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LIFTING OF THE HYPERSPACE TRIPLE TO THE CATEGORY OF (STRONGLY) SEMICONVEX COMPACTA

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It is proved that the (strongly) semiconvex hyperspace functor is a unique lifting of the hyperspace functor onto the category of (strongly) semiconvex compacta.

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Доказано, что функтор (сильно) полувыпуклого гиперпространства является единственным поднятием функтора гиперпространства на категорию (сильно) полувыпуклых компактов.

We use the following terminology and notation: $I = [0,1]$ is *a unit segment*, *a compactum* is a (not necessarily metrizable) (bi)compact Hausdorff topological space. *A semiconvex compactum* [4] is a compactum X with a continuous ternary operation $c\colon X \times X \times I \to X$ (we usually call it the semiconvex combination and write $\lambda(x,y)$ instead of $c(x,y,\lambda)$) for which the following axioms hold:

(1) for all $x, y \in X$, $\lambda \in I$: $\lambda(x,y) = (1-\lambda)(y,x)$ (commutative law);

(2) for all $x, y, z \in X$, $\lambda, \mu, \nu \in I$, $\lambda + \mu + \nu = 1$, $\mu \neq 0$:

$$\lambda\Big(x, \frac{\mu}{\mu+\lambda}(y,z)\Big) = (\lambda+\mu)\Big(\frac{\lambda}{\lambda+\mu}(x,y), z\Big)$$

(associative law);

(3) for all $x, y \in X$: $1(x,y) = x$;

(4) there exists a base β of the unique uniformity inducing the topology on X [2] such that $B \in \beta$, $(x,y),(z,t) \in B$, $\lambda \in I$ implies $(\lambda(x,z), \lambda(y,t)) \in B$.

The last axiom is equivalent [4] to the following:

(4′) the topology on X is generated by a saturated family of pseudometrics $(d_\alpha)_{\alpha\in\mathcal{A}}$ such that $x, y, z, t \in X$, $\varepsilon > 0$, $\alpha \in \mathcal{A}$, $d_\alpha(x,y) < \varepsilon$, $d_\alpha(z,t) < \varepsilon$, $\lambda \in I$ implies $d_\alpha(\lambda(x,z), \lambda(y,t)) < \varepsilon$.

Extend the notion of semiconvex combination onto a finite number of elements of X. Let $\lambda_1, \dots, \lambda_n \geq 0$, $\lambda_1 + \dots + \lambda_n = 1$ and $x_1, \dots, x_n \in X$,

$$(\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n) = \begin{cases} x_1, & \text{if } \lambda_1 = 1; \\ \lambda_1(x_1, (\frac{\lambda_2}{1-\lambda_1}, \dots, \frac{\lambda_n}{1-\lambda_1})(x_2, \dots, x_n)), & \text{if } \lambda_1 \neq 1. \end{cases}$$

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The *arguments* $x_1, \dots, x_n$ of semiconvex combination can be permuted (simultaneously with the *coefficients* $\lambda_1, \dots, \lambda_n$) without changing its value.

There exists the largest closed under semiconvex combination closed subset $A \subset X$ such that $\lambda\colon A^2 \to A$ is surjective for any $\lambda \in I$. It is called *the weak center* of X and denoted $W\,\mathrm{Ctr}(X)$. *The center* $\mathrm{Ctr}(X)$ of the semiconvex compactum X is a closed subset consisting of all points $b \in X$ such that $\lambda(b,b) = b$ for any $\lambda \in I$. Always $\mathrm{Ctr}(X) \subset W\,\mathrm{Ctr}(X)$, and

$$W\,\mathrm{Ctr}(X) = \bigcap_{\substack{n\in\mathbb{N},\,\lambda_1,\dots,\lambda_n\in I,\\ \lambda_1+\dots+\lambda_n=1}} (\lambda_1,\dots,\lambda_n)(x_1,\dots,x_n) | x_1,\dots,x_n \in X,$$

$$\mathrm{Ctr}(X) = \bigcap_{\substack{n\in\mathbb{N},\,\lambda_1,\dots,\lambda_n\in I,\\ \lambda_1+\dots+\lambda_n=1}} (\lambda_1,\dots,\lambda_n)(x,\dots,x) | x \in X.$$

The center of X is closed under semiconvex combination and with induced operation becomes a convex compactum. Always $\mathrm{Ctr}(X) \subset W\,\mathrm{Ctr}(X)$. If $\mathrm{Ctr}(X) = W\,\mathrm{Ctr}(X)$, then X is called a strongly semiconvex compactum.

Semiconvex compacta and their continuous mappings which preserve semiconvex combination form a category denoted by $\mathcal{S}Conv$. Strongly semiconvex compacta form a full subcategory $\mathcal{S}s\mathcal{C}onv \subset \mathcal{S}Conv$. By $\mathcal{C}omp$ we denote the category of compacta.

Note that if X is a (strongly) semiconvex compactum, then so is $\exp X$ (the space of nonempty closed subsets of X, endowed with the Vietoris topology, see [3]). If there exists a saturated separating family of continuous pseudometrics $(d_\alpha)_{\alpha\in\mathcal{A}}$ on X, then define a family $(\bar{d}_\alpha)_{\alpha\in\mathcal{A}}$ by the formulae:

$$\bar{d}_\alpha(F,G) = \max(\inf\{\delta > 0 \,|\, \forall x \in F\, \exists y \in G\, d_\alpha(x,y) < \delta\},$$
$$\inf\{\delta > 0 \,|\, \forall y \in G\, \exists x \in F\, d_\alpha(x,y) < \delta\}), \quad F,G \in \exp X.$$

A saturated separating family of continuous pseudometrics on $\exp X$ is obtained. Define the semiconvex combination on $\exp X$ as $\lambda(F,G) = \{\lambda(x,y) \,|\, x \in F, y \in G\}$. Then (4′) holds. Denote $\exp X$ with the defined operation by $\exp_s X$.

It is even easier to show that the product $\prod_{i\in\mathcal{I}} X_i$ of (strongly) semiconvex compacta X_i, $i \in \mathcal{I}$, is a (strongly) semiconvex compactum.

In the sequel we use the denotation I for the unit segment with the usual convex combination. Another semiconvex combination defined by the formula $\lambda(x,y) = \max(\lambda x,\, (1-\lambda)y)$, $x, y \in [0;1]$, makes I a strongly semiconvex compactum which we denote by I^*.

Note that for a continuous mapping of semiconvex compacta $f\colon X \to Y$ that preserves semiconvex combination, $\exp f$ is a continuous and preserving semiconvex combination mapping $\exp_s X \to \exp_s Y$. Thus, $\exp_s$ extends to an endofunctor $\mathcal{S}Conv \to \mathcal{S}Conv$. Since $\exp_s$ preserves the class of strongly semiconvex compacta, $\exp_s|\mathcal{S}s\mathcal{C}onv$ can be considered as an endofunctor in $\mathcal{S}s\mathcal{C}onv$. Denote this restriction by $\exp_{ss}$. Therefore, *liftings* $\exp_s$ and $\exp_{ss}$ of exp from the category of compacta to the categories of (strongly) semiconvex compacta are obtained (this means that $U' \circ \exp_s = \exp \circ U'$, $U \circ \exp_{ss} = \exp \circ U$, U and U' are the underlying functors from $\mathcal{S}Conv$ and $\mathcal{S}s\mathcal{C}onv$ to $\mathcal{C}omp$). The singleton map $sX\colon X \to \exp_s(X)$, $sX(x) = \{x\}$, $x \in X$, preserves the semiconvex combination for a semiconvex compactum X. Thus, it is a component of a natural transformation [1] $s\colon \mathbf{1}_{\mathcal{S}Conv} \to \exp_s$.

Theorem 1. *The functor* $\exp_s$ *is the unique lifting of* $\exp$ *to* $\mathcal{S}Conv$ *such that* s *is a natural transformation from* $\mathbf{1}_{\mathcal{S}Conv}$ *to it.*

Suppose that $\overline{\exp}\colon \mathcal{S}Conv \to \mathcal{S}Conv$ is a lifting that satisfies the conditions of the theorem. Prove that $\overline{\exp} = \exp_s$. We need auxiliary results.

Lemma 1. *Let* F *be a subset of a semiconvex compactum* X*. The least closed subset* $X_0 \subset X$*,* $X_0 \supset F$*, closed under semiconvex combination (i. e., the least semiconvex compactum that contains* F*), coincides with* $\mathrm{Cl}\{(\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n) | \lambda_1, \dots, \lambda_n \leq 0, \lambda_1 + \dots + \lambda_n = 1, x_1, \dots, x_n \in F\}$.

The proof is straightforward. This X_0 is called *the semiconvex hull* of F in X.

Lemma 2. *Let* F, G, H *be closed subsets of a semiconvex compactum* X*,* X_0 *be the semiconvex hull of* $F \times \{1\} \times \{0\} \cup G \times \{0\} \times \{1\} \cup H \times \{0\} \times \{0\}$ *in* $X \times I^* \times I^*$*. Then for any* $\lambda \in (0,1)$ *we have* $X_0 \cap (X \times \{\lambda\} \times \{1-\lambda\} = \{(\lambda(x,y), \lambda, 1-\lambda) \,|\, x \in F, y \in G\}$.

This implies:

Lemma 3. *Suppose* F, G *are closed nonempty subsets of a semiconvex compactum* X*. Then in* $\overline{\exp} X$ *for any* $\lambda \in I$ *we have:* $\lambda(F,G) \subset \{\lambda(x,y) \,|\, x \in F, y \in G\}$.

Proof. If $\lambda = 0$ or $\lambda = 1$, then the statement is obvious. Assume that $\lambda \in (0,1)$. Denote $F' = F \times \{1\} \times \{0\}$, $G' = G \times \{0\} \times \{1\}$, $F, G \subset X \times I^* \times I^*$. Let X_0 be the semiconvex hull of $F' \cup G'$ and $\mathrm{pr}_{23}\colon X_0 \to I^* \times I^*$ the projection. Then $\overline{\exp}\,\mathrm{pr}_{23}(F') = \{(1;0)\}$, $\overline{\exp}\,\mathrm{pr}_{23}(G') = \{(0;1)\}$. By the assumption in $\overline{\exp}(I^* \times I^*)$ we have $\lambda(\overline{\exp}\,\mathrm{pr}_{23}(F'), \overline{\exp}\,\mathrm{pr}_{23}(G')) = \{(\lambda, 1-\lambda)\}$ and therefore $\lambda(F', G') \subset \mathrm{pr}_{23}^{-1}(\lambda, 1-\lambda) = \{(\lambda(x,y), \lambda, 1-\lambda) \,|\, x \in F, y \in G\}$ in $\overline{\exp}(X \times I^* \times I^*)$. If $\mathrm{pr}_1\colon X_0 \to X$ is the projection, then

$$\lambda(F,G) = \lambda(\overline{\exp}\,\mathrm{pr}_1(F'), \overline{\exp}\,\mathrm{pr}_1(G')) = \overline{\exp}\,\mathrm{pr}_1(\lambda(F', G')) \subset \{\lambda(x,y) \,|\, x \in F, y \in G\}. \quad \square$$

Prove the inverse inclusion.

Lemma 4. *Suppose* $x \neq y$*,* $z \neq t$ *are points of a semiconvex compactum* X*,* $\lambda \in I$*. Then in* $\overline{\exp} X$ *we have* $\lambda(\{x,y\}, \{z,t\}) = \{\lambda(x,z), \lambda(x,t), \lambda(y,z), \lambda(y,t)\}$.

Proof. It makes no sense to examine the case $\lambda \in \{0,1\}$. Assume that the inverse holds and all points $\lambda(x,z), \lambda(x,t), \lambda(y,z), \lambda(y,t)$ are distinct. Put $F, G \subset X \times X$, $F = \{(x,y),(y,x)\}$, $G = \{(z,z),(t,t)\}$, $i\colon X \times X \to X \times X$, $i(a,b) = (b,a)$ for any $a, b \in X$. Note that $i(F) = F$, $i(G) = G$. Thus in $\overline{\exp}(X \times X)$: $\overline{\exp}\, i(\lambda(F,G)) = \lambda(\overline{\exp}\, i(F), \overline{\exp}\, i(G)) = \lambda(F,G)$. By previous lemma $\lambda(F,G) \subset \{(\lambda(x,z), \lambda(y,z)), (\lambda(y,z), \lambda(x,z)), (\lambda(x,t), \lambda(y,t)), (\lambda(y,t), \lambda(x,t))\}$ Taking into consideration

$$i((\lambda(x,z), \lambda(y,z))) = (\lambda(y,z), \lambda(x,z)), \quad i((\lambda(y,z), \lambda(x,z))) = (\lambda(x,z), \lambda(y,z)),$$
$$i((\lambda(x,t), \lambda(y,t))) = (\lambda(y,t), \lambda(x,t)), \quad i((\lambda(y,t), \lambda(x,t))) = (\lambda(x,t), \lambda(y,t)),$$

we obtain that $\lambda(F,G)$ can contain or not contain $(\lambda(x,z), \lambda(y,z))$ and $(\lambda(y,z), \lambda(x,z))$ only simultaneously, and the same holds for $(\lambda(x,t), \lambda(y,t))$ and $(\lambda(y,t), \lambda(x,t))$. Applying $\mathrm{pr}_1\colon X \times X \to X$, we see that $\lambda(\{x,y\}, \{z,t\})$ contains or does not contain both points $\lambda(x,z)$ and $\lambda(y,z))$ (just as $\lambda(x,t), \lambda(y,t)$).

In the same way we can prove that $\lambda(\{x,y\}, \{z,t\})$ contains or does not contain both points $\lambda(x,z)$ and $\lambda(x,t))$ (and $\lambda(y,z), \lambda(y,t)$). Since $\lambda(\{x,y\}, \{z,t\})$ is nonempty, $\lambda(\{x,y\}, \{z,t\}) = \{\lambda(x,z), \lambda(x,t), \lambda(y,z), \lambda(y,t)\}$.

To prove this lemma in general case, let Δ^3 be the 3-dimensional simplex with vertices e_0, e_2, e_2, e_3 and usual convex combination. Then by the above in $\overline{\exp}(X \times \Delta^3)$

$$\lambda(\{(x,e_0),(y,e_1)\},\{(z,e_2),(t,e_3)\} =$$
$$= \{(\lambda(x,z),\lambda(e_0,e_2)),(\lambda(x,t),\lambda(e_0,e_3)),(\lambda(y,z),\lambda(e_1,e_2)),(\lambda(y,t),\lambda(e_1,e_3))\}.$$

Applying the projection $\mathrm{pr}_1\colon X \times \Delta^3 \to X$, we obtain the required equality. □

Lemma 5. *Let $x \in F$, $y \in G$, $F, G \in \overline{\exp} X$, $\lambda \in I$. Then $\lambda(x,y) \in \lambda(F,G)$.*

Proof. Put $F' = F \times \{0\} \times \{0\} \times \{0\} \cup \{x\} \times \{1\} \times \{0\} \times \{0\}$, $G' = G \times \{0\} \times \{0\} \times \{1\} \cup \{y\} \times \{0\} \times \{1\} \times \{1\}$, $F', G' \in X \times I^* \times I^* \times I$, and let $X = 0$ be the semiconvex hull of $F' \cup G'$. Then by lemma 2 $X_0 \cap X \times \{\lambda\} \times \{1-\lambda\} \times \{\lambda\} = \{(\lambda(x,y),\lambda,1-\lambda,\lambda)\}$. Denoting by $\mathrm{pr}_1\colon X_0 \to X$ and $\mathrm{pr}_{234}\colon X_0 \to I^* \times I^* \times I$ the projections, by previous lemma we obtain $\lambda(\overline{\exp}\,\mathrm{pr}_{234}(F'), \overline{\exp}\,\mathrm{pr}_{234}(G')) = \lambda(\{(0,0,0),(1,0,0)\},\{(0,0,1),(0,1,1)\} \ni (\lambda, 1-\lambda, \lambda)$, and $\lambda(F',G') \ni (\lambda(x,y),\lambda,1-\lambda,\lambda)$. Applying pr_1, we obtain the required inclusion. □

This implies $\lambda(F,G) \supset \{\lambda(x,y) \mid x \in F, y \in G\}$. Thus, $\lambda(F,G) = \{\lambda(x,y) \mid x \in F, y \in G\}$, $\overline{\exp} = \exp_s$, and the theorem is proved.

Note that if X is a strongly semiconvex compactum, then all constructions used in the proof of the theorem produce strongly semiconvex compacta. Therefore we obtain a counterpart in $\mathcal{S}s\mathcal{C}onv$:

Theorem 2. *The functor $\exp_{ss}$ is the unique lifting of $\exp$ to $\mathcal{S}s\mathcal{C}onv$ such that s is a natural transformation from $\mathbf{1}_{\mathcal{S}s\mathcal{C}onv}$ to it.*

If T is an endofunctor in a category $\mathcal{C}$ and $\eta\colon \mathbf{1}_{\mathcal{C}} \to T$ and $\mu\colon T^2 \equiv TT \to T$ are natural transformations, then $\mathbb{T} = (T,\eta,\mu)$ is called a *triple* if $\mu \circ \eta T = \mu \circ T\eta = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$

For an arbitrary triple $\mathbb{T} = (T,\eta,\mu)$ in $\mathcal{C}$ a pair (X,f), where $f\colon TX \to X$ is a morphism in $\mathcal{C}$, is called a $\mathbb{T}$-*algebra* if $f \circ \eta X = \mathbf{1}_X$, $f \circ \mu X = f \circ Tf$.

An arrow $\varphi\colon X \to Y$ is called *a map of algebras* $(X,f) \to (Y,g)$ if and only if $\varphi \circ f = g \circ T\varphi$.

The algebras of a triple $\mathbb{T}$ in $\mathcal{C}$ and their maps form a category $\mathcal{C}^{\mathbb{T}}$. There exists a pair of adjoint functors $F^{\mathbb{T}}\colon \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ and $U^{\mathbb{T}}\colon \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$, $F^{\mathbb{T}}X = (TX, \mu X)$, $F^{\mathbb{T}}\varphi = T\varphi$, $U^{\mathbb{T}}(X,f) = X$, $U^{\mathbb{T}}\varphi = \varphi$. The triple $\mathbb{T}$ arises from this pair in a way described in [1].

The functor $\exp\colon \mathcal{C}omp \to \mathcal{C}omp$ and natural transformations $s\colon \mathbf{1}_{\mathcal{C}omp} \to \exp$ and $u\colon \exp^2 \to \exp$, s is defined above, $uX(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} F$, produce a *hyperspace triple* $\mathbb{H}$ [3].

If X is a (strongly) semiconvex compactum, then the map $uX\colon {\exp_s}^2 X \to \exp_s X$ preserves the semiconvex combinations. Thus, sX and uX are components of natural transformations $s\colon \mathbf{1}_{\mathcal{S}s\mathcal{C}onv} \to \exp_{ss}$, $u\colon {\exp_{ss}}^2 \to \exp_{ss}$ that form the triple $ss\mathbb{H} = (\exp_{ss}, s, u)$. This triple is a lifting [6] of the hyperspace triple to $\mathcal{S}s\mathcal{C}onv$, and by previous theorem this lifting is unique.

Recall [6] that $(X, c\colon \exp X \to X)$ is an $\mathbb{H}$-algebra if and only if X is a Lawson $\wedge$-semilattice, i.e. a compact complete $\wedge$-semilattice such that the map $c = \inf_X\colon \exp X \to X$ is continuous. The morphisms of $\mathbb{H}$-algebras are continuous complete homomorphisms of $\wedge$-semilattices. Thus, $ss\mathbb{H}$-algebras are *strongly semiconvex Lawson $\wedge$-semilattices*, i. e. Lawson $\wedge$-semilattices that are strongly semiconvex compacta with inf-mappings preserving the semiconvex combination. The morphisms of $ss\mathbb{H}$-algebras are continuous complete homomorphisms of $\wedge$-semilattices

that preserve semiconvex combination. We can describe the $ss\mathbb{H}$-algebras more precisely: (X, c) is an $ss\mathbb{H}$-algebra iff X is a strongly semiconvex compactum that is a complete $\wedge$-semilattice with continuous mapping $c = \inf_X \colon \exp X \to X$, and for any $x, y, z \in X$, $\lambda \in I$ we have $\lambda(x, z) \wedge \lambda(y, z) = \lambda(x \wedge y, z)$.

Suppose that $I \colon \mathcal{B} \to \mathcal{C}$ and $R \colon \mathcal{C} \to \mathcal{B}$ are functors, and $r \colon \mathbf{1}_{\mathcal{B}} \to RI$, $i \colon IR \to \mathbf{1}_{\mathcal{C}}$, $q \colon RI \to \mathbf{1}_{\mathcal{B}}$, $j \colon \mathbf{1}_{\mathcal{C}} \to IR$ are natural transformations such that $r \circ q = \mathbf{1}_{RI}$, $j \circ i = \mathbf{1}_{IR}$. Then the following statements are equivalent:

(1) I is left adjoint to R with unit r and counit i (i. e., $iI \circ Ir = \mathbf{1}_I$, $Ri \circ rR = \mathbf{1}_R$);
(2) R is left adjoint to I with unit j and counit q (i. e., $qR \circ Rj = \mathbf{1}_R$, $Iq \circ jI = \mathbf{1}_I$).

Both statements hold if $\mathcal{C} = \mathcal{S}s\mathcal{C}onv$, $\mathcal{B} = \mathcal{C}onv$, I is the inclusion of full subcategories, $RX = \mathrm{Ctr}(X)$ for a strongly semiconvex compactum X, Rf for an arrow $f \colon X \to Y$ in $\mathcal{S}s\mathcal{C}onv$ is the restriction of f onto the center of X, considered as a map into $\mathrm{Ctr}(Y)$, r and q are identities (because $\mathrm{Ctr}(X) = X$ for a convex compactum X), iX is the inclusion of $\mathrm{Ctr}(X)$ into X, and $jX \colon X \to \mathrm{Ctr}(X)$ is the natural retraction [4]. Moreover, r and q are natural isomorphisms, therefore $iI = Iq$ and $jI = Ir$ are inverse natural isomorphisms (as well as $rR = Rj$ and $qR = Ri$).

In the sequel suppose that for $\mathcal{C}$, $\mathcal{B}$, I, R, i, j, r, q the above hold (so $\mathcal{B}$ has to be reflexive [1] subcategory of $\mathcal{C}$). Let $T \colon \mathcal{C} \to \mathcal{C}$ be a functor. We say that T *preserves reflection* if for any $X \in Ob\,\mathcal{C}$ the arrow iTX can be factored through TiX. This is equivalent to $iTX = TiX \circ TjX \circ iTX$ for any $X \in Ob\,\mathcal{C}$.

Suppose that $\mathbb{T} = (T, \eta, \mu)$ is a triple in $\mathcal{C}$, and T preserves reflection. The functors $F' = F^{\mathbb{T}} I \colon \mathcal{B} \to \mathcal{C}^{\mathbb{T}}$ and $U' = RU^{\mathbb{T}} \colon \mathcal{C}^{\mathbb{T}} \to \mathcal{B}$ are left and right adjoint and produce the triple $\tilde{\mathbb{T}} = (\tilde{T}, \tilde{\eta}, \tilde{\mu})$, $\tilde{T} = RU^{\mathbb{T}} F^{\mathbb{T}} I = RTI$, $\tilde{\eta} = R\eta I \circ r \colon \mathbf{1}_{\mathcal{B}} \to \tilde{T}$, $\tilde{\mu} = R\mu I \circ RTiTI \colon \tilde{T}^2 \to \tilde{T}$. Denote by $\mathcal{B}^{\tilde{\mathbb{T}}}$ the category of $\tilde{\mathbb{T}}$-algebras, and by $F^{\tilde{\mathbb{T}}} \colon \mathcal{B} \to \mathcal{B}^{\tilde{\mathbb{T}}}$ and $U^{\tilde{\mathbb{T}}} \colon \mathcal{B}^{\tilde{\mathbb{T}}} \to \mathcal{B}$ the pair of adjoint functors that arises. We have the Eilenberg-Moore comparison functor [1] $\Phi \colon \mathcal{C}^{\mathbb{T}} \to \mathcal{B}^{\tilde{\mathbb{T}}}$ that makes the diagram

$$\begin{tikzcd}
\mathcal{C}^{\mathbb{T}} \arrow[r, "U^{\mathbb{T}}"] \arrow[ddr, "\Phi"] & \mathcal{C} \arrow[r, "R"] & \mathcal{B} \\
\mathcal{C} \arrow[u, "F^{\mathbb{T}}"] & & \\
\mathcal{B} \arrow[u, "I"] \arrow[rr, "F^{\tilde{\mathbb{T}}}"'] & & \mathcal{B}^{\tilde{\mathbb{T}}} \arrow[uu, "U^{\tilde{\mathbb{T}}}"']
\end{tikzcd}$$

commute. It is easy to check that $\Phi(C, c) = (RC, Rc \circ RTiC \colon RTIRC \to RC)$, $(C, c) \in Ob\,\mathcal{C}^{\mathbb{T}}$, $\Phi f = Rf$, $f \in Ar\,\mathcal{C}^{\mathbb{T}}$.

Put for $(B, b \colon RTIB \to B) \in Ob\,\mathcal{B}^{\tilde{\mathbb{T}}}$ $\Psi(B, b) = (IB, c = Ib \circ jTIB \colon TIB \to IB)$, and for $f \colon (B, b) \to (B', b') \in Ar\,\mathcal{B}^{\tilde{\mathbb{T}}}$ $\Psi f = If \colon \Psi(B, b) \to \Psi(B', b')$.

Theorem 3. *$\Psi \colon \mathcal{B}^{\tilde{\mathbb{T}}} \to \mathcal{C}^{\mathbb{T}}$ is a functor such that $\Phi \circ \Psi \cong \mathbf{1}_{\mathcal{B}^{\tilde{\mathbb{T}}}}$.*

Proof. First prove that $\Psi(B, b) = (IB, c)$ is a $\mathbb{T}$-algebra for $\tilde{\mathbb{T}}$-algebra (B, b). The diagram

$$\begin{tikzcd}
TIB \arrow[r, "jTIB"] & IRTIB \arrow[r, "Ib"] & IB \\
IB \arrow[u, "\eta IB"] \arrow[ur, "I\tilde{\eta}B"] \arrow[urr, "\mathbf{1}_{IB}"] \arrow[r, "jIB=IrB"'] & IRIB \arrow[u, "IR\eta IB"'] &
\end{tikzcd}$$

shows that $c \circ \eta IB = \mathbf{1}_{IB}$. To prove that $c \circ Tc = c \circ \mu IB$, note that $f \circ iX = g \circ iX$, then $f = g$. The diagram below is the proof :

$$\begin{tikzcd}
 & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & IY \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "jX"'] & IY \arrow[ur, "\mathbf{1}_{IY}"] \arrow[dr, "jIY"] & \\
 & IRX \arrow[uu, "iX"] \arrow[r, shift left, "IRf"] \arrow[r, shift right, "IRg"'] & IRIY \arrow[uu, "iIY"']
\end{tikzcd}$$

(recall that $iIY \circ jIY = \mathbf{1}_{IY}$).

Thus prove that

$$\begin{tikzcd}
IRT^2IB \arrow[r, "iT^2IB"] \arrow[d, "iT^2IB"'] & T^2IB \arrow[r, "Tc"] & TIB \arrow[d, "c"] \\
T^2IB \arrow[r, "\mu IB"'] & TIB \arrow[r, "c"'] & IB
\end{tikzcd}$$

commutes. This is a consequence of the commutativity of

$$\begin{tikzcd}
IRT^2IB \arrow[r, "iT^2IB"] \arrow[dr, "iT^2IB"'] & T^2IB \arrow[r, "TjTIB"] & TIRTIB \arrow[r, "TIb"] \arrow[dl, "TiTIB"'] \arrow[d, "jTIRTIB"] & TIB \arrow[d, "jTIB"] \\
 & T^2IB \arrow[d, "\mu IB"] & IRTIRTIB \arrow[r, "IRTIb"] \arrow[d, "IRTiTIB"] & IRTIB \arrow[dd, "Ib"] \\
 & TIB \arrow[dr, "jTIB"'] & IRT^2IB \arrow[d, "IR\mu IB"] & \\
 & & IRTIB \arrow[r, "Ib"'] & IB
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
TIB \arrow[r, "TIf"] \arrow[d, "jTIB"'] & TIB' \arrow[d, "jTIB'"] \\
IRTIB \arrow[r, "IRTIf"] \arrow[d, "Ib"'] & IRTIB' \arrow[d, "Ib'"] \\
IB \arrow[r, "If"'] & IB'
\end{tikzcd}$$

commutes, whenever $b' \circ RTIf = If \circ b$. This means that Ψ works on arrows properly.

Calculate $\Psi\circ\Phi$: $\Psi\circ\Phi(f) = RIf$, $\Psi\circ\Phi(B,b) = (RIB, RIb\circ RjTIB\circ RTiIB)$. Then $rB\colon B\to RIB$ is a component of natural isomorphism $\mathbf{1}_{\mathcal{B}^{\tilde{\mathbb{T}}}}\to\Psi\circ\Phi$:

$$\begin{tikzcd}
RTIB \arrow[r, shift left, "RTIrB"] \arrow[r, shift right, "RTjIB"'] \arrow[dr, "\mathbf{1}_{RTIB}"'] \arrow[dd, "b"'] & RTIRIB \arrow[d, "RTiRIB"] \\
& RTIB \arrow[d, shift right, "rRTIB"'] \arrow[d, shift left, "RjTIB"] \\
& RIRTIB \arrow[d, "RIb"] \\
B \arrow[r, "rB"'] & RIB
\end{tikzcd}$$

□

Consider the full subcategory $C_I^{\mathbb{T}}$ of $\mathcal{C}^{\mathbb{T}}$ with objects (IB,c), $B\in Ob\,\mathcal{B}$. It is easy to check that Φ is faithful on $C_I^{\mathbb{T}}$. Since $\Psi(\mathcal{B}^{\tilde{\mathbb{T}}})$ lies in $C_I^{\mathbb{T}}$, the previous theorem implies that Ψ and $\Phi|C_I^{\mathbb{T}}$ are adjoint equivalences between $\mathcal{B}^{\tilde{\mathbb{T}}}$ and $C_I^{\mathbb{T}}$. This means that $\tilde{\mathbb{T}}$-algebras and their morphisms and $\mathbb{T}$-algebras of the form (IB,c), $B\in\mathcal{B}$, and their morphisms are "the same".

Now return to the case of functors in $\mathcal{S}s\mathcal{C}onv$. Let I, R, i, r, j, q be as above.

Lemma 6. *Let $T\colon\mathcal{S}s\mathcal{C}onv\to\mathcal{S}s\mathcal{C}onv$ be a functor that preserves singleton, preimages and descending intersections, and let the singleton map $sX\colon X\to TX$, $\{sX(x)\} = T(\{x\})\subset TX$, be embedding that preserves semiconvex combination for any $X\in Ob\,\mathcal{S}s\mathcal{C}onv$. Then T preserves reflection.*

Proof. Prove first that if $F,G\subset X$ are closed semiconvex subsets of strongly semiconvex compactum X, $a,b\in TX$, $a\in TF$, $b\in TG$ (this means that $a\in Ti_F(TF)$, $b\in Ti_G(TG)$, $i_F\colon F\hookrightarrow X$, $i_G\colon G\hookrightarrow X$ are embeddings), then for any $\lambda\in I$ we have $\lambda(a,b)\in T(\{\,\lambda(x,y)\,|\,x\in F, y\in G\,\})$. (Note that $\{\,\lambda(x,y)\,|\,x\in F, y\in G\,\}$ is a strongly semiconvex compactum.)

Put $F' = F\times\{0\}, G' = G\times\{1\}\subset X\times I$, and let Z be the semiconvex hull of $F'\cup G'$ in $X\times I$. Then $Z\cap X\times\{\lambda\} = \{\,\lambda(x,y)\,|\,x\in F, y\in G\,\}\times\{\lambda\}$. Denote by $i'_F\colon F\cong F'\hookrightarrow Z$, $i'_G\colon G\cong G'\hookrightarrow Z$ the embeddings, by $\mathrm{pr}_1\colon Z\to X$ and $\mathrm{pr}_2\colon Z\to I$ the projections, and put $a' = Ti'_F(a)$, $b' = Ti'_G(b)$, $a'' = T\,\mathrm{pr}_2(a')$, $b'' = T\,\mathrm{pr}_2(b')$, Then $\{a''\} = T(\{0\}) = sX(0)$, $\{b''\} = T(\{1\}) = sX(1)$, thus $\lambda(a'',b'') = T(\{\lambda\}) = sX(\lambda)$. As T preserves preimages, $\lambda(a',b)\in(T\,\mathrm{pr}_2)^{-1}(T(\{\lambda\})) = T((\mathrm{pr}_2)^{-1}(\{\lambda\}) = T(\{\,\lambda(x,y)\,|\,x\in F, y\in G\,\}\times\{\lambda\})$. Then $\lambda(a,b) = T\,\mathrm{pr}_1(\lambda(a',b')\in T(\{\,\lambda(x,y)\,|\,x\in F, y\in G\,\})$.

This implies that for $a_1,\dots,a_n\in TX$, $\lambda_1,\dots,\lambda_n\in I$, $\lambda_1+\dots+\lambda_n = 1$ we have: $\lambda(a_1,\dots,\lambda_n)\in T(\{\,(\lambda_1,\dots,\lambda_n)(x_1,\dots,x_n)\,|\,x_1,\dots,x_n\in X\,\})$. Since T preserves descending intersections,

$$\mathrm{Ctr}(TX)\subset\bigcap_{\substack{n\in\mathbb{N},\,\lambda_1,\dots,\lambda_n\in I,\\ \lambda_1+\dots+\lambda_n=1}} T((\lambda_1,\dots,\lambda_n)(x_1,\dots,x_n)\ x_1,\dots,x_n\in X) =$$

$$T\Bigg(\bigcap_{\substack{n\in\mathbb{N},\,\lambda_1,\dots,\lambda_n\in I,\\ \lambda_1+\dots+\lambda_n=1}} (\lambda_1,\dots,\lambda_n)(x_1,\dots,x_n)\,|\,x_1,\dots,x_n\in X\Bigg) = T(\mathrm{Ctr}(X)).\quad\square$$

Since for $ss\mathbb{H}$, $\exp_{ss}\colon \mathcal{S}s\mathcal{C}onv \to \mathcal{S}s\mathcal{C}onv$ the conditions of the previous lemma and theorem hold, we can apply them to obtain the triple $c\mathbb{H}$ in $\mathcal{C}onv$. One can check that this triple is of the form (cc, s, u), $\mathrm{cc}\, X$ for a convex compactum X is a subspace of $\exp X$ that consists of all *convex* closed subsets. The mapping $\mathrm{cc}\, f$ is a restriction of $\exp f$ for an affine continuous map f of convex compacta. Then s is the singleton natural transformation, and $uX(F) = \bigcup_{F\in F} F$ for $F \in \mathrm{cc}^2\, X$. The last theorem gives immediately that:

Theorem 4. *A pair (X, f), where X is a convex compactum and $f\colon \mathrm{cc}\, X \to X$ is a continuous affine map, is a $c\mathbb{H}$-algebra if and only if:*

(1) there exists a convex closed partial order $\le$ on X such that for arbitrary $x, y, z \in X$,
$\lambda \in I \setminus \{0\}$ we have: $x \le y \iff \lambda x + (1-\lambda)z \le \lambda y + (1-\lambda)z$;

(2) $(X,\le)$ is a Lawson $\wedge$-semilattice [6] such that continuous map $\inf\colon \exp X \to X$ coincides with $f \circ \mathrm{Cl} \circ \mathrm{conv}$.

Embed X affinely into arbitrary vector space L in a way such that $\mathbf{0}$ coincides with the least element of X. The linear span of X in L is the set of vectors which can be obtained as $\lambda(x-y)$, where $x, y \in X$, $\lambda \in \mathbb{R}$ (not necessary uniquely). Without loss of generality assume that the linear span equals the whole L.

Let $a \le b$ for $a, b \in L$ if and only if $a - b = \lambda(x-y), \lambda > 0$, $x, y \in X$ implies $x \le y$. Obviously it is sufficient to check $x \le y$ for one such x, y, λ. Prove that L then becomes a vector lattice.

Assume $a \le b$, $b \le a$. Then

$$a - b = \lambda_1(x_1 - y_1) = -\lambda_2(x_2 - y_2),\ \lambda_i > 0,\ x_i \le y_i,\ i = 1, 2.$$

Put for $i = 1, 2:\ x_i' = \frac{\lambda_i}{\lambda} x_i,\ y_i' = \frac{\lambda_i}{\lambda} y_i,\ \lambda = \lambda_1 + \lambda_2$. Then

$$\lambda > 0,\ a - b = \lambda(x_1' - y_1') = -\lambda(x_1' - y_1'),\ x_i, y_i \in X,\ x_i' \le y_i',\ i = 1, 2.$$

Thus, $x_1' - y_1' = y_2' - x_2'$.

Put $u = \frac{x_1'+y_2'}{2}$, $v = \frac{x_1'+x_2'}{2} = \frac{y_1'+y_2'}{2}$. We have: $x_1' \le y_1' \iff u \le v$, $x_2' \le y_2' \iff v \le u$. Thus, $u = v$ and therefore $x_1' - y_1' = x_2' - y_2' = \mathbf{0}$ and $a = b$.

Assume then that $a \le b$, $b \le c$. Thus

$$a - b = \lambda_1(x - y),\ b - c = \lambda_2(z - t),\ x, y, z, t \in X, x \le y,\ z \le t,\ \lambda_1, \lambda_2 > 0.$$

Obtain:

$$\frac{a-c}{\lambda_1+\lambda_2} = \frac{\lambda_1 x + \lambda_2 z}{\lambda_1+\lambda_2} - \frac{\lambda_1 y + \lambda_2 t}{\lambda_1+\lambda_2} = (u - v) + (v - w),$$
$$\text{where } u = \frac{\lambda_1 x + \lambda_2 z}{\lambda_1+\lambda_2},\ v = \frac{\lambda_1 x + \lambda_2 t}{\lambda_1+\lambda_2},\ w = \frac{\lambda_1 y + \lambda_2 t}{\lambda_1+\lambda_2}.$$

Thus: $u, v, w \in X$, $u \le v$, $v \le w$ and $u \le w$. Therefore, $a \le c$ and $\le$ is a partial order on L. It is easy to check that the restriction of $\le$ on X coincides with the original partial order. Preservation of the relation under translations and multiplications by non-negative coefficients is obvious.

Prove that L is a vector lattice. Assume

$$x, y, z, t \in X,\ \lambda, \mu \ge 0,\ \lambda + \mu > 0, u = \lambda(x - y),\ v = \mu(z - t).$$

Find $u \wedge v$. Put

$$\theta = \lambda + \mu,\ x' = (\lambda/\theta)x,\ y' = (\lambda/\theta)x,\ z' = (\mu/\theta)x,\ t' = (\mu/\theta)t.$$

Then $x', y', z', t' \in X,\ u = \theta(x' - y'),\ v = \theta(z' - t')$. The infimum is well defined by the formula $u \wedge v = \theta(2(\frac{x'+t'}{2} \wedge \frac{y'+z'}{2}) - y' - t')$. This implies the existence of supremums.

Let $F \in X$ and $x_0 = \inf F$ in X. Prove that $x_0 = \inf F$ in L. Suppose that $a \in L$ and $a \leq x$ for all $x \in F$. Then a equals $\lambda(u - v)$ for some $u, v \in X,\ \lambda > 0$. Thus, $\lambda(u - v) - x \leq 0,\ \frac{\lambda}{1+\lambda}u \leq \frac{\lambda}{1+\lambda}v + \frac{1}{1+\lambda}x$ for all $x \in F$. Therefore,

$$\frac{\lambda}{1+\lambda}u \leq \inf\Big(\frac{\lambda}{1+\lambda}v + \frac{1}{1+\lambda}F\Big) = \frac{\lambda}{1+\lambda}v + \frac{1}{1+\lambda}\inf F = \frac{\lambda}{1+\lambda}v + \frac{1}{1+\lambda}x_0,$$

and $a = \lambda(u - v) \leq x_0$.

Consequently, the embedding of X into L preserves infimums of sets.

Theorem 5. *Any convex Lawson $\wedge$-semilatice X can be affinely embedded into a locally convex vector lattice L as a compact convex subset of positive cone which contains zero the embedding is a complete homomorphism of $\wedge$-semilattices.*

Proof. An appropriate topology on L is the least vector topology which induces the original topology on X. For details see [5]. $\square$

Moreover, existence of such embedding is sufficient for $(X, \inf)$ being a $c\mathbb{H}$-algebra.

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