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ANALYTIC FUNCTIONS OF BOUNDED l-INDEX

V.O. Kushnir, M.M. Sheremeta

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A notion of bounded l-index for an arbitrary domain is introduced and investigated.

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Для произвольной области вводится и исследуется понятие ограниченного l-индекса.

1°. Introduction. Let at first l be a positive, continuous function in $[0,+\infty)$. An entire function f is called of bounded l-index [1] if there exists $N\in\mathbb{Z}_+$ such that

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)}\le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)}:0\le k\le N\right\}. \tag{1}$$

for all $n\in\mathbb{Z}_+$ and $z\in\mathbb{C}$ The least such integer N is called the l-index of f.

If $l(x)\equiv 1$ then (1) gives the definition of the entire function of bounded index introduced by B. Lepson in 1969 [2]. The notion of the entire function of bounded index (bounded l-index) turned out to be rather useful, because, firstly, the derivative of such a function is the function of bounded value distribution (of l-bounded value distribution [1]). Besides, the entire solutions of ordinary differential equations of the specific structure are the functions of bounded index (l-index). We note that together with this properties the entire functions of bounded index (l-index) demonstrate some certain regularity in behaviour of maximum modulus, logarithmic derivative, distribution of zeros ([3, 4, 6] and the bibliography in [3]).

Thereby, the problem of introducing a notion of bounded l-index of analytic in an arbitrary complex domain function f and exploration the properties of such functions arises. So, the aim of this paper is a definition of bounded l-index of analytic in an arbitrary complex domain function f and to explore the properties of analytic in an arbitrary fixed complex domain function f. We notice that in [5] a concept of l-index was introduced in the class of analytic in the disc $\mathbb{D}=\{z:|z|<1\}$ functions and some analogues of theorems from [3, 4] were considered.

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Let G be an arbitrary complex domain, f be an analytic function in G and l be positive, continuous function in G. We assume that there exists $N \in \mathbb{Z}_+$ such that

$$\frac{|f^{(n)}(z)|}{n!l^n(z)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(z)} : 0 \le k \le N\right\} \tag{2}$$

for all $n \in \mathbb{Z}_+$ and $z \in G$. We notice that the concepts introduced in [5] is obtained with $l(z) = l_1\left(\frac{1}{1-|z|}\right)$.

Let us expand the function f in the Taylor series $f(w) = \sum_{n=0}^{\infty} f^{(n)}(z)/n!(w-z)^n$. Since from (2) it follows that $|f^{(n)}(z)|/n! \le K(z)l^n(z)$, where $K(z) = \max\{|f^{(k)}(z)|/(k!l^k(z)) : 0 \le k \le N\}$, the radius of convergence of the series $R \ge 1/l(z)$, and the function f is analytic in every disc $\{w : |w-z| < 1/l(z)\}$. If now $l(z) < (\mathrm{dist}\,(z, \partial G))^{-1}$ for all $z \in G$, then it follows that f can be analytically continued through ∂G. So, henceforth we will consider the case, where the function f has the irregular points on ∂G (perhaps only one point, or perhaps ∂G is the natural border for f), and we require that

$$l(z) > \frac{\beta}{\mathrm{dist}\,(z, \partial G)}, \quad z \in G, \tag{3}$$

where $\beta > 1$ is a fixed number. So, for the continuous, positive in G function l that satisfies condition (3), an analytic in G function f we will be called a function of bounded l-index, if there exists $N \in \mathbb{Z}_+$ such that inequality (2) holds for all $n \in \mathbb{Z}_+$ and $z \in G$. The least such integer N will be called the l-index of f and denoted by $N(f; l)$.

Let us notice that if $G = \mathbb{C}$ and f is an entire function, then the validity of inequality (3) for every positive function $l(z)$ is obvious. From (3) it follows that if $z_0 \in G$, then $\{z : |z - z_0| \le \beta/l(z_0)\} \subset G$. We will often use this fact in the sequel.

For $r \in [0, \beta]$ let

$$\lambda_1(r) = \inf\left\{\frac{l(z)}{l(z_0)} : |z - z_0| \le \frac{r}{l(z_0)},\ z_0 \in G\right\}$$

and

$$\lambda_2(r) = \sup\left\{\frac{l(z)}{l(z_0)} : |z - z_0| \le \frac{r}{l(z_0)},\ z_0 \in G\right\}.$$

It is obvious that $\lambda_1(r) \le 1 \le \lambda_2(r)$. We denote by $Q_\beta(G)$ the class of positive continuous in G functions satisfying (3) and the conditions $0 < \lambda_1(r) \le \lambda_2(r) < +\infty$ for all $r \in [0, \beta]$.

Let us notice that if $l \in Q_\beta(G)$ and $z_0 \in G$, then from the inequality $|z - z_0| \le \frac{r}{l(z_0)}$ it follows

$$\lambda_1(r)l(z_0) \le l(z) \le \lambda_2(r)l(z_0) \tag{4}$$

for all $r \in [0, \beta]$.

Here we prove the analogues of three main criteria of boundedness of l-ÿndex which were obtained for the entire functions in [4].

2°. The first criterion. Let us begin with the following theorem that points to the behaviour of the derivatives of the analytic in G function of bounded l-ÿndex and is essentially used in the proofs of other theorems.

Theorem 1. *Let $\beta > 1$ and $l \in Q_\beta(G)$. An analytic in G function $f(z)$ is of bounded l-index if and only if for all $0 < \eta \le \beta$ there exist numbers $n_0 \in \mathbb{Z}_+$ and $P_0 \ge 1$ such that for all $z_0 \in S$ there exists $k_0 = k_0(z_0) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$ such that*

$$\max\left\{|f^{(k_0)}(z)| : |z - z_0| \le \frac{\eta}{l(z_0)}\right\} \le P_0|f^{(k_0)}(z_0)|. \tag{5}$$

Proof. Let f have l-index $N(f;l) = N < \infty$. Let $q(\eta) = [2\eta(N+1)\lambda_2^{N+1}(\eta)\lambda_1^{-N}(\eta)] + 1$, and for $z_0 \in G$ and $n \in \{0, 1, \dots, q(\eta)\}$ let

$$R_n(z_0,\eta) = \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(z)} : |z - z_0| \le \frac{n\eta}{q(\eta)l(z_0)},\ \ 0 \le k \le N\right\},$$
$$R_n^*(z_0,\eta) = \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(z_0)} : |z - z_0| \le \frac{n\eta}{q(\eta)l(z_0)},\ \ 0 \le k \le N\right\}.$$

Since $|z - z_0| \le \frac{n\eta}{q(\eta)l(z_0)} \le \frac{\eta}{l(z_0)} \le \frac{\beta}{l(z_0)}$, in virtue of condition (3), the values $R_n(z_0,\eta)$ and $R_n^*(z_0,\eta)$ are defined, and in virtue of (4) we have $R_n(z_0,\eta) \le R_n^*(z_0,\eta)\lambda_1^{-N}(\eta)$ and $R_n^*(z_0,\eta) \le R_n(z_0,\eta)\lambda_2^N(\eta)$. Further, repeating literally the reasoning from [4], it is possible to show that $R_n^*(z_0,\eta) \le 2R_{n-1}^*(z_0,\eta)$. From the last three inequalities it follows that $R_n(z_0,\eta) \le 2\lambda_2^N(\eta)\lambda_1^{-N}(\eta)R_{n-1}(z_0,\eta)$. Therefore,

$$\max\left\{\frac{|f^{(k)}(z)|}{k!l^k(z)} : |z - z_0| \le \frac{\eta}{l(z_0)},\ \ 0 \le k \le N\right\} = R_{q(\eta)}(z_0,\eta) \le$$
$$\le 2\lambda_2^N(\eta)\lambda_1^{-N}(\eta)R_{q(\eta)-1}(z_0,\eta) \le (2\lambda_2^N(\eta)\lambda_1^{-N}(\eta))^2 R_{q(\eta)-2}(z_0,\eta) \le \dots$$
$$\le P_1R_0(z_0,\eta) = P_1\max\left\{\frac{|f^{(k)}(z_0)|}{k!l^k(z_0)} : 0 \le k \le N\right\}, \quad P_1 = (2\lambda_2^N(\eta)\lambda_1^{-N}(\eta))^{q(\eta)}.$$

Further, the proof of the necessity is the same as that of the similar statement in [4].

Now suppose that for all η, $0 < \eta \le \beta$ there exist numbers $n_0 \in \mathbb{Z}_+$ and $P_0 \ge 1$ such that for all $z_0 \in G$ there exists k_0, $0 \le k_0 \le n_0$ such that (5) holds. Let $\eta = \beta$ and choose j_0 such that $P_0 \le \beta^{j_0}$. Since, in virtue of condition (3), we have $\{z : |z - z_0| \le \beta/l(z_0)\} \subset G$ for all $z \in G$, hence in virtue of (5) for all $z_0 \in G$ and corresponding $k_0 = k_0(z_0)$

$$\frac{|f^{(k_0+j)}(z_0)|}{j!} \le \frac{l^j(z_0)}{\beta^j}\max\left\{|f^{(k_0)}(z)| : |z - z_0| = \frac{\beta}{l(z_0)}\right\} \le P_0\frac{l^j(z_0)}{\beta^j}|f^{(k_0)}(z_0)|,$$

so

$$\frac{|f^{(k_0+j)}(z_0)|}{(k_0+j)!l^{k_0+j}(z_0)} \le \frac{j!k_0!}{(j+k_0)!}\frac{P_0}{\beta^j}\frac{|f^{(k_0)}(z_0)|}{k_0!l^{k_0}(z_0)} \le$$
$$\le \beta^{j_0-j}\frac{|f^{(k_0)}(z_0)|}{k_0!l^{k_0}(z_0)} \le \frac{|f^{(k_0)}(z_0)|}{k_0!l^{k_0}(z_0)},\ j \ge j_0. \tag{6}$$

Since $k_0 \le n_0$, and the numbers $n_0 = n_0(\beta)$ and $j_0 = j_0(\beta)$ do not depend on z_0, inequality (6) means that $N(f;l) \le n_0 + j_0$. □

The Fricke technique [6] is used in the proofs of two following theorems similarly as in [4].

3°. The second criterion. Maximum modulus of function of bounded l-index.

Theorem 2. *Let $\beta > 1$ and $l \in Q_\beta(G)$. An analytic in G function $f(z)$ is of bounded l-index if and only if for all $0 < r_1 < r_2 \le \beta$ there exists $P_1(r_1, r_2) \ge 1$, such that for all $z_0 \in G$*

$$\max\left\{|f(z)| : |z-z_0| = \frac{r_2}{l(z_0)}\right\} \le P_1(r_1, r_2)\max\left\{|f(z)| : |z-z_0| = \frac{r_1}{l(z_0)}\right\}. \quad (7)$$

Proof. Using Theorem 1 we prove the necessity of condition (7). An analogous statements are proved like in [4] and [6]. Let us prove the adequacy of condition (7). Let $z_0 \in G$ be an arbitrary point. We expand the function f into the power series $f(z) = \sum_{m=0}^{\infty} b_m(z-z_0)^m$, $b_m = f^{(m)}(z_0)/m!$, in the circle $\{z : |z-z_0| \le \beta/l(z_0)\} \subset G$. For $r \le \beta/l(z_0)$ we denote $M(r, z_0, f) = \max\{|f(z)| : |z - z_0| = r\}$, and let $\mu(r, z_0, F) = \max\{|b_m|r^m : m \ge 0\}$ be the maximum term of series (18), and $\nu(r, z_0, f) = \max\{|b_m|r^m : |b_m|r^m = \mu(r, z_0, f)\}$ be its central index. According to the Cauchy inequality $\mu(r, z_0, f) \le M(r, z_0, f)$. On the other hand, if $r \le 1/l(z_0)$ then $M(r, z_0, f) \le \sum_{m=0}^{\infty} |b_m|(\beta r)^m 2^{-m} \le \frac{\beta}{\beta-1}\mu(\beta r, z_0, f)$ and $\ln\mu(\beta r, z_0, f) - \ln\mu(r, z_0, f) = \int_r^{\beta r} \nu(t, z_0, f)t^{-1}dt \ge \nu(r, z_0, f)\ln\beta$, so

$$\begin{aligned}\nu(r, z_0, f) &\le \frac{1}{\ln\beta}(\ln\mu(\beta r, z_0, f) - \ln\mu(r, z_0, f)) \le \\ &\le \frac{1}{\ln\beta}\Big\{\ln M(\beta r, z_0, f) - \ln\Big(\frac{\beta-1}{\beta}M(r/\beta, z_0, f)\Big)\Big\} = \\ &= \frac{\ln\beta - \ln(\beta-1)}{\ln\beta} + \frac{1}{\ln\beta}\{\ln M(\beta r, z_0, f) - \ln M(r/\beta, z_0, f)\}. \quad (9)\end{aligned}$$

Let $N(f; l, z_0)$ be the l-index of function f in z_0, i.e. the least of numbers N for which inequality (2) holds under $z = z_0$. It is easy to see that $N(f; l, z_0) \le \nu(1/l(z_0), z_0, f)$. On the other hand, putting $r_2 = \beta$ and $r_1 = 1/\beta$ in (7) we obtain $M(\beta/l(z_0), z_0, f) \le P_1^* M(1/\beta l(z_0), z_0, f)$, where $P_1^* = P_1(1/\beta, \beta)$. Thus from (9) we obtain the inequality

$$N(f; l, z_0) \le N(\beta) = \frac{\ln\beta - \ln(\beta-1)}{\ln\beta} + \frac{\ln P_1^*}{\ln\beta}$$

for all $z_0 \in G$, hence $N(f; l) \le N(\beta)$. □

From the proof of Theorem 2 we can easily see that the following theorem is valid.

Theorem 2′. *Let $\beta > 1$ and $l \in Q_\beta(G)$. An analytic in G function $f(z)$ is of bounded l-index if and only if there exist $0 < r_1 < r_2 \le \beta$ and $P_1(r_1, r_2) \ge 1$ such that for all $z_0 \in G$ inequality (7) holds.*

4°. The third criterion. Maximum and minimum modulus.

Theorem 3. *Let $\beta > 1$, $l \in Q_\beta$. An analytic in G function $f(z)$ is of bounded l-index if and only if for all $0 < R \le \beta$ there exist $P_2(R) \ge 1$ and $\eta(R) \in (0, R)$ such that for all $z_0 \in G$ there exists $r(z_0) = r \in [\eta(R), R]$ such that*

$$\max\left\{|f(z)|:\,|z-z_0| = \frac{r}{l(z_0)}\right\} \le P_2 \min\left\{|f(z)|:\,|z-z_0| = \frac{r}{l(z_0)}\right\}. \tag{10}$$

Proof. The necessity of condition (10) can be proved literally as in [4]. For the proof of the adequacy, according to Theorem 2′ we have to show that there exists a number P_1 such that

$$\max\left\{|f(z)|:\,|z-z_0| = \frac{\beta+1}{2l(z_0)}\right\} \le P_1 \max\left\{|f(z)|:\,|z-z_0| = \frac{\beta-1}{4\beta l(z_0)}\right\} \tag{11}$$

for all $z_0 \in S$. Let $R = \frac{\beta-1}{4\beta}$. Then there exist $P_2^* = P_2\left(\frac{\beta-1}{4\beta}\right)$ and $\eta = \eta\left(\frac{\beta-1}{4\beta}\right) \in \left(0, \frac{\beta-1}{4\beta}\right)$, such that for all $z^* \in G$ and some $r \in \left[\eta, \frac{\beta-1}{4\beta}\right]$

$$\max\left\{|f(z)|:\,|z-z^*| = \frac{r}{l(z^*)}\right\} \le P_2^* \min\left\{|f(z)|:\,|z-z^*| = \frac{r}{l(z^*)}\right\}. \tag{12}$$

Let us denote $l^* = \max\left\{l(z):\,|z-z_0| \le \frac{\beta}{l(z_0)}\right\}$, $\rho_0 = \frac{\beta-1}{4\beta l(z_0)}$ and $\rho_k = \rho_0 + \frac{k\eta}{l^*}$, $k \in \mathbb{Z}_+$. Since $\frac{\eta}{l^*} < \frac{\beta-1}{4\beta l(z_0)} < \frac{\beta}{l(z_0)} - \frac{\beta+1}{2l(z_0)}$, there exists $n^* \in \mathbb{N}$ that does not depend on z_0 such that $\rho_{n-1} < \frac{\beta+1}{2l(z_0)} \le \rho_n \le \frac{\beta}{l(z_0)}$ for some $n = n(z_0) \le n^*$.

Let $C_k = \{z:\,|z-z_0| = \rho_k\}$, $|f(z_k^{**})| = \max\{|f(z)|:\,z \in C_k\}$, and z_k^* be the point of intersection of the segment $[z_0, z_k^{**}]$ with the circle C_{k-1}. Then for $r < \eta$ $|z_k^{**} - z_k^*| = \eta l^* \le r l(z_k^*)$ and taking into consideration (12) there exists $r \in \left[\eta, \frac{\beta-1}{4\beta}\right]$ such that

$$|f(z_k^{**})| \le \max\left\{|f(z)|:\,|z-z_k^*| = \frac{r}{l(z_k^*)}\right\} \le$$
$$\le P_2^* \min\left\{|f(z)|:\,|z-z_k^*| = \frac{r}{l(z_k^*)}\right\} \le P_2^* \max\left\{|f(z)|:\,z \in C_{k-1}\right\}.$$

Hence

$$\max\left\{|f(z)|:\,|z-z_0| = \frac{\beta+1}{2l(z_0)}\right\} \le \max\left\{|f(z)|:\,z \in C_n\right\} \le$$
$$\le P_2^* \max\left\{|f(z)|:\,z \in C_{n-1}\right\} \le \dots \le (P_2^*)^n \max\left\{|f(z)|:\,z \in C_0\right\} \le$$
$$\le (P_2^*)^{n^*} \max\left\{|f(z)|:\,|z-z_0| = \frac{\beta-1}{4\beta l(z_0)}\right\},$$

i. e. (11) holds with $P_1^* = (P_2^*)^{n^*}$. $\square$

5°. The fifth criterion. The bounded zeros distribution of function of bounded l-index. Let a_k be the zeros of the function f analytic in G,

$$n(r, z_0, 1/f) = \sum_{|a_k - z_0| \le r,\, a_k \in G} 1, \quad G_r = \bigcup_k \{z:|z-a_k| \le r/l(a_k)\}.$$

Theorem 4. *Let $\beta > 1$, $l \in Q_{\beta(G)}$ and $G \setminus G_r \neq \varnothing$. A function f is of bounded l-index if and only if*

1) for all $r \in (0, \beta]$ there exists $P = P(r)$ such that $|f'(z_0)/f(z_0)| \leq Pl(z_0)$ for all $z_0 \in G \setminus G_r$; 2) for all $r \in (0, \beta]$ there exists $\tilde{n} = \tilde{n}(r) \in Z_+$ such that $n(r/l(z_0), z_0, 1/f) \leq \tilde{n}$ for all $z_0 \in G$.

Proof. We prove the necessity of conditions 1) and 2). Firstly, we show that if f is of bounded l-index then for all $z_0 \in G \setminus G_r$ ($r \in (0, \beta]$) and for all $k \in Z_+$

$$|z_0 - a_k| > \frac{r}{2\lambda_2(r)l(z_0)}. \tag{13}$$

Let us assume, on the contrary, that there exist $z_0 \in G \setminus G_r$ and $k \in Z_+$ such that $|z_0 - a_k| \leq r/(2\lambda_2(r)l(z_0)) < r/l(z_0)$. Then from (4) it follows that $l(a_k) \leq \lambda_2(r)l(z_0)$, hence $|z_0 - a_k| \leq r/(2l(a_k)) < r/l(a_k)$ that contradicts to the condition $z_0 \in G \setminus G_r$.

Let us put in Theorem 3 $R = r/(2\lambda_2(r))$. Then there exist $P_2 \geq 1$ and $\eta \in (0, r/2\lambda_2(r))$ such that for all $z_0 \in G$ and some $r^* \in [\eta, r/2\lambda_2(r)]$ (10) holds with r^* instead of r. Therefore, according to the Cauchy inequality

$$|f'(z_0)| \leq (l(z_0)/r^*) \max\{|f(z)| : |z - z_0| = r^*/l(z_0)\} \leq$$
$$\leq (l(z_0)/\eta) P_2 \min\{|f(z)| : |z - z_0| = r^*/l(z_0)\}.$$

But according to inequality (13) for all $z_0 \in G \setminus G_r$ the disc $\{z : |z - z_0| \leq r/(2\lambda_2(r)l(z_0))\}$ does not contain the zeros of function f. So, applying to the function $1/f$ the maximum modulus principle we obtain $|f(z_0)| \geq \min\{|f(z)| : |z - z_0| = r^*/l(z_0)\}$ so $|f'(z_0)/f(z_0)| \leq (P_2/\eta)l(z_0)$, i. e. property 1) is proved with $P_3 = P_2/\eta$.

Now we show that if f is of bounded l-index then there exists $P_4 > 0$ such that for all $z_0 \in G$ and $r \in (0, 1]$

$$n(r/l(z_0), z_0, 1/f) \min\{|f(z)| : |z - z_0| = r/l(z_0)\} \leq$$
$$\leq P_4 \max\{|f(z)| : |z - z_0| = 1/l(z_0)\}. \tag{14}$$

Indeed, according to the Cauchy inequality for all z, $|z - z_0| = 1/l(z_0)$. Using Theorem 2 we obtain

$$|f'(z)| \leq (l(z_0)/(\beta - 1)) \max\{|f(\tau)| : |\tau - z| = (\beta - 1)/l(z_0)\} \leq$$
$$\leq (l(z_0)/(\beta - 1)) \max\{|f(z)| : |z - z_0| = \beta/l(z_0)\} \leq$$
$$\leq (P_1(1, \beta - 1)l(z_0)/(\beta - 1)) \max\{|f(z)| : |z - z_0| = 1/l(z_0)\}.$$

If $f(z) \neq 0$ on the circumference $\{z : |z - z_0| = r/l(z_0)\}$ then

$$n\left(\frac{r}{l(z_0)}, z_0, \frac{1}{f}\right) = \left|\frac{1}{2\pi i} \int\limits_{|z - z_0| = r/l(z_0)} \frac{f'(z)}{f(z)} dz\right| \leq$$
$$\leq \frac{r}{l(z_0)} \max\left\{|f'(z)| : |z - z_0| = \frac{r}{l(z_0)}\right\} \Big/ \min\left\{|f(z)| : |z - z_0| = \frac{r}{l(z_0)}\right\}.$$

From the last two inequalities we obtain

$$n(r/l(z_0), z_0, 1/f) \min\{|f(z)| : |z-z_0| = r/l(z_0)\} \leq$$
$$\leq (1/l(z_0)) \max\{|f'(z)| : |z-z_0| = 1/l(z_0)\} \leq$$
$$\leq (P_1(1,\beta-1)/(\beta-1)) \max\{|f(z)| : |z-z_0| = 1/l(z_0)\},$$

i. e. (14) with $P_4 = P_1(1,\beta-1)/(\beta-1)$. If there are zeros of f on the circle $\{z : |z-z_0| = r/l(z_0)\}$ then inequality (14) is obvious.

Now let us choose $R = 1$ in Theorem 3. Then there exist $P_2 = P_2(1) \geq 1$ and $\eta \in (0,1)$ such that for all $z_0 \in G$ and some $r^* \in (\eta, 1)$

$$\max\left\{|f(z)| : |z-z_0| = \frac{r^*}{l(z_0)}\right\} \leq P_2 \min\left\{|f(z)| : |z-z_0| = \frac{r^*}{l(z_0)}\right\}.$$

Therefore, according to Theorem 2 there exists $P_1(1,\eta)$ such that

$$\max\left\{|f(z)| : |z-z_0| = \frac{1}{l(z_0)}\right\} \leq P_1(1,\eta) \max\left\{|f(z)| : |z-z_0| = \frac{\eta}{l(z_0)}\right\} \leq$$
$$\leq P_1(1,\eta) \max\left\{|f(z)| : |z-z_0| = \frac{r^*}{l(z_0)}\right\} \leq P_1(1,\eta) P_2 \min\left\{|f(z)| : |z-z_0| = \frac{r^*}{l(z_0)}\right\}$$

and according to (14),

$$n(r^*/l(z_0), z_0, 1/f) \min\{|f(z)| : |z-z_0| = r^*/l(z_0)\} \leq$$
$$\leq P_1(1,\eta) P_2 P_4 \max\{|f(z)| : |z-z_0| = 1/l(z_0)\},$$

i. e.

$$n\left(\frac{\eta}{l(z_0)}, z_0, \frac{1}{f}\right) \leq n\left(\frac{r^*}{l(z_0)}, z_0, \frac{1}{f}\right) \leq P_5 =$$
$$= P_1(1,\eta) P_2 P_4 = P_1(1,\eta) P_2(1) P_1(1,\beta-1)/(\beta-1).$$

Now let $r \in (\eta, \beta]$ be an arbitrary number and $l_* = \max\{l(z) : |z-z_0| = r/l(z_0)\}$. Then $l_* \leq \lambda_2(r) l(z_0)$. Let $\rho = r/(\lambda_2(r) l(z_0))$ and $R = r/l(z_0)$. An arbitrary closed disc $\overline{K}$ of radius R can be covered with a finite quantity m of closed discs $\overline{K_j}$ of radius ρ with the centers in $\overline{K}$. Since $\eta/l(z_j) \geq \eta/l_* \geq \eta/(\lambda_2(r) l(z_0))$, in every $\overline{K_j}$ are no more then P_5 zeros of function f. Therefore there are no more then mP_5 zeros of f in $\overline{K}$. Hence, $n(r/l(z_0), z_0, 1/f) \leq \tilde{n}(r) = [mP_5] + 1$, and property 2) is proved.

On the contrary, let conditions 1) and 2) hold. According to 2) for any $R \in (0, \beta]$ there exists $\tilde{n}(R) \in \mathbb{Z}_+$ such that there are no more than $\tilde{n}(R)$ zeros of f in an arbitrary disc $\overline{K} = \{z : |z-z_0| \leq R/l(z_0)\}$. Let $a = a(R) = R\lambda_1(R)/(2(\tilde{n}(R)+1))$. According to condition 1) there exists $P = P(R) \geq 1$ such that $|f'(z)/f(z)| \leq Pl(z)$ for all $z \in G \backslash G_a$, i.e. for all z that does not lie in the discs $\{z : |z-a_n| \leq a(R)/l(a_n)\}$ with the centers $a_n \in \overline{K}$. But $\lambda_1(R) l(z_0) \leq l(a_n)$. Therefore, $|f'(z)/f(z)| \leq Pl(z)$ for all z that does not lie in the discs $\{z : |z-a_n| \leq a(R)/(\lambda_1(R) l(z_0)) = R/(2(\tilde{n}(R)+1) l(z_0))$. The total length of the diameters of these discs does not exceed $R\tilde{n}(R)/((\tilde{n}(R)+1)l(z_0)) < R/l(z_0)$. So there exists the circle $\{z: |z-z_0| = r/l(z_0)\}$ with $R\lambda_1(R)(4(\tilde{n}(R)+1)) = \eta(R) < r < R$, on which $|f'(z)/f(z)| \le Pl(z) \le P\lambda_2(R)l(z_0)$. For the arbitrary points z_1 and z_2 that lie on this circle

$$\ln\left|\frac{f(z_1)}{f(z_2)}\right| \le \int_{z_1}^{z_2}\left|\frac{f'(z)}{f(z)}\right||dz| \le P\lambda_2(R)l(z_0)\frac{2r}{l(z_0)} \le 2R\lambda_2(R)P(R).$$

Hence

$$\max\{|f(z)|: |z-z_0| = r/l(z_0)\} \le P_2 \min\{|f(z)|: |z-z_0| = r/l(z_0)\},$$

where $P_2 = \exp\{2R\lambda_2(R)P(R)\}$, and according to Theorem 2 the function f is of bounded l-index. □

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Department of Mechanics and Mathematics, Lviv State University

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