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ON THE ALGEBRAIC TORI OVER SOME FUNCTION FIELDS

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Using the fact that the global class field theory can be generalized to the case of an algebraic function field in one variable over a pseudofinite constant field (such a field is called pseudoglobal), we show that some theorems on Tate-Shafarevich groups of algebraic tori proved by J. Tate for tori over global fields hold also for tori over pseudoglobal fields.

Besides it is proved that there exists an exact sequence (discovered by V. Voskresenskiy in the case of a global basic field) which connects the Tate-Shafarevich group of a torus defined over a pseudoglobal field and the obstruction to the property of weak approximation of this torus. Some results concerning R-equivalence on an algebraic torus over a global field are transferred to the case of a pseudoglobal basic field.

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Используя тот факт, что глобальная теория полей классов обобщается на случай полей алгебраических функций от одной переменной с псевдоконечными полями констант (такие поля называются псевдоглобальными), мы показываем, что некоторые теоремы, доказанные Дж. Тейтом для торов над глобальными полями, имеют место и для торов над псевдоглобальными полями.

Кроме того, доказано, что существует точная последовательность (открытая В. Воскресенским в случае глобального основного поля), которая связывает групу Тейта-Шафаревича тора, определенного над псевдоглобальным полем, с препятствием к свойству слабой апроксимации этого тора. На случай псевдоглобального основного поля перенесены также некоторые результаты об R-эквивалентности на алгебраическом торе над глобальным полем.

Let K be an algebraic function field in one variable over a pseudofinite [1] constant field k. Recall that a field k is called pseudofinite if k satisfies the following three properties:

1) k is perfect;

2) k has a unique extension of each degree;

3) k is pseudo-algebraically closed, i.e., every absolutely irreducible variety over k has a k-rational point.

We call K *a pseudoglobal* field.

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The pseudofinite constant fields of these pseudoglobal fields were introduced by J. Ax in his paper on the elementary theory of finite fields [1]. These fields are infinite fields which are similar to finite ones. It follows from the Riemann hypothesis for curves over finite fields that every field of non-zero characteristic p which is algebraic over the prime subfield $\mathbb{Z}/p\mathbb{Z}$ and having finite q-primary degree for all prime q (i.e. $[k:\mathbb{Z}/p\mathbb{Z}] = \prod_q q^{v_q}$ with $v_q < \infty$ for all q) and every non-principal ultraproduct of non-isomorphic finite fields are pseudofinite.

The purpose of this paper is to extend some results on algebraic tori over global fields to the case of tori over pseudoglobal fields.

In Section 1 we shall prove, using a result of D.S. Rim and G. Whaples [2], that the global class field theory can be generalized to the pseudoglobal fields.

In Section 2 we shall show that the Tate-Nakayama theorem holds for algebraic tori over pseudoglobal fields. Using this result we shall prove in Section 3 that some theorems on the Tate-Shafarevich groups of algebraic tori proved by J.Tate for tori over global fields hold also for tori over pseudoglobal fields.

In Section 4 we shall establish for algebraic tori over pseudoglobal fields the exact sequence which connects the Tate-Shafarevich group and the obstruction of the weak approximation. In the case of tori over global fields this exact sequence was discovered by V. Voskresenskii [3].

In Section 5 we shall consider the Manin group of norm tori over pseudoglobal fields.

1. On the class field theory for pseudoglobal fields

Let K be an algebraic function field in one variable over a quasifinite constant field k. Let X be a smooth, absolutely irreducible curve over k with function field K, $\bar{k}$ the algebraic closure of k, $G_k = \mathrm{Gal}(\bar{k}/k), \bar{X} = X \times \bar{k}, \mathrm{Pic}\,\bar{X}$ the divisor class group of $\bar{X}$.

Let V be the set of all valuations of K (which are trivial on k). We write K_v for the completion of K under the valuation v and $\mathrm{Br}\,K, \mathrm{Br}\,K_v$ for the Brauer groups of K and K_v respectively. In the sequel $H^i(G,M)$ or $H^i(L/K,M)$ denote the Galois cohomology of a G-module M, where L/K is a Galois extension with $G = \mathrm{Gal}(L/K)$.

D.S. Rim and G. Whaples have shown [2] that there is the following exact sequence

$$0 \longrightarrow H^1(G_k, \mathrm{Pic}\,\bar{X}) \longrightarrow \mathrm{Br}\,K \longrightarrow \textstyle\sum_{v\in V} \mathrm{Br}\,K_v \xrightarrow{\mathrm{inv}_K} \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \quad (1)$$

where $\mathrm{inv}_K = \sum_v \mathrm{inv}_{K_v}$, $\mathrm{inv}_{K_v}\colon \mathrm{Br}\,K_v \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ are the local invariants. They have shown also that if $H^1(G_k, \mathrm{Pic}\,\bar{X}) = 0$ then the reciprocity law holds over k, i.e. for any function field K in one variable over k the norm residue map induces the isomorphism $C_K/N_{L/K}C_L \longrightarrow \mathrm{Gal}(L/K)$ for all finite abelian extensions L/K, where C_K and C_L are the idele class groups of K and L.

Let us first prove that the reciprocity law holds over any pseudofinite field k.

Proposition 1. *Let k be a pseudofinite field. Then, in the above notations,*

$$H^1(G_k, \mathrm{Pic}\,\bar{X}) = 0.$$

Proof. One has the exact sequence $0 \longrightarrow \mathrm{Pic}_0\,\bar{X} \longrightarrow \mathrm{Pic}\,\bar{X} \longrightarrow \mathbb{Z} \longrightarrow 0$ of G_k-modules. It gives us the exact sequence

$$H^1(G_k, \mathrm{Pic}_0\,\bar{X}) \longrightarrow H^1(G_k, \mathrm{Pic}\,\bar{X}) \longrightarrow H^1(G_k, \mathbb{Z}) = 0. \tag{2}$$

Let J be the Jacobian variety of X. The G_k-modules $\mathrm{Pic}_0\,\bar{X}$ and $J(\bar{k})$ are isomorphic, therefore $H^1(G_k, \mathrm{Pic}_0\,\bar{X}) = H^1(G_k, J(\bar{k})) = 0$ because k is pseudo-algebraically closed.

It follows from (2) that $H^1(G_k, \mathrm{Pic}\,\bar{X}) = 0$.

In [2] D.S.Rim and G.Whaples proved also that the condition $H^1(G_k, \mathrm{Pic}\,\bar{X}) = 0$ implies $H^1(G_k, C_L) = 0$ and $\left|H^2(G, C_L)\right| = [L : K]$ for all cyclic extensions L/K, $G = \mathrm{Gal}(L/K)$. Therefore we have as usual (see, e.g., [4], Ch.XIV) that $H^1(G, C_L) = 0$ and $|H^2(G, C_L)|$ divides $[L : K]$ for all Galois extensions.

Our aim now is to prove that $H^2(G, C_L)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, where $n = [L : K]$. For this, we consider the following commutative diagram which was used by J.Tate in [4, Ch.VII] for the global field case.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & 0 \arrow[d] & 0 \arrow[d] & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \arrow[dd, "i"] & 0 \arrow[d] & \\
0 \arrow[r] & H^2(G,L^*) \arrow[r, "\gamma_1"] \arrow[dd, "\mathrm{inf}"] & H^2(G,J_L) \arrow[rr, "\varepsilon_1"] \arrow[ur, "\mathrm{inv}_{L/K}"] \arrow[dd, "\mathrm{inf}"] & & H^2(G,C_L) \arrow[r] \arrow[ul, "\beta_1"'] \arrow[dd, "\mathrm{inf}"] & 0 \\
 & & & \mathbb{Q}/\mathbb{Z} \arrow[dd, "n"] & & \\
0 \arrow[r] & \mathrm{Br}\,K \arrow[r, "\gamma_K"] \arrow[dd, "\mathrm{res}"] & \sum_v \mathrm{Br}\,K_v \arrow[rr, "\varepsilon_K"] \arrow[ur, "\mathrm{inv}_K"] \arrow[dd, "\mathrm{res}"] & & H^2(K,C_{\bar{K}}) \arrow[r] \arrow[ul, "\beta_K"'] \arrow[dd, "\mathrm{res}"] & 0 \\
 & & & \mathbb{Q}/\mathbb{Z} & & \\
0 \arrow[r] & \mathrm{Br}\,L \arrow[r, "\gamma_L"] & \sum_w \mathrm{Br}\,L_w \arrow[rr, "\varepsilon_L"] \arrow[ur, "\mathrm{inv}_L"] & & H^2(L,C_{\bar{L}}) \arrow[r] \arrow[ul, "\beta_L"'] & 0
\end{tikzcd} \tag{3}$$

Here J_L is the idele group of L, w runs through all valuations of L, the homomorphisms β_1, β_K and β_L are induced by $\mathrm{inv}_{L/K}$, inv_K and inv_L, $\bar{K}$ and $\bar{L}$ are the separable closures of K and L respectively.

All the rows, columns and "bent" sequences with γ and inv in (3) are exact. The exactness of "bent" sequences follows from (1) and the above proposition. ε_K and ε_L are surjective because $\mathrm{c.d.}\,K = 2$ [5], β_K and β_L are surjective, because so are inv_K and inv_L. Further, β_K and β_L are injective, because $\mathrm{inv} = \beta\varepsilon$, so β_K and β_L are isomorphisms. It follows that β_1 is an isomorphism and $H^2(G, C_L) \simeq \mathbb{Z}/n\mathbb{Z}$.

Let us denote β_1 again by $\mathrm{inv}_{L/K}$. Two lower rows of (3) remain commutative, if one replace L by K', where K' is an arbitrary separable extension of K. Taking $K \subset K' \subset L$ with L/K Galois we get the following commutative diagram

$$
\begin{tikzcd}
H^2(G, J_L) \arrow[rr, "\varepsilon_2"] \arrow[dr] \arrow[dd, "\mathrm{res}"] & & H^2(G, C_L) \arrow[dl, "\mathrm{inv}_{L/K}"'] \arrow[dd, "\mathrm{res}"] \\
& \mathbb{Z}/mn\mathbb{Z} \arrow[dd, "m"] & \\
H^2(H, J_L) \arrow[rr, "\varepsilon_3"] \arrow[dr] & & H^2(H, C_L) \arrow[dl, "\mathrm{inv}_{L/K'}"'] \\
& \mathbb{Z}/n\mathbb{Z} &
\end{tikzcd}
$$

where $H = \mathrm{Gal}(L/K')$, $m = [K' : K]$, $n = [L : K']$. So $\mathrm{inv}_{L/K'} \mathrm{res} = [K' : K] \,\mathrm{inv}_{L/K}$ and $\mathrm{inv}_{L/K'} = \mathrm{inv}_{L/K} |_{H^2(H, C_L)}$. $\square$

Finally, we have the following theorem.

Theorem 1. *Let K be a pseudoglobal field. Then the idele-classes of K form the class formation.*

Recall that an element $u_{L/K} \in H^2(G, C_L)$ is said to be the fundamental class if $\mathrm{inv}_{L/K} u_{L/K} = 1/n$.

2. Tate-Nakayama theorem for pseudoglobal fields

The preceding theorem allows us to prove the Tate-Nakayama theorem for tori over pseudoglobal fields. First, let us fix some necessary notation which will be used later.

Let T be an algebraic torus over the field K, i.e. an algebraic group isomorphic to $\mathbb{G}_m^d$. We denote by $\hat{T}$ and $\hat{T}^*$ the character group and the cocharacter group of T respectively: $\hat{T} = \mathrm{Hom}(T, \mathbb{G}_m)$, $\hat{T}^* = \mathrm{Hom}(\mathbb{G}_m, T)$. Recall that a torus T is said to be split over an extension L/K, if an isomorphism $T \simeq \mathbb{G}_m^d$ can be defined over L, the field L is said to be the splitting field of T. All the characters of T are defined over any splitting field. One can choose a splitting field L so that L is the finite Galois extension of the basic field K. We denote by A_L the ring of adeles of the pseudoglobal field L. $T(L) = \mathrm{Hom}(\hat{T}, L)$ and $T(A_L) = \mathrm{Hom}(\hat{T}, A_L)$ are the groups of points of T over L and A_L respectively, $C_L(T) = \mathrm{Hom}(T, C_L) \simeq T(A_L)/T(L)$. In what follows we shall denote by $H^n(,)$ the Galois cohomology modified by Tate.

We shall need later both local and global versions of the Tate-Nakayama theorem, so let us formulate the following theorem.

Theorem 2. *Let T be an algebraic torus defined over field K. Suppose that T splits over a finite Galois extension L/K, $G = \mathrm{Gal}(L/K)$. The following holds:*

a) If K is a general local field, then

$$H^n(G, T) \simeq H^{2-n}(G, \hat{T})$$

for all $n \in \mathbb{Z}$.

b) If K *is a pseudoglobal field, then*

$$H^n(G, C_L(T)) \simeq H^{2-n}(G, \hat{T})$$

for all $n \in \mathbb{Z}$.

Proof. The proofs of both versions a) and b) are essentially the same as in the cases of local or global fields respectively (see, e.g. [3] or [6]). For a) we note only that one must take into account that the local class field theory has been generalized for the case of general local fields [5].

We make the proof of b) by imitation of the proof used in [6] for the global field case.

Using Tate theorem [4, Ch. IV] and multiplying by the fundamental class $u_{L/K} \in H^2(G, C_L)$, we obtain $H^n(G, \hat{T}^*) \simeq H^{n+2}(G, \hat{T}^* \otimes C_L)$. Since $\hat{T}^* \otimes C_L$ and $C_L(T)$ are G-isomorphic, we have $H^n(G, \hat{T}^*) \simeq H^{n+2}(G, C_L(T))$. Now, using the duality of $H^n(G, \hat{T}^*)$ and $H^{-n}(G, \hat{T})$ [7], we have $H^{2-n}(G, \hat{T}) \simeq H^n(G, C_L(T))$, as was to be proved. □

3. Tate-Shafarevich group

In this section we shall show that well-known theorems on the Tate-Shafarevich groups of algebraic tori over global fields can be stated for tori over pseudoglobal fields. The proofs we present are essentially those appearing in [6] for the case of global basic field.

For the finite Galois extension L/K of pseudoglobal field K and for the valuation w of L which prolong a valuation v of K, we shall denote by K_v and L_w the corresponding completions of the fields K and L, O_w the valuation ring of the field L_w, $G_w = \mathrm{Gal}(L_w/K_v)$ the decomposition group of w. All the groups G_w are isomorphic for $w \mid v$, so we shall write sometimes G_v instead of G_w. We denote by V the set of all the valuations of K.

Theorem 3. *Let* T *be an algebraic torus defined over a field* K *which splits over finite Galois extension* $L/K, G = \mathrm{Gal}(L/K)$. *Then*

a) The groups $H^n(G, T(L))$ *are finite if* K *is a general local field.*

b) The groups $Ш^n(L/K, T) = \mathrm{Ker}(H^n(G, T(L)) \to \prod_{v\in V} H^n(G_w, T(L_w))$ *are finite if* K *is a pseudoglobal field. Here* w *is one of the valuations which prolong* v.

Proof. a) The character group $\hat{T}$ is a finitely generated G-module. Thus all groups $H^n(G, \hat{T})$ are finite for all $n \in \mathbb{Z}$. It follows from Theorem 2a) that $H^n(G, T(L))$ are finite.

b) The exact sequence

$$1 \longrightarrow T(L) \longrightarrow T(A_L) \longrightarrow C_L(T) \longrightarrow 1$$

yields, on passage to cohomology, the exact sequence

$$H^{n-1}(G, T(A_L)) \xrightarrow{g} H^{n-1}(G, C_L(T)) \longrightarrow H^n(G, T(L)) \xrightarrow{f} H^n(G, T(A_L)), \tag{4}$$

from which we see that $Ш^n(L/K, T)$ is isomorphic to a quotient group of $H^{n-1}(G, C_L(T))$ which is finite by Theorem 2b). Hence $Ш^n(L/K, T)$ are finite for all n. □

Corollary. *The Tate-Shafarevich group*

$$\text{ш}(T) = \operatorname{Ker}(H^1(K,T) \to \prod_{v\in V} H^1(K_v,T))$$

is finite for a torus T defined over a pseudoglobal field.

Proof. We use the fact that for a torus defined over K and splitting over the finite Galois extension L/K, we have $H^1(K,T) \cong H^1(\mathrm{Gal}(L/K),T)$. Then the corollary follows from the preceding theorem. □

To prove the following theorem we shall need two lemmas.

Lemma 1. *In the above notations, we have*

$$H^n(G,T(A_L)) = \sum_{v\in V} H^n(G_w,T(L_w)),$$

where w is some valuation of L which prolong $v \in V$.

Proof. Let S be a finite set of valuations of the field K, S_L the set of valuations of L which prolong the valuations from S. Let $A_L^{S_L}$ be the ring of S_L - adeles, i. e.,

$$A_L^{S_L} = \prod_{w\in S_L} L_w \times \prod_{w\notin S_L} O_w, \quad T(A_L^{S_L}) = \prod_{w\in S_L} T(L_w) \times \prod_{w\notin S_L} T(O_w).$$

$$H^n(G,T(A_L^{S_L})) = H^n(G,\prod_{v\in S}\prod_{w|v} T(L_w)) \times H^n(G,\prod_{v\notin S}\prod_{w|v} T(O_w)) =$$

$$= \sum_{v\in S} H^n(G_w,T(L_w)) \times \prod_{v\notin S} H^n(G_w,T(O_w)).$$

The last equality follows from Shapiro's lemma. Suppose now that S contains all the valuations ramified in L. Then $\prod_{v\notin S} H^n(G_w,T(O_w)) = 0$. Indeed, let U_w be the group of units of O_w, and w is unramified. There is a filtration

$$U_{L_w} = U^0_{L_w} \supset U^1_{L_w} \supset \cdots \supset U^n_{L_w} \supset \cdots,$$

where $U^i_{L_w} = \{a : w(a-1) \geq i\}$. We have $U_{L_w}/U^1_{L_w} \simeq l^*, U^i_{L_w}/U^{i+1}_{L_w} \simeq l^+$ for $i > 0$, where l is the residue field of L_w, moreover, under our assumption, l is the Galois extension of the constant field k with Galois group G_w. Now, $H^n(G_w,l^+) = 0, H^1(G_w,l^*) = 0$ by Hilbert's 90 theorem, $H^2(G_w,l^*) = 0$, because c.d. $k = 1$. Hence $H^n(G_w,U^i/U^{i+1}) = 0$ and it follows that $O^*_w = U_{L_w}$ is G_w-cohomologically trivial, therefore $T(O_w) = \mathrm{Hom}(\hat{T},O_w)$ is G_w-cohomologically trivial as well. The lemma follows by passing to the limit over S. □

Lemma 2. *The diagram*

$$\begin{tikzcd}
H^{n-2}(G_w,\hat{T}^*) \arrow[r, "u_{L_w/K_v}"] \arrow[d, equal] \arrow[dr, phantom, "\boxed{1}"] & H^n(G_w,T(L_w)) \arrow[r, "i"] \arrow[d, "j"] & H^n(G,\prod_{w|v} T(L_w)) \arrow[dd, equal] \\
H^{n-2}(G_w,\hat{T}^*) \arrow[r, "u_{L_w/K_v}"] \arrow[d, "\mathrm{cor}"] \arrow[dr, phantom, "\boxed{2}"] & H^n(G_w,C_L(T)) \arrow[d, "\mathrm{cor}"] \arrow[r, phantom, "\boxed{3}"] & {} \\
H^{n-2}(G,\hat{T}^*) \arrow[r, "u_{L/K}"'] & H^n(G,C_L(T)) & H^n(G,\prod_{w|v} T(L_w)) \arrow[l, "h"']
\end{tikzcd}$$

commutes. The arrows u_{L_w/K_v} and $u_{L/K}$ are induced by multiplication by corresponding fundamental classes, i is the isomorphism from Shapiro's lemma, j is induced by the map $L_w \to C_L$ and $h = \operatorname{cor} \circ j \circ i^{-1}$.

Proof. The lemma follows from the reasoning analogous to that of [6, Ch. VI]. It is obvious that the square $\boxed{1}$ commutes. For the square $\boxed{2}$ this follows from the equality $\operatorname{cor}(\operatorname{res} x \cup y) = x \cup \operatorname{cor} y$ and the square $\boxed{3}$ commutes by construction. □

Theorem 4. *In the notations of Theorem 3 there are the isomorphisms*

$$\text{Ш}^n(L/K, T) \cong \operatorname{Ker}(H^{3-n}(G, \hat{T}) \longrightarrow \prod_{v \in V} H^{3-n}(G_v, \hat{T})).$$

Proof. The exact sequence (4) yields $\text{Ш}^n(L/K, T) = \operatorname{Ker} f = \operatorname{Coker} g$. Using Lemma 1, we obtain that g is a homomorphism

$$\sum_{v \in V} H^{n-1}(G_w, T(L_w)) \longrightarrow H^{n-1}(G, C_L(T)).$$

Consider the commutative diagram

$$\begin{tikzcd}
\sum_{v\in V} H^{n-1}(G_w, T(L_w)) \arrow[r, "g=\sum_v h\circ i"] \arrow[d, "u^{-1}_{L_w/K_v}"] & H^{n-1}(G, C_L(T)) \arrow[d, "u^{-1}_{L/K}"] \\
\sum_{v\in V} H^{n-3}(G_v, \hat{T}^*) \arrow[r, "\sum_v \operatorname{cor}"] \arrow[d] & H^{n-3}(G, \hat{T}^*) \arrow[d] \\
\prod_{v\in V} H^{3-n}(G_v, \hat{T}) & H^{3-n}(G, \hat{T}) \arrow[l, "\operatorname{res}"']
\end{tikzcd}$$

in which the top square commutes by Lemma 2 and the lower square commutes by the duality theorem $H^i(H, \hat{T}) \cong H^{-i}(H, \hat{T}^*)$ (see [7], Ch. XII). We have

$$\text{Ш}^n(L/K, T) \cong \operatorname{Coker} g \cong \operatorname{Ker}(H^{3-n}(G, \hat{T}) \xrightarrow{\operatorname{res}} \prod_{v\in V} H^{3-n}(G_v, \hat{T}))$$

as was to be proved. □

As in the case of tori over global fields, this theorem provides an effective method of calculation of Tate-Shafarevich group for the norm tori.

Theorem 5. *Let $T = R^1_{L/K}(\mathbb{G}_m)$ be a norm torus over a pseudoglobal field K, L/K being the finite Galois extension with Galois group G. The Tate-Shafarevich group $\text{Ш}(T)$ is isomorphic to the kernel of the canonical isomorphism*

$$H^3(G, \mathbb{Z}) \longrightarrow \prod_v H^3(G_v, \mathbb{Z}),$$

where G_v is the decomposition group of v.

Proof. The character group $\hat{T}$ of T can be included in the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \longrightarrow \hat{T} \longrightarrow 0.$$

Therefore, we have $H^n(G, \hat{T}) \cong H^{n+1}(G, \mathbb{Z})$ and in the same way

$$H^n(G_v, \hat{T}) \cong H^{n+1}(G_v, \mathbb{Z}).$$

Using Theorem 4 we obtain $\text{Ш}(T) \cong \operatorname{Ker}(H^3(G, \mathbb{Z}) \longrightarrow \prod_{v\in V} H^3(G_v, \mathbb{Z}))$. □

4. The Voskresenskii exact sequence

First let us recall some necessary notation. Let T be a torus (or more generally an algebraic group), defined over a global or pseudoglobal field K. For every valuation v the completion K_v is endowed with the topology defined by valuation v, therefore, all the groups $T(K_v)$ are topological groups. The group $T(K)$ is included in the direct product $\prod_{v\in S} T(K_v)$ of topological groups $T(K_v)$, where $S \subset V, V$ the set of all the valuations of K. Denote $A(T,S) = \prod_{v\in S} T(K_v)/\overline{T(K)}$, where $\overline{T(K)}$ is the closure of $T(K)$ in $\prod_{v\in S} T(K_v), A(T,V) = A(T)$. $A(T,S)$ is called the obstruction to weak approximation relatively to S. Note that $A(T) = 0$ if and only if $A(T,S) = 0$ for all finite sets $S \subset V$.

There is a canonical exact sequence investigated by V.Voskresenskii [3]

$$0 \longrightarrow \hat{T} \longrightarrow \hat{M} \longrightarrow \operatorname{Pic} V_L(T) \longrightarrow 0 \tag{5}$$

for a torus T defined over an arbitrary field K and splitting over a finite Galois extension L/K with Galois group G. Here $\hat{T}$ is the character group of T, $\hat{M}$ a permutational

G-module, $\operatorname{Pic} V_L(T)$ is characterized by $H^{-1}(H, \operatorname{Pic} V_L(T)) = 0$ for all subgroups $H \subset G$. Note that the notation $\operatorname{Pic} V_L(T)$ reflects the geometric meaning of the module $\operatorname{Pic} V_L(T)$.

Theorem 6. *Let T be a torus defined over a pseudoglobal field K and splitting over a finite Galois extension L/K. Let S be a finite set of valuations of K for which all the decomposition groups are cyclic. Then $A(T,S) = 0$.*

Proof. We follow the reasonings used by Voskresenskii [3] for the case of global ground field. Consider the exact sequence of tori corresponding to (5)

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0. \tag{6}$$

Taking cohomology of decomposition group G_v with coefficients in the exact sequence (6), we obtain

$$M(K_v) \longrightarrow T(K_v) \longrightarrow H^1(G_w, N(L_w)) \longrightarrow H^1(G_w, M(L_w)). \tag{7}$$

We have $H^1(G_w, M(L_w)) = 0$ and $H^1(G_w, N(L_w)) = H^1(G_w, \hat{N})$ by Theorem 2a).

Let S_0 be a finite set of valuations of K with non-cyclic decomposition groups. Using the characterization of $\hat{N} = \operatorname{Pic} V_L(T)$, we obtain

$$H^1(G_w, N(L_w)) = H^1(G_w, \hat{N}) = 0$$

for all $v \notin S_0$.

Thus the exact sequence (7) shows that there is an epimorphism

$$\prod_{v\in S} M(K_v) \longrightarrow \prod_{v\in S} T(K_v) \longrightarrow 0,$$

where S is a finite set of valuations, $S \cap S_0 = \varnothing$.

The group $M(K)$ is rational, therefore $A(M,S) = 0$ and $A(T,S) = 0$. $\square$

Theorem 7. *In the above notation there is the exact sequence*

$$0 \longrightarrow A(T) \longrightarrow (H^1(L/K, \operatorname{Pic} V_L(T)))^* \longrightarrow Ш(T) \longrightarrow 0,$$

where $(H^1(L/K, \operatorname{Pic} V_L(T)))^* = \operatorname{Hom}(H^1(L/K, \operatorname{Pic} V_L(T)), \mathbb{Q}/\mathbb{Z})$.

Proof. Again we follow [3]. The exact sequence (6) yields the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & N(L) \arrow[r] \arrow[d] & M(L) \arrow[r] \arrow[d] & T(L) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N(A_L) \arrow[r] \arrow[d] & M(A_L) \arrow[r] \arrow[d] & T(A_L) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & C_L(N) \arrow[r] \arrow[d] & C_L(M) \arrow[r] \arrow[d] & C_L(T) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Taking cohomology we obtain the commutative diagram with exact rows and columns (we write $H^n(X)$ instead of $H^n(G,X) = H^n(L/K,X)$) :

$$\begin{tikzcd}
H^0(M(A_L)) \arrow[r, "\alpha_1"] \arrow[d, "\gamma_2"] & H^0(T(A_L)) \arrow[r, "\alpha"] \arrow[d, "\gamma_1"] & H^1(N(A_L)) \arrow[r] \arrow[d] & H^1(M(A_L)) = 0 \\
H^0(C_L(M)) \arrow[r, "\beta_1"] \arrow[d] & H^0(C_L(T)) \arrow[r, "\beta"] \arrow[d, "\delta"] & H^1(C_L(N)) \arrow[r] & H^1(C_L(M)) = 0 \\
0 = H^1(M(L)) \arrow[r] \arrow[d] & H^1(T(L)) \arrow[d, "\varepsilon"] & & \\
0 \arrow[r] & H^1(T(A_L)). & &
\end{tikzcd}$$

We have $H^1(G, C_L(M)) = H^1(G, \hat{M}) = 0$ by Theorem 2b), because $\hat{M}$ is a permutational G-module, $H^1(G, C_L(N)) = (H^1(G, \operatorname{Pic} V_L(T)))^*$ by Theorem 2b). $Ш(T) = \operatorname{Ker} \varepsilon = \operatorname{Im} \delta \cong H^0(C_L(T))/\operatorname{Ker} \delta$. However $\operatorname{Im} \beta_1 = \operatorname{Ker} \beta \subset \operatorname{Ker} \delta$, thus there is an epimorphism $f\colon H^1(C_L(N)) \to Ш(T)$,

$$\operatorname{Ker} f = \operatorname{Ker} \delta/\operatorname{Ker} \beta = \operatorname{Im} \gamma_1/\operatorname{Im} \beta_1 = \operatorname{Im} \gamma_1/\operatorname{Im} \beta_1\gamma_2 = \operatorname{Im} \gamma_1/\operatorname{Im} \gamma_1\alpha_1.$$

If S_0 is a finite set of valuations with non-cyclic decomposition groups, then the map $M(K_v) \xrightarrow{\mu} T(K_v)$ is an epimorphism for all $v \notin S_0$ as in the preceding theorem, therefore

$$\operatorname{Ker} f = \prod_{v \in S_0} T(K_v)/\mu(\prod_{v \in S_0} (M(K_v))T(K) = A(T, S_0) = A(T),$$

as was to be proved. □

5. The Manin group

Let A be an algebraic variety over an arbitrary field K such that $A(K) \neq \varnothing$. Yu. Manin [8] introduced the notion of R-equivalence on A and showed its importance for study of algebraic varieties (in particular, cubic surfaces). Let us recall the definition.

Two points $a_1, a_2 \in A(K)$ are said to be strongly R-equivalent, if there exist a rational map $f\colon \mathbb{P}^1_K \to A(K)$ of projective line over K to $A(K)$ and $x_1, x_2 \in \mathbb{P}^1_K$ such that $f(x_1) = a_1, f(x_2) = a_2$. The points $a, b \in A(K)$ are called R-equivalent, if there exists a sequence $a = a_1, a_2, ..., a_n = b$ such that a_i and a_{i+1} are strongly R-equivalent.

If A is a connected linear algebraic group then the R-equivalence is compatible with the group law and, therefore, the quotient group $A(K)/R$ is defined. This latter group is called the Manin group.

The Manin group $T(K)/R$ of an algebraic torus T was investigated by J.-L. Colliot-Thelen and J.-J. Sansuc in [9].

In this section we shall prove that some results on $T(K)/R$ for local or global field K can be extended to the cases where K is a general local or pseudoglobal field. We shall keep the notations of preceding sections. Let us denote $\operatorname{Pic} V_L(T)$ in the canonical exact sequence (5) by $\hat{N}$.

The following theorems are the counterparts of Theorem 14 and Corollary to Theorem 15 in [3, p.204].

Theorem 8. *Let T be a torus over a general local field K and splitting over a Galois extension L/K. Then $T(K)/R \cong H^1(L/K, \operatorname{Pic} V_L(T))$. In particular, $T(K)/R \cong H^3(L/K, \mathbb{Z})$ if $T = R^1_{L/K}(\mathbb{G}_m)$.*

Proof. J.L. Colliot-Thelen and J.-J. Sansuc proved [9] that $T(K)/R \cong H^1(K, N(K_s)) = H^1(L/K, N(L))$, where N is the torus corresponding to $\hat{N}$. By Tate-Nakayama theorem 2a) we have $H^1(L/K, N(L)) = H^1(L/K, \hat{N})$.

If $T = R^1_{L/K}(\mathbb{G}_m)$ then, as was shown in [3, p.157] $H^1(L/K, \hat{N}) = H^3(L/K, \mathbb{Z})$. □

Theorem 9. *Let T be a torus over a pseudoglobal field K, T splits over a finite Galois extension L/K, $G = \operatorname{Gal}(L/K)$, $\hat{N} = \operatorname{Pic} V_L(T)$ is a module from canonical exact sequence (5).*

a) There is the exact sequence

$$0 \longrightarrow Ш(N) \longrightarrow T(K)/R \longrightarrow \textstyle\sum_{v \in S} H^1(G_v, \hat{N}) \longrightarrow H^1(G, \hat{N}),$$

where S is the finite set of valuations v for which the decomposition group G_v is not metacyclic, i.e., its Sylow subgroups are not all cyclic.

b) Let $T = R^1_{L/K}(\mathbb{G}_m)$. Suppose that there exists a valuation v for which $G_v = G$. Then there is the exact sequence

$$0 \longrightarrow T(K)/R \longrightarrow \textstyle\sum_{v \in S} H^3(G_v, \mathbb{Z}) \longrightarrow H^3(G, \mathbb{Z}) \longrightarrow 0,$$

where S is as above. If $S = \varnothing$, then all groups of this sequence are trivial.

Proof. a) Consider the exact sequence

$$0 \longrightarrow N(L) \longrightarrow N(A_L) \longrightarrow C_L(N) \longrightarrow 0.$$

Passing to cohomology, we have

$$0 \longrightarrow \text{Ш}(N) \longrightarrow H^1(L/K, N(L)) \longrightarrow H^1(L/K, N(A_L)) \longrightarrow H^1(L/K, C_L(N)).$$

Applying Lemma 1, Theorem 2 and the theorem of J.L. Colliot-Thelen and J.-J. Sansuc mentioned in the course of the proof of Theorem 8, we obtain the required exact sequence.

b) Under our assumption for all $n \in \mathbb{Z}$ we have

$$\text{Ш}^n(L/K) = \operatorname{Ker}(H^n(L/K, L^*) \longrightarrow H^n(L/K, J_L) = 0.$$

It follows that the sequence

$$0 \longrightarrow H^{-1}(L/K, L^*) \longrightarrow H^{-1}(L/K, J_L) \longrightarrow H^{-1}(L/K, C_L) \longrightarrow 0$$

is exact.

J.L. Colliot-Thelen and J.-J. Sansuc proved [9] that $H^{-1}(L/K, L^*) = T(K)/R$ for $T = R^1_{L/K}(\mathbb{G}_m)$. From Tate-Nakayama theorem we have (w is a valuation of L which prolong $v \in V$)

$$\begin{aligned} H^{-1}(L/K, J_L) &\cong \sum_{v \in V} H^{-1}(L_w/K_v, L_w^*) \cong \sum_{v \in S} H^{-1}(L_w/K_v, L_w^*) \cong \\ &\cong \sum_{v \in S} H^3(L_w/K_v, \hat{L}_v^*) \cong \sum_{v \in S} H^3(L_w/K_v, \mathbb{Z}) \end{aligned}$$

and by the same way $H^{-1}(L/K, C_L) = H^3(L/K, \mathbb{Z})$. This completes the proof. □

We conclude this Section with an example.

Example. Let K be a rational function field over a pseudofinite constant field k, $\operatorname{char} k \neq 2$, $\alpha \in k$, $\alpha \notin k^2$.

Let $T = R^1_{L/K}(\mathbb{G}_m)$, where $L = K(\sqrt{\alpha}, \sqrt{x})$. Then $G = \operatorname{Gal}(L/K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H^3(G, \mathbb{Z}) \cong \mathbb{Z}_2, G_x = \operatorname{Gal}(L_{\sqrt{x}}/K_x) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It follows from Theorem 5, that $\text{Ш}(T) = 0$. It is known [3] that $H^1(G, \operatorname{Pic} V_L(T)) = H^3(G, \mathbb{Z})$. So Theorem 7 says that T is not rational and $A(T) = \mathbb{Z}_2$ and Theorem 9 says that $T(K)/R = 0$.

Now let $T = R^1_{L/K}(\mathbb{G}_m)$ where $L = K(\sqrt{x}, \sqrt{x+1})$. Suppose that $-1 \in k^2$. Then $\sqrt{1+x} \in K_x$ and $\sqrt{x} \in K_{1+x}$. Therefore, all decomposition groups are cyclic and $\prod_v H^3(G_v, \mathbb{Z}) = 0, H^3(G, \mathbb{Z}) = \mathbb{Z}_2$. Then Theorem 5 says that $\text{Ш}(T) = \mathbb{Z}_2$ and Theorem 7 yields that T is not rational and $A(T) = 0$.

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