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# AUTOMATIC CONTINUITY OF MULTIPLICATIVE POLYNOMIAL OPERATORS ON BANACH ALGEBRAS

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In this paper we prove the continuity of multiplicative polynomial operators from a complex Banach algebra with unit to a complex uniform Banach algebra.

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Доказана непрерывность мультипликативных полиномиальных операторов из комплексной банаховой алгебры с единицей в комплексную равномерную банахову алгебру.

## 1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach algebras. The basic automatic continuity problem is to give algebraic conditions on $\mathcal{X}$ and $\mathcal{Y}$ which ensure that every homomorphism $F: \mathcal{X} \to \mathcal{Y}$ is necessarily continuous. It is well known that a multiplicative linear homomorphism from a complex Banach algebra $\mathcal{X}$ to $\mathbb{C}$ is continuous. Moreover in [3], [2] it is proved that every homomorphism from a Banach algebra $\mathcal{X}$ into a commutative semisimple Banach algebra $\mathcal{Y}$ is continuous.

The purpose of this paper is an extension of this result onto automatic continuity multiplicative polynomial operators between Banach algebras. Conditions of continuity of polynomial and multilinear operators on topological linear spaces has been investigated by Bochnak and Siciak [1], Drużkowski [4], Fernandes [5], Gajda [6], Plichko and Zagorodnyuk [8].

## 2. Main results

Throughout this note $\mathcal{X}$ and $\mathcal{Y}$ denote the complex Banach algebras with unit, $\mathcal{X}^{-1}$ denotes the group of invertible elements of $\mathcal{X}$. A *homomorphism* $F: \mathcal{X} \to \mathcal{Y}$ is a complex-linear map such that $F(x_1x_2) = F(x_1)F(x_2)$, $x_1, x_2 \in \mathcal{X}$ and $F(1) = 1$. An operator $P: \mathcal{X} \to \mathcal{Y}$ is called a *polynomial operator* of degree $n$ if $P(x) = \sum_{k=0}^{n} P_k(x)$, where $P_k(x)$ are restriction of some $n$-linear symmetric mappings $\bar{P}(x_1, ..., x_n)$ on $\mathcal{X} = \mathcal{X} \times ... \times \mathcal{X}$ ($k$ times) to the diagonal $(x, ..., x)$ (i. e.

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$P_k(x) = \bar{P}_k(x, ..., x))$ and $P_n(x) \not\equiv 0$, $P_0$ is constant in $\mathcal{Y}$. This is equivalent to the fact that for each fixed $x$ and $h$ in $\mathcal{X}$ $P(x + th) = \sum_{k=0}^{n} b_k t^k$, where $t \in \mathbb{C}$, and $b_k \in \mathcal{Y}$ is independ of $t$.

We call an operator $P: \mathcal{X} \to \mathcal{Y}$ to be *multiplicative polynomial operator* if it is a polynomial operator and $P(x_1 x_2) = P(x_1)P(x_2)$ for each $x_1, x_2 \in \mathcal{X}$.

**Theorem 1.** *Let $p: \mathcal{X} \to \mathbb{C}$ be a multiplicative polynomial functional. Then $p$ is continuous.*

*Proof.* Suppose the contrary. Then from Theorem 1 [4] it follows that $\ker p = \{x \in \mathcal{X} : p(x) = 0\}$ is a dense set in $\mathcal{X}$. Hence, there is an element $a \in \mathcal{X}$ such that $||1 - a|| < 1$ and $p(a) = 0$. Thus, the element $a$ is invertible. Indeed, by Theorem 1.2 from [6], the number 1 is contained in the resolvent set of the element $1 - a$ and we see that $a = 1 - (1 - a) \in \mathcal{X}^{-1}$. Therefore $p(1) = p(aa^{-1}) = p(a)p(a^{-1}) = 0$. So $p(x) = p(1)p(x) \equiv 0$. This contradicts the assumption that $p$ is discontinuous. $\square$

Recall that a Banach algebra is called *uniform* if it is commutative and $||y|| = \sup_{l \in M} |l(y)|$, where $y \in \mathcal{Y}$ and $M$ is the set of homomorphisms from $\mathcal{Y}$ to $\mathbb{C}$.

**Theorem 2.** *Let $p: \mathcal{X} \to \mathcal{Y}$ be a multiplicative polynomial operator and $\mathcal{Y}$ a uniform Banach algebra. Then $p$ is continuous and either $||p|| = 1$ or $p \equiv 0$.*

*Proof.* Let $l$ be a homomorphism from $\mathcal{Y}$ to $\mathbb{C}$. Then $l(p(x))$ is a multiplicative polynomial functional. From Theorem 1 it follows that $l(p(x))$ is continuous. Suppose that there is $x \in \mathcal{X}$ such that $||x|| < 1$ and $||p(x)|| > 1$. Since $||p(x)|| = \sup_{l \in M} |l(p(x))|$, there exists a homomorphism $l_0 \in M$ such that $|l_0(p(x))| = c > 1$. So $|l_0(p(x^n))| = |l_0(p(x))|^n = c^n \to \infty$ as $n \to \infty$. On the other hand, $||x^n|| \le ||x||^n \to 0$ as $n \to \infty$. But this contradicts to continuity of $l(p(x))$. This means that $p$ is bounded on the unit ball t and therefore $p$ is continuous (see [1]) and $||p(x)|| \le 1$ if $||x|| < 1$. Then $||p|| = \sup_{||x|| \le 1} ||p(x)|| \le 1$. But if $p \not\equiv 0$ then $p(1) = 1$. Thus, $||p|| = 1$. $\square$

**Example 1.** A finite product of homomorphisms from a Banach algebra $\mathcal{X}$ to a commutative Banach algebra $\mathcal{Y}$ is a multiplicative polynomial operator.

**Example 2.** Let $H$ be a Hilbert space and $V$ be an $n$-dimensional subspace of $H$. Let $\mathcal{X}_V$ denote the subalgebra of algebra $\mathcal{L}(H)$ of linear operators on $H$ such that every $A \in \mathcal{L}(H)$ has decomposition $A = A_1 + A_2$, where $A_1 \in \mathcal{L}(V)$ and $A_2 \in \mathcal{L}(H \ominus V)$. Let us denote by $p$ the polynomial functional on $\mathcal{X}_V$ defined by $p(A) = \det(\mathbb{A}_1)$, where $\mathbb{A}_1$ is the matrix of the linear operator $A_1: V \to V$ in some fixed basis. It is clear that $p$ is multiplicative polynomial.

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