УДК 517.98

ON THE ANALYTICITY OF THE SOLUTIONS OF EVOLUTIONARY EQUATIONS GENERATED BY ELLIPTIC OPERATOR PERTURBATIONS

A.O. Lopushansky

A.O. Lopushansky. *On the analyticity of the solutions of evolutionary equations generated by elliptic operators perturbations*, Matematychni Studii, **12**(1999) 145–148.

Was generalize a theorem of S.Agmon on analytic continuation from the theory of evolutionary equations, generated by elliptic operators, onto classes of perturbed operators, is reduced. The perturbation is described by interpolation of domain of definition of the given elliptic operator and input space.

А.О. Лопушанский. *Об аналитичности решений эволюционных решений, порожденных возмущениями эллиптических операторов* // Математичнi Студiї. – 1999. – Т.12, № 2. – С.145–148.

В работе приведено обобщение теоремы об аналитическом продолжении С. Агмона из теории эволюционных уравнений, порожденных эллиптическими операторами. Возмущение описано путем интерполяции области определения заданного эллиптического оператора и исходного пространства.

Let a *strong elliptic* operator of the order $2m$ $\ell \equiv \sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha$, $a_\alpha(x)\in L^\infty(\Omega)$, $D^\alpha \equiv \frac{1}{i^{|\alpha|}}\frac{\partial^\alpha}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ be given in a bounded domain $\Omega\subset\mathbb{R}^n$ of the class C^∞ such that $\operatorname{Re} a\,(x,\,\xi) > 0$ for all $\xi\in\mathbb{R}^n\backslash\{0\}$ and all $x\in\overline{\Omega}$. For such an operator there exists an angle $\omega_0\in[0\,,\,\pi/2)$ such that $a(x,\,\xi)\neq e^{i\omega}$ for each $\omega\in[\omega_0\,,\,2\pi-\omega_0]$. Moreover, we assume that the coefficients $a_\alpha(x)$ for $|\alpha|=2m$ are continuous in $\overline{\Omega}$.

Above $a\,(x,\,\xi)\equiv\sum_{|\alpha|=2m}a_\alpha(x)\,\xi^\alpha$, $L^\infty(\Omega)$ is the complex space of essentially bounded measurable functions, $|\alpha|=\alpha_1+\ldots+\alpha_n$ is the order of multiindex $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\xi^\alpha=\xi^{\alpha_1}\cdot\ldots\cdot\xi^{\alpha_n}$.

Let differential operators $B_j\equiv\sum_{|\alpha|\le k_j}b_{j,\alpha}(x)D^\alpha$, $b_{j,\alpha}(x)\in C^\infty(\partial\Omega)$, $j=\overline{1,m}$, be defined on the boundary $\partial\Omega$ of the domain Ω, where $C^\infty(\partial\Omega)$ is the complex space of infinitely smooth functions on $\partial\Omega$.

Next, we assume, that $\big\{B_j\big\}_{j=1}^m$ is a *normal system*, that is $0\le k_1<k_2<\ldots<k_m$ and $b_j\,(x,\,\nu_x)\equiv\sum_{|\alpha|=k_j}b_{j,\alpha}(x)\nu_x^\alpha\neq 0$ for all $j=\overline{1,m}$ and for each normal vector ν_x at the point $x\in\partial\Omega$ $\big([1],\ \text{n}^\circ\ 3.7\big)$.

In the Banach space of summable functions $L_p(\Omega)$ $(1<p<\infty)$ we consider the linear operator A, defined by the relation

$$Au=\ell u\,,\qquad\text{where}\quad \mathfrak{D}(A)\equiv\Big\{u(x)\in W_p^{2m}(\Omega):\ \ B_ju|_{\partial\Omega}=0,\quad j=\overline{1,m}\Big\}$$

1991 *Mathematics Subject Classification.* 35A20, 35J99, 47A55.

is its domain of definition and $W_p^{2m}(\Omega)$ is the classical Sobolev space of order $2m$. Operator A is closed over $L_p(\Omega)$ and its domain of definition is dense in $L_p(\Omega)$ $\big($[1], n° 3.8$\big)$. We endow the subspace $\mathfrak{D}(A)$ with the norm of the graph of A so that $\mathfrak{D}(A)$ becomes a Banach space.

We shall describe functional spaces, necessary for construction of perturbations of an operator A. For $0 < s < \infty$, $1 < p < \infty$ the space $B^s_{p,q}(\Omega)$ is defined as the restriction of the corresponding Besov space $B^s_{p,q}(\mathbb{R}^n) \equiv \Big\{f \in S'(\mathbb{R}^n) : f =_{S'} \sum_{j=0}^{\infty} f_j(x)\, ;\ \operatorname{supp} \widehat{f_j} \subset M_j\, ;\ \|f\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty\Big\}$ onto Ω, where $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \equiv \Big[\sum_{j=0}^{\infty} 2^{qsj} \|f_j\|^q_{L_p(\mathbb{R}^n)}\Big]^{1/q}$ for $1 \le q < \infty$ and $\|f\|_{B^s_{p,\infty}(\mathbb{R}^n)} \equiv \sup_j 2^{sj} \|f_j\|_{L_p(\mathbb{R}^n)}$, $S'(\mathbb{R}^n)$ is the space of generalized functions of slow growth, $M_0 \equiv \big\{\xi \in \mathbb{R}^n : |\xi| \le 2\big\}$, $M_j \equiv \big\{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1};\ j \ge 1\big\}$, $\widehat{f_j}$ is the Fourier transform of a function $f_j(x) \in S'(\mathbb{R}^n)$ with support $\operatorname{supp} \widehat{f_j}$. We denote the convergence in $S'(\mathbb{R}^n)$ by "$\underset{S'}{=}$". It is defined by the factor-norm $\|g\|_{B^s_{p,q}(\Omega)} \equiv \inf\big\{\|f\|_{B^s_{p,q}(\mathbb{R}^n)} :\ f \in B^s_{p,q}(\mathbb{R}^n);\ \ f|_{\Omega} = g\big\}$ in the space $B^s_{p,q}(\Omega)$. In particular, $B^s_{p,p}(\Omega) \equiv W^s_p(\Omega)$ are Sobolev spaces for $s \neq$ *integer*.

For arbitrary number θ $(0 < \theta < 1)$ for $1 < p < \infty$ and $1 \le q \le \infty$ in the space $B^{2m\theta}_{p,q}(\Omega)$ consider the closed subspace $B^{2m\theta}_{p,q,B_j}(\Omega) \equiv \Big\{g(x) \in B^{2m\theta}_{p,q}(\Omega) : B_j g_{\,|\,\partial\Omega} = 0\, ;\ \ k_j < 2m\theta - 1/p\, ;\ \ j = \overline{1,m}\Big\}$ and put in correspondence to it the space of linear bounded operators $\mathfrak{L}\big[B^{2m\theta}_{p,q,B_j}(\Omega);\ L_p(\Omega)\big]$. We assume that $T \in \mathfrak{L}\big[B^{2m\theta}_{p,q,B_j}(\Omega); L_p(\Omega)\big]$.

On the space of $\mathfrak{D}(A)$-valued smooth functions $t \to w(t,x) \in C^1\big[\,(0,\infty]; \mathfrak{D}(A)\big]$ we consider the Cauchy problem for the evolutionary equation

$$w'_t(t,x) = -\big(A+T\big)\, w(t,x), \quad \lim_{t\to 0} \|w(t,x) - w(0,x)\|_{L_p(\Omega)} = 0, \quad w(0,x) \in L_p(\Omega),$$

generated by the perturbation of the elliptic operator A by the operator (T) Domain of $\mathfrak{D}(A)$ lies in the space $B^{2m\theta}_{p,q,B_j}(\Omega)$ in which operator T may be given. Therefore the equation in space of functions $w(t,x) \in C^1\big[\,(0,\infty]; \mathfrak{D}(A)\big]$ is correctly defined.

Theorem 1. *Let operator A satisfy the conditions:*

(i) for arbitrary $j = \overline{1,m}$ the inequalities $k_j < 2m - 1/p$ hold;

(ii) for arbitrary $\omega \in [\omega_0\,,\, 2\pi - \omega_0]$ and a tangential in the point $x \in \partial\Omega$ nonzero vector μ_x each polynomial $\mathbb{C} \ni z \to a\,(x,\, \mu_x + z\,\nu_x) - \lambda$, such that complex number λ satisfies the condition $\arg \lambda = \omega$, *has exactly m roots $z_1(\lambda,\mu_x)\,, \dots, z_m(\lambda,\mu_x)$ with positive imaginary part and the polinomials $b_j\,(x,\, \mu_x + z\,\nu_x)$, $(j = \overline{1,m}\,)$ are linearly independent in modulus $\prod_{j=1}^m\, \big[z - z_j(\lambda,\mu_x)\big]$.*

Then for each operator $T \in \mathfrak{L}\big[B^{2m\theta}_{p,q,B_j}(\Omega); L_p(\Omega)\big]$ there exists a unique solution of the Cauchy problem (1) and it admits a $\mathfrak{D}(A)$-valued analytical continuation $z \to w(z,x)$ to some sector of positive axis $\big\{z \in \mathbb{C} :\ |\arg z| < \delta < \pi/2\big\}$.

Proof. It is sufficient to show, that the perturbed operator $-\big(A + T_{\,|\,\mathfrak{D}(A)}\big)$ generates the analytical semigroup $z \to G(z)$ on the space $L_p(\Omega)$ in the sector $\big\{z \in \mathbb{C} : |\arg z| < \delta\big\}$. Then the analytical function $z \to w(z,x) = G(z)w(0,x)$ will be a desired solution of problem (1).

Next, E: $\mathfrak{D}(A) \to L_p(\Omega)$ is the operator enclosure, $(\lambda E - A)^{-1}$: $L_p(\Omega) \to \mathfrak{D}(A)$ is the resolvent, $\rho(A) \equiv \{\lambda \in \mathbb{C} : (\lambda E - A)^{-1} \in \mathfrak{L}[L_p(\Omega); \mathfrak{D}(A)]\}$ the resolvent set (everywhere $\mathfrak{L}[\cdot]$ is a space of linear bounded operators). Since semigroups in the sector $\{z \in \mathbb{C} : |\arg z| < \delta\}$ generated by the operator $-(\beta E + A + T|_{\mathfrak{D}(A)})$ as $\beta \in \mathbb{R}$ and $-(A + T|_{\mathfrak{D}(A)})$ analytical simultaneously, we may assume without loss of generality that $0 \in \rho(A)$.

According to known S. Agmon's theorem ([2]) for strongly elliptic operators ℓ with normal system of boundary operators B_j it follows from condition (ii) of Theorem 1, that there exists a constant $M > 0$ such that for each $\omega \in [\omega_0\,, 2\pi - \omega_0]$ the ray $l_{\omega,M} \equiv \{\lambda : |\lambda| > M, \ \arg \lambda = \omega\}$ is contained in $\rho(A)$ and for all $\lambda \in l_{\omega,M}$ the inequality $\|\lambda E(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} \le M$ holds.

We shall use the identity $\lambda E(\lambda E - A)^{-1} = I_0 + A(\lambda E - A)^{-1}$, where I_0 is the unit operator in $\mathfrak{L}[L_p(\Omega)]$. This equality as well as the above holding in the sector $\{ l_{\omega,M} : \omega \in [\omega_0\,, 2\pi - \omega_0]\}$ imply that for all $\{\lambda : \operatorname{Re}\lambda \le -M\}$ the inequality $\|I_0 + A(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} \le M$ holds. After its estimation we obtain $\big|1 - \|A(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]}\big| \le M$, hence $\|A(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} \le 1 + M$. Applying to this inequality the bounded inverse operator A^{-1} we obtain

$$\|(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]} \le (1 + M)\|A^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]} \quad (\operatorname{Re}\lambda \le -M). \qquad (2)$$

Now let $T \in \mathfrak{L}\big[B^{2m\theta}_{p,q,B_j}(\Omega); L_p(\Omega)\big]$. As P. Grisvard [3] noted, under conditions of normality of system of boundary operators $\{B_j\}_{j=1}^m$ and fulfillment of inequality (i), for each $0 < \theta < 1$ space $B^{2m\theta}_{p,q,B_j}(\Omega)$ coincides with the space $\big(L_p(\Omega), \mathfrak{D}(A)\big)_{\theta,q}$, $(1 \le q \le \infty)$, constructed by one of equivalent methods of real interpolation of pair of spaces $\{L_p(\Omega), \mathfrak{D}(A)\}$. We shall use J-method ([4] n° 3.2). For arbitrary $\tau > 0$ functional $J(\tau, f) = \max\{\|f\|_{L_p(\Omega)};\ \tau\,\|f\|_{\mathfrak{D}(A)}\}$, where $f \in \mathfrak{D}(A)$, defines an equivalent norm in $\mathfrak{D}(A)$. The norm of space $\big(L_p(\Omega), \mathfrak{D}(A)\big)_{\theta,q}$, in terms of $J(\tau, f)$, is expressed by

$$\|f\|_{\theta,q} = \inf_w \Big[\int_0^\infty \tau^{-q\theta} J^q\big(\tau, w(\tau,x)\big)\, d\tau/\tau\Big]^{1/q},$$

where inf is taken with respect to all $\mathfrak{D}(A)$-valued functions $w(\tau, x)$, such that $f(x) = \int_0^\infty w(\tau, x)\tau^{-1}\, d\tau$ in norm $L_p(\Omega)$. As it is known ([4], Theorem 3.2.2), for such a norm the inequality $\|f\|_{\theta,q} \le C\tau^{-\theta} J(\tau, f)$ holds, where the constant C is not dependent of θ and q. From here for all $\tau > 0$, we have $\big(f \in \mathfrak{D}(A)\big)$

$$\|f\|_{\theta,q} \le C \max\{\tau^{-\theta}\|f\|_{L_p(\Omega)};\ \tau^{1-\theta}\|f\|_{\mathfrak{D}(A)}\} \le C\tau^{-\theta}\|f\|_{L_p(\Omega)} + C\tau^{1-\theta}\|f\|_{\mathfrak{D}(A)}. \qquad (3)$$

Denoting by $I: \mathfrak{D}(A) \to B^{2m\theta}_{p,q,B_j}(\Omega)$ the operator injection, we obtain from (3) the inequality $\|I(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega);B^{2m\theta}_{p,q,B_j}(\Omega)]} \le C\tau^{-\theta}\|E(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} + C\tau^{1-\theta}\|(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]}$. Using (2), we get $\|I(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega);B^{2m\theta}_{p,q,B_j}(\Omega)]} \le M_1\big(\tau^{1-\theta} + \tau^{-\theta}|\lambda|^{-1}\big)$ for some constant M_1 and $\operatorname{Re}\lambda \le -M$. Therefore $\|TI(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} \le M_1\big(\tau^{1-\theta} + \tau^{-\theta}|\lambda|^{-1}\big)\|T\|_{\mathfrak{L}[B^{2m\theta}_{p,q,B_j}(\Omega);L_p(\Omega)]}$. From the last inequality it follows that for arbitrary $\varepsilon > 0$ it can be chosen such τ and $M_0 \ge M$ that $\|TI(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} \le \varepsilon\,\|T\|_{\mathfrak{L}[B^{2m\theta}_{p,q,B_j}(\Omega);L_p(\Omega)]}$ for all $\operatorname{Re}\lambda \le -M_0$. Setting $2\varepsilon = \|T\|^{-1}_{\mathfrak{L}[B^{2m\theta}_{p,q,B_j}(\Omega);L_p(\Omega)]}$, we obtain the inequality $\|TI(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} \le 1/2$. Therefore the series $(\lambda E - A - TI)^{-1} = (\lambda E - A)^{-1}\sum_{k=0}^\infty \big[TI(\lambda E - A)^{-1}\big]^k$ is absolutely convergent and estimate $\|(\lambda E - A - TI)^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]} \le 2\,\|(\lambda E - A)^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]}$ is correct. From here and from inequality (2), we have $\|(\lambda E - A - TI)^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]} \le 2(1+M)\|A^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]}$ for all $\{\lambda : \operatorname{Re}\lambda \le -M_0\}$. We shall use the identity $\lambda E(\lambda E - A - TI)^{-1} = I_0 + (A+TI)(\lambda E - A - TI)^{-1}$. It follows from it and the last inequality, that there exists C_0 such that

$$\|E(\lambda E - A - TI)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]} \le C_0\,|\lambda|^{-1} \quad \big(\operatorname{Re}\lambda \le -M\big). \tag{4}$$

We choose $\lambda = \lambda_0$ on the vertical line $\operatorname{Re}\lambda_0 = -M_0$. According to the identity $(\lambda_0+\xi)E - A - TI = (\lambda_0 E - A - TI)\big[I_0 + \xi\,E(\lambda_0 E - A - TI)^{-1}\big]$, the operator from the left hand and $I_0 + \xi\,E(\lambda_0 E - A - TI)^{-1}$ are invertible simultaneously. Therefore for all $\big\{\xi : |\xi| < \|E(\lambda_0 E - A - TI)^{-1}\|^{-1}_{\mathfrak{L}[L_p(\Omega)]}\big\}$ we have $\lambda_0 + \xi \in \rho(A+TI)$. By absolutely summing the series $\big[(\lambda_0+\xi)E - A - TI\big]^{-1} = (\lambda_0 E - A - TI)^{-1}\sum_{k=0}^{\infty}(-\xi)^k\big[E(\lambda_0 E - A - TI)^{-1}\big]^k$, we obtain the inequality $\big\|\big[(\lambda_0+\xi)E - A - TI\big]^{-1}\big\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]} \le \|(\lambda_0 E - A - TI)^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]}\big[1 - |\xi|\,\|E(\lambda_0 E - A - TI)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]}\big]^{-1}$. To estimate $\|E(\lambda_0 E - A - TI)^{-1}\|_{\mathfrak{L}[L_p(\Omega)]}$ we use inequality (4). Then from the last inequality we obtain the upper bound $\big\|\big[(\lambda_0+\xi)E - A - TI\big]^{-1}\big\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]} \le \|(\lambda_0 E - A - TI)^{-1}\|_{\mathfrak{L}[L_p(\Omega);\mathfrak{D}(A)]}\big(1 - C_0\,|\xi|\,|\operatorname{Im}\lambda_0|^{-1}\big)^{-1}$, uniformly in the disk $\{\xi \in \mathbb{C} : |\xi| \le |\operatorname{Im}\lambda_0|/2C_0\}$. Hence, it exists such an angle $\omega_0 \in \big(0,\pi/2\big)$, that $\Lambda \equiv \big\{\lambda_0 + \xi \in \mathbb{C} : \operatorname{Re}\lambda_0 = -M_0;\ |\xi| \le |\operatorname{Im}\lambda_0|/2C_0\big\}$ does not intersect the sector $\big\{\lambda \in \mathbb{C} : |\arg(\lambda - M_0)| < \omega_0\big\}$, which lies in the half-plane $\operatorname{Re}\lambda \ge -M_0$.

Thus, inequality (4) is correct for all $\lambda \in \mathbb{C}\setminus\Lambda$ and we can apply the known criterion (see [1], Theorem 3.3.1) for generator of analytical semigroups. According to it there exists some sector $\big\{z \in \mathbb{C} : |\arg z| < \delta\big\}$ in which operator $-(A+TI)$ generates analytical semigroup $G(z) = e^{-z(A+TI)} \in \mathfrak{L}\big[L_p(\Omega)\big]$. □

Remark 1. Conditions (i)–(ii) of Theorem 1 are fulfilled in the case, when boundary operators are of the type $\big\{B_j\big\}_{j=1}^{m} = \big\{(\partial/\partial\nu_x)^{j-1}\big\}_{j=1}^{m}$ ([1], theorem 3.8.2). Fractional power of operator $T = A^{\theta}$, where $0 < \theta < 1$, may be as example of a perturbation.

Remarks 2. Under the conditions of Theorem 1 the elliptic operator A is regular in the sense of Lopatynsky [5].

REFERENCES

[1] Tanabe H., *Equations of evolution*, Pitman, 1979.

[2] Agmon S., *On the eigenfunctions and on the eigenvalues of elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.

[3] Grisvard P, *Équations operationnelles abstraites et problémes aux limites dans des domaines non réguliers*, Actes. Congrés Intern. Math., 1970. **2** (1971), 731–736.

[4] Bergh J., Löfström J., *Interpolation spaces*, Springer, 1978.

[5] Лопатинский Я.Б., *Об одном способе приведения граничных задач для системы дифференциальных уравнений эллиптического типа к регулярным интегральным уравнениям*, Укр. мат. журн. **5** (1953), 123–151.

State University "Lviv Politechnics"

Received 22.04.1999
Revised 7.07.1999