *Алгебра: кольца, модули и категории. Т. 1*, М.: Мир, 1972, 688 с.

24. Hartley B. *Periodic locally soluble groups containing an element of prime order with Černikov centralizer*, Quart. J. Math. Oxford. Ser.2 **33** (1982), 309–322.

25. Черников С.Н. *Дополнения к статье "К теории полных групп"* Матем. сб. **22(64)** (1948), 455–456.

26. Asar A.O. *On nonnilpotent p-groups and the normalizer condition,* Tr. J. Math. **18** (1994), 114–129.

27. Warfield R.B. *Nilpotent groups*, Lect. Notes in Math, 513, 1976, 115 p.

28. Kurdachenko L.A., Subbotin I.Ya. *Modules over Dedekind domains*, Los Angeles: National University, 1996, 76 p.

29. Кон П. *Свободные кольца и их связи,* М.: Мир, 1975, 424 с.

30. Matlis E. *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528.

31. Атья М., Макдональд И. *Введение в коммутативную алгебру.* М.: Мир, 1972, 160 с.

32. Фукс Л. *Бесконечные абелевы группы. Т. II*, М.: Мир, 1977, 416 с.

Department of Algebra and Mathematical Logic, Faculty of Mathematics and Mechanics
Kyiv Taras Shevchenko University, Volodymyrs'ka St 64 252033 Kyiv, Ukraine
topos@franko.lviv.ua

Received 20.05.1999