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ON AREA OF 2-SPHERES IN EUCLIDEAN SPACE

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It is proved that for a 2-sphere embedded in Euclidean 3-space the infimum of the area limits of approximating PL 2-spheres is equal to the Lebesgue area of the 2-sphere.

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Доказано,что для двумерной топологической сферы, вложенной в трехмерное евклидово пространство, точная нижняя грань пределов площадей полиэдральных сфер, аппроксимирующих топологическую сферу, равна площади Лебега топологической сферы.

1. Introduction. Suppose F is a continuous piecewise linear map of a boundary B of a 3-simplex into the Euclidean space $\mathbb{R}^3$. Let $e(F)$ be the sum of areas of faces obtained under the linear maps of separate pieces of B by F. If $F\colon B \to \mathbb{R}^3$ is a continuous (not piecewise linear) map, the value $L(F) = \inf \varliminf_{n\to\infty} e(F_n)$, where the infimum is taken over the set of all sequences $\{F_n\}$ of piecewise linear continuous maps $F_n\colon B \to \mathbb{R}^3$ converging uniformly to F, is called the Lebesgue area of F [1, p. 37].

If $H\colon B \to \mathbb{R}^3$ is an embedding, it is natural to define the area of H by means of sequences of piecewise linear embeddings. Let $A(H) = \inf \varliminf_{n\to\infty} e(H_n)$, where the infimum is taken over the set of all sequences $\{H_n\}$ of piecewise linear embeddings $H_n\colon B \to \mathbb{R}^3$ converging uniformly to H (such sequences exist in view of Bing's approximation theorem [2]). It is clear that $L(H) \le A(H)$. If H' is another homeomorphism of B onto the 2-sphere $S = H(B)$, then $L(H) = L(H')$ and $A(H) = A(H')$; so we can speak of the Lebesgue area $L(S) = L(H)$ and the area $A(S) = A(H)$ of S; $L(S) \le A(S)$. We prove the following result.

Theorem 1.1. *If S is a 2-sphere in $\mathbb{R}^3$, then $L(S) = A(S)$.*

Some relations between the areas $A(S)$ and $L(S)$ (but under some restrictions on S) were obtained in [3] and [4]. Since $L(S) \le A(S)$, Theorem 1.1 should be proved for the case $L(S) < \infty$. In Section 5 we shall construct polyhedral 2-spheres with areas differing from $L(S)$ by a quantity as small as we please and approximating S

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to within any desired precision. For this purpose we first derive some properties of surfaces of finite Lebesgue area and then use the results of Bing [2].

2. Preliminaries. 2.1. Level sets of continuous functions. Let f be a real continuous function defined on a 2-sphere S. The set of all $x \in S$ such that $f(x) = t$ is called the level $E_t(f)$ of f (or t-level). If there would not be confusion, we denote $E_t = E_t(f)$.

First we cite after [5, §1] some topological properties of the E_t's similar to the properties of t-levels of functions defined on the standard 2-sphere.

The E_t's are closed sets for all $t \in f(S)$ and $E_t \cap E_{t'} = \varnothing$ if $t \neq t'$. A component K of E_t is called regular if K separates S into two parts, otherwise it is called irregular. The regular irreducible components of E_t are called completely regular. The regular component K of E_t separating S into two domains S_1 and S_2 is called quasiextremal if there are two continua P_1 and P_2 meeting K and both S_1 and S_2 respectively such that either $f(x) \leq t$ for all $x \in P_1 \cup P_2$ or $f(x) \geq t$ for all $x \in P_1 \cup P_2$.

Theorem 2.1 ([5, Theorem 3]). *If f is a real continuous function defined on S, then for all but at most a countable subset t of $f(S)$ each E_t contains either*

1) completely regular components, or

2) together with every irregular component nonseparating S, E_t contains also an infinite collection of completely regular components;

and at the same time E_t contains neither

3) extremal (nonstrictly) points of f, nor

4) completely regular quasiextremal components.

Theorem 2.2. *Every locally connected completely regular component of the t-level of a continuous function on S is a simple closed curve.*

Theorem 2.2 is a consequence of Theorem VII of [6] which asserts that a locally connected irreducible bounded separator of a plane between two points is a simple closed curve.

As for irregular components of E_t, they either do not separate S at all or do separate S into more than two parts (the last components may occur on at most countable collection of t-levels). A component K of E_t nonseparating S is called a component of concentric singularity if for each neighborhood U of K there is a component K' of E_t inside U that separates K from the complement of U (with respect to S). A component K of E_t nonseparating S is called a component of semiextremum if there is a continuum P, $P \cap K \neq \varnothing$, $P \setminus K = P \setminus E_t \neq \varnothing$, such that either $f(x) \geq t$ for all $x \in P$ (then K is a semiminimum component) or $f(x) \leq t$ for all $x \in P$ (then K is a semimaximum component).

Theorem 2.3 ([5, Theorem 1]). *Each component K of E_t nonseparating S is 1) a concentric singularity component, or 2) a semimaximum component, or 3) a semiminimum component, moreover, cases 1), 2), and 3) exclude one another.*

Metric properties of t-levels of functions defined on S are somewhat different from the properties of t-levels given in [5, Part I, §4] for functions defined on the standard 2-sphere.

A length of a set is the outer one-dimensional Hausdorff measure of the set unless the contrary is indicated.

Lemma 2.1. *Suppose S is a 2-sphere and $\beta > 0$. Then there is $\delta > 0$ such that none of two disjoint continua on S each of diameter larger than β are separated on S by a continuum of a diameter less than δ.*

Lemma 2.2. *Suppose f is a continuous function defined on S such that a continuum nonseparating S is a concentric singularity component of a t-level. Then the sum of lengths of t-level components separating S is unbounded.*

Lemma 2.3. *Suppose f is a continuous function such that there is an uncountable subcollection of t-levels each containing a continuum nonseparating S as its semiextremum component. Then the sum of lengths of t-level components separating S is unbounded for all but at most a countable subcollection of the collection of the t-levels.*

Theorem 2.4. *Suppose f is a continuous function defined on S and T is a subset of $f(S)$ such that for each t of T the sum of lengths of t-level components separating S is finite. Then for all but at most a countable number of elements t of T no t-levels contain continua as irregular components (that is all the irregular components of the t-levels are single points).*

We sketch proofs of last three lemmas and the theorem. The stronger variants of them were proved by Kronrod in case S is a square or a standard 2-sphere (see Lemmas 20, 21, 25, and Theorem 10 of [5]).

Lemma 2.1 is obvious.

Let a continuum K be the concentric singularity component of the t-level in Lemma 2.2. Then there exists a sequence of t-level components separating S and contracting to K. Lemma 2.2 follows from this.

Proof of Lemma 2.3. Let T be an uncountable subset of $f(S)$ such that each E_t, $t \in T$, contains some continuum (denote it by C_t) being the component of semimaximum (if C_t is a component of semiminimum, the proof will be similar). Since $\operatorname{diam} C_t > 0$, there is $\beta > 0$ and an uncountable subset T_1 of T such that for all $t \in T_1$ we have $\operatorname{diam} C_t \geq 3\beta/2$. We use $U_\alpha(C_t)$, $\alpha > 0$, to denote an α-neighborhood of C_t in S. Let T_2 be a subset of T_1 such that for any $\varepsilon > 0$ and t of T_2 the collection of C_t, $C_t \in U_\varepsilon(C_\tau)$, and that of C_τ, $C_\tau \subset U_\varepsilon(C_t)$, both for $\tau > t$ and for $\tau < t$ are uncountable. According to Lemma 23 of [5], $T_1 \setminus T_2$ is at most countable. We have also $\operatorname{diam} C_\tau \geq \beta$ if ε is rather small. We choose any t of T_2 and $\alpha > 0$. Let a continuum C_τ, $\tau > t$, be a component of semimaximum of τ-level such that $C_\tau \subset U_\alpha(C_t)$ and $\operatorname{diam} C_\tau \geq \beta$. There is a component K_t of E_t separating C_t and C_τ in $U_\alpha(C_t)$ (otherwise C_t would not be the semimaximum component of E_t). According to Lemma 2.1, there is $\delta > 0$ such that $\operatorname{diam} K_t \geq \delta$. Since $\alpha > 0$ may be taken as small as we please, Lemma 2.3 is proved.

Theorem 2.4 is a consequence of Lemmas 2.2 and 2.3.

2.2. The length of the boundary of an open set and the Cavalieri inequality (after the results of Cesari [1, §§19, 20]). Suppose $F\colon B \to \mathbb{R}^3$ is a continuous map of the boundary B of a 3-simplex into $\mathbb{R}^3$ and G is a simply connected domain on B. We fix one of two cyclic orderings in the set of prime ends of the boundary ∂G. Let $x_1, \dots, x_n, x_{n+1} = x_1$, be a finite collection of points in ∂G which are accessible from G and cyclicly ordered on ∂G.

The quantity

$$l_\partial(F(\partial G)) = \sup \sum_{i=1}^{n} |F(x_i) - F(x_{i+1}|,$$

where the supremum is taken over the set of all such finite collections of $x_i's$ on ∂G, is called the l_∂-length of F-image of ∂G (or briefly the l_∂-length of $F(\partial G)$). In [1, p.317] the l_∂-length is called generalized.

Let G be any domain on B. To define $l_\partial(F(\partial G))$ we first introduce some sets as follows ([1, p.312]). Let γ be a component of ∂G. If γ is the only component of ∂G, we set $A(G,\gamma) = G$. Otherwise, for each component γ' of ∂G, $\gamma' \neq \gamma$, we denote by $a(\gamma',\gamma)$ the set of all points in B which are separated by γ' from γ in B. The set $a(\gamma',\gamma)$ may be empty and, if not, is open in B and not necessarily connected. Let

$$A = A(G,\gamma) = G \cup (\bigcup(\gamma' \cup a(\gamma',\gamma))),$$

where the union $\bigcup(\gamma' \cup a(\gamma',\gamma))$ is extended over all the components γ' of ∂G, $\gamma' \neq \gamma$. Since $A(G,\gamma)$ is a simply connected domain with γ as a boundary, we use the notation $l_\partial(F(\gamma))$ for the l_∂-length of $F(\partial A(G,\gamma))$. The l_∂-length of $F(\partial G)$ is defined as the sum

$$l_\partial(F(\partial G)) = \sum l_\partial(F(\gamma))$$

being extended over all the components γ of ∂G.

If G is an open set in B having the components G_i, $i = 1, 2, \dots$, the number

$$l_\partial(F(\partial G)) = \sum_{i=1}^{\infty} l_\partial(F(\partial G_i))$$

is called the l_∂-length of $F(\partial G)$.

If G is an open set in B and H_1, H_2 are two embeddings of B into $\mathbb{R}^3$ such that $H_1(G) = H_2(G) = D$, then the number $l_\partial(H_1(\partial G)) = l_\partial(H_2(\partial G))$ is denoted by $l_\partial(\partial D)$ and called the l_∂-length of the boundary ∂D of the open set D (lying on the 2-sphere $H_1(B)$ (or $H_2(B)$)).

Suppose D is an open set on a 2-sphere S. Then $l_\partial(\partial D) = 0$ if all the components of ∂D are single points, and $l_\partial(\partial D) \neq 0$ if there is at least one continuum as a component of ∂D.

Theorem 2.5. *Suppose D is a domain on a 2-sphere S and $l_\partial(\partial D)$ is finite and not equal to zero. Then all but at most a countable number of components of ∂D are single points (and therefore their l_∂-lengths are zeros); the other components of ∂D are continuous curves such that their Jordan lengths are finite and equal to their l_∂-lengths. Moreover, $l_\partial(\partial D) = \sum l_\partial(\gamma)$, where the sum is extended over all the nondegenerate components γ of ∂D.*

Theorem 2.5 follows from the properties of the l_∂-length (see [1, p.317–320]). Suppose F is a continuous map of a boundary B of a 3-simplex into $\mathbb{R}^3$ and f is a real continuous function from $\mathbb{R}^3$ into $\mathbb{R}$. For any $t \in \mathbb{R}$ let D_t be the set of all $x \in B$, where $f(F(x)) < t$ and let $l_\partial(t)$ be the l_∂-length of $F(\partial D_t)$ (we put $l_\partial(t) = 0$ if $D_t = \varnothing$). The following theorem is a slight obvious modification of Theorem of [1, p.327].

Theorem 2.6. *Suppose F is a continuous map of B into $\mathbb{R}^3$, $L(F)$ is the Lebesgue area of F, and f is a real continuous function from $\mathbb{R}^3$ into $\mathbb{R}$ such that $|f(y) - f(y')| \leq K|y - y'|$ for all y, y' of $\mathbb{R}^3$ (K is a constant, $K > 0$). Then*

$$K \cdot L(F) \geq \int_{-\infty}^{+\infty} l_\partial(t)\, dt. \tag{2.1}$$

If in Theorem 2.6 $F{:}B \to \mathbb{R}^3$ is an embedding, $S = F(B)$, and $L(S) = L(F)$, then the Cavalieri inequality (2.1) may be written in the form

$$K \cdot L(S) \geq \int_{-\infty}^{+\infty} l_\partial(t)\, dt. \tag{2.2}$$

2.3. Minimal surfaces. A C^2-map from a domain of $\mathbb{R}^2$ into $\mathbb{R}^3$ is minimal (map or surface) if it has the mean curvature zero at all points. Let Γ be a simple closed curve in $\mathbb{R}^3$. The Plateau problem for Γ is to find a continuous map F from the closed unit disk $\overline{G}_0$ into $\mathbb{R}^3$ such that F is minimal in $\operatorname{Int} G_0$ and maps ∂G_0 homeomorphically onto Γ. A minimal disk of least area spanning Γ is the embedded solution of the Plateau problem for Γ which has the least area (in the sense of Lebesgue) of all areas of continuous maps from $\overline{G}_0$ into $\mathbb{R}^3$ mapping ∂G_0 homeomorphically onto Γ. While the Plateau problem was solved at the beginning of the thirties, the structure and the properties of minimal surfaces were studied in the seventies-eighties.

Theorem 2.7. *Suppose M is a compact convex subset of $\mathbb{R}^3$ and Γ is a simple closed curve of finite length on the boundary of M. Then there is a minimal disk of least area spanning Γ and lying in M.*

Theorem 2.8. *Suppose M is a compact convex subset of $\mathbb{R}^3$ and Γ_1, Γ_2 are two disjoint simple closed curves on the boundary ∂M. If D_1 and D_2 are two minimal disks of least areas spanning Γ_1 and Γ_2 respectively such that $\operatorname{Int} D_i \cap \partial M = \varnothing$, $i = 1, 2$, then $\overline{D}_1 \cap \overline{D}_2 = \varnothing$.*

Theorem 2.7 follows from Theorem 2 of [7, p.421]; an alternative proof is given also in [8]. Theorem 2.8 is a consequence of the results of Meeks and Yau on the local structure of minimal surfaces (see [7, §2]) or it may be proved like a second part of Theorem 2.9 of [9, p.95].

3. The consequences of the restrictions on S. We shall assume throughout that S is a 2-sphere of finite Lebesgue area $L(S)$. The construction of a polyhedral 2-sphere approximating S depends on the properties of intersections of S with lines and some surfaces in $\mathbb{R}^3$. Because of restrictions on S we can find among the intersections those of rather simple structure.

Lemma 3.1. *Suppose S is a 2-sphere of finite Lebesgue area in $\mathbb{R}^3$ and f is a Lipschitzian function defined on S. Then for almost all t of $f(S)$ the components of levels $E_t(f)$ separating S are simple closed curves having the finite sum of their Jordan lengths.*

Proof. We specify the following steps.

1. In this step we show that some components of $E_t = E_t(f)$ are simple closed curves of finite lengths.

For each $t \in f(S)$, let $G_t = G_t(f)$ be a set of x of S such that $f(x) < t$ and let $l_\partial(t) = l_\partial(\partial G_t)$ be the l_∂-length of the boundary ∂G_t (section 2), clearly $\partial G_t \subset E_t$. We see that for all but at most a countable number t of $f(S)$ all the components (completely regular and irregular non separating S) of E_t have the properties stated in Theorem 2.1. Moreover, we also see (by the Cavalieri inequality (2.2)) that the length $l_\partial(t)$ is finite for almost all t of $f(S)$. Let τ be one of such t. By Theorem 2.5, every nondegenerate component of the boundary of every component of the open set G_τ has a finite Jordan length as a continuous curve. The sum of their lengths is equal to $l_\partial(\tau)$. If J is a component of ∂G_τ which separates S, then $J \subset E_\tau$ and therefore J is a completely regular (by Theorem 2.1) locally connected (because of $l_\partial(J) < \infty$) component of E_τ. According to Theorem 2.2, J is a simple closed curve. The sum of the lengths of all such J's does not exceed $l_\partial(\tau) < \infty$.

2. Let τ be as in previous step and let K be a component of $E_\tau(f)$ which separates S. By Theorem 2.1, K is a completely regular component of $E_\tau(f)$ and let S_1 and S_2 be two components of $S \setminus K$. We define (just as in [5, p.30]) two sets

$K_\tau^i = K \cup S_i,\ i = 1, 2$, and two functions

$$f_\tau^i(x) = \begin{cases} \tau, & \text{if } x \in S_i, \\ f(x), & \text{if } x \in S \setminus S_i. \end{cases}$$

Since no $K_\tau^i,\ i = 1, 2$, separates S, K_τ^i is (by Theorem 2.3) either a component of concentric singularity or a component of semiextremum of $E_\tau(f_\tau^i)$.

First suppose K_τ^1 is a component of concentric singularity of $E_\tau(f_\tau^1)$.

We show that for any neighborhood U of K_τ^1 in S there is a component of $G_\tau(f)$ lying in U and separating K_τ^1 from $S \setminus U$. For if not, we find a neighborhood U_0' of K_τ^1 such that no component of $G_\tau(f)$ separates K_τ^1 from $S \setminus U_0'$. Let $\tilde{K}$ be a component of $E_\tau(f_\tau^1)$ which lies in U_0' and separates K_τ^1 from $S \setminus U_0'$. Then $\tilde{K}$ is a completely regular component of $E_\tau(f)$ which bounds a neighborhood U_0 of K_τ^1, $U_0 \subset U_0'$.

Since some components of $E_\tau(f)$ separate K_τ^1 from $S \setminus U_0$ and $E_\tau(f)$ does not contain the extremal points of f (by Theorem 2.1), $G_\tau(f) \cap (U_0 \setminus K_\tau^1) \neq \varnothing$. Let $G_1, G_2, \dots$ be the components of $G_\tau(f)$ in $U_0 \setminus K_\tau^1$. Since K_τ^1 is not a semiextremum component of $E_\tau(f_\tau^1)$ (because it is the component of concentric singularity), $K_\tau^1 \cap \partial G_n = \varnothing$ for $n = 1, 2, \dots$ and therefore each ∂G_n separates K_τ^1 from G_n. Let D_n be a simply connected domain in S containing G_n and such that ∂D_n is a component of ∂G_n. By Step 1, ∂D_n is a simple closed curve and D_n is a disk. Two different D_n's intersect only if one contains the other. So the union $\bigcup_{n=1}^\infty \overline{D}_n$ does not separate the annulus $\overline{U}_0 \setminus \operatorname{Int} K_\tau^1$ and the continuum $P = (\overline{U}_0 \setminus \operatorname{Int} K_\tau^1) \setminus (\bigcup_{n=1}^\infty \operatorname{Int} D_n)$ includes $\partial(\overline{U}_0 \setminus \operatorname{Int} K_\tau^1)$. Since K_τ^1 is the component of concentric singularity of $E_\tau(f_\tau^1)$, there is a component K' of $E_\tau(f_\tau^1)$ in $\operatorname{Int}(\overline{U}_0 \setminus K_\tau^1)$ such that K' separates K_τ^1 from $S \setminus U_0$. Clearly K' is a completely regular component of $E_\tau(f)$, the continuum P meets K' and both components of $S \setminus K'$, and $f(x) \geq \tau$ for $x \in P$. Therefore, K' is a quasiextremal component of $E_\tau(f)$ (see Section 2.1). This contradicts to the choice of τ (Step 1 and Theorem 2.1). Hence for any neighborhood U of K_τ^1 there is a component of $G_\tau(f)$ lying in U and separating K_τ^1 from $S \setminus U$.

It follows from this that for each neighborhood U of K_τ^1 there exists a sequence of components $\tilde{G}_n$ of $G_\tau(f)$, $n = 1, 2, \dots$, each lying in U and separating K_τ^1 from $S \setminus U$. If γ_n is a component of $\partial\tilde{G}_n$ which separates K_τ^1 from $S \setminus U$, then $\sum_{n=1}^\infty l_\partial(\gamma_n) = \infty$. This contradicts to the inequality $l_\partial(\tau) < \infty$ stated in Step 1. Hence K_τ^1 cannot be a component of concentric singularity of $E_\tau(f_\tau^1)$ and likewise K_τ^2 cannot be a component of concentric singularity of $E_\tau(f_\tau^2)$.

Therefore both K_τ^1 and K_τ^2 are the semiextremum components of $E_\tau(f_\tau^1)$ and $E_\tau(f_\tau^2)$ respectively. But since K is not a quasiextremum component of $E_\tau(f)$, one of two sets K_τ^1, K_τ^2 is a semiminimum component and the other is a semimaximum one of $E_\tau(f_\tau^1)$ and $E_\tau(f_\tau^2)$.

3. Suppose K_τ^1 is the semimaximum component of $E_\tau(f_\tau^1)$. To prove the lemma we show that there exists a component G of $G_\tau(f)$ such that K is a component of ∂G. In case if K_τ^2 were a semimaximum component of $E_\tau(f_\tau^2)$, the proof would be similar. Since $\overline{K}_\tau^1$ is the semimaximum component of $E_\tau(f_\tau^1)$, there is a continuum P meeting $\overline{K}_\tau^1$, not lying wholly in $\overline{K}_\tau^1$, not meeting $E_\tau(f_\tau^1) \setminus \overline{K}_\tau^1$ and such that $f(x) \leq \tau$ for $x \in P$. Let P_1 be any component of $P \setminus K_\tau^1$. Then $f(x) < \tau$ for $x \in P_1$ and hence P_1 is a subset of some component G of $G_\tau(f)$. Clearly $P_1 \subset G \subset S \setminus \overline{K}_\tau^1$, $\overline{P}_1 \cap \partial(S \setminus \overline{K}_\tau^1) \neq \varnothing$, and therefore the set $K = \partial(S \setminus \overline{K}_\tau^1)$ meets ∂G. In particular,

K meets the component γ of ∂G separating G from $\overline{K}^1_\tau$. But the component γ of ∂G is a separator of S, the component K of $E_\tau(f)$ is an irreducible separator of S, $K \cap \gamma \neq \varnothing$, and $\partial G \subset E_\tau(f)$. We deduce from this that $\gamma = K$. So if the component K of $E_\tau(f)$ separates S, then there exists a component G of $G_\tau(f)$ with K as a component of its boundary. Then by Step 1, K is a simple closed curve of finite length that does not exceed $l_\partial(\tau)$. This proves Lemma 3.1.

Theorem 3.1. *Suppose S is a 2-sphere of finite Lebesgue area in $\mathbb{R}^3$ and $f{:}S \to \mathbb{R}$ is a Lipschitzian function. Then for almost all t of $f(S)$, $E_\tau(f)$ contains only simple closed curves (as its nondegenerate components) which have the finite sum of their lengths, and also one-point components.*

Theorem 3.1 follows immediately from Lemma 3.1 and Theorem 2.4.

Corollary 3.1. *Let S be a 2-sphere of finite Lebesgue area in $\mathbb{R}^3$, let m be a line, and let $C^2(t)$ be a circular right cylinder with the axis on m and of radius t. Then for almost all t of which $C^2(t)$ meet S, $S \cap C^2(t)$ contains only simple closed curves (as its nondegenerate components) which have the finite sum of their lengths, and also one-point components.*

Proof. We denote by $f(x)$ the distance of $x \in S$ from m. Clearly $f(x)$ is a Lipschitzian function on S with constant 1. Since $S \cap C^2(t)$ is a t-level of f, Corollary 3.1 follows from Theorem 3.1.

4. Substituting S by a simpler surface. For any rectilinear triangulation $\mathcal{Q}$ of $\mathbb{R}^3$ let $\mathcal{Q}^i$, $i = 0, 1, 2, 3$, be the i-skeleton of $\mathcal{Q}$ that is, the union of simplices of $\mathcal{Q}$ of dimensions $\leq i$. An arc is said to pierce a 2-sphere in $\mathbb{R}^3$ ([10, p.148]) if the arc intersects the 2-sphere at only one point and the ends of the arc lie in different components of the complement of the 2-sphere.

Lemma 4.1. *Suppose S is a 2-sphere in $\mathbb{R}^3$ of finite Lebesgue area $L(S)$ and $\varepsilon > 0$. Then there are a 2-sphere S', a homeomorphism $F{:}S \to S'$, and a triangulation $\mathcal{Q}$ of $\mathbb{R}^3$ such that*

(i) $|F(x) - x| < \varepsilon$ for $x \in S$;

(ii) $L(S') < L(S) + \varepsilon$;

(iii) $S' \cap \mathcal{Q}^0 = \varnothing$; $S' \cap \mathcal{Q}^1$ is a finite set;

(iv) S' is pierced by an arc of $\mathcal{Q}^1$ at each point of $S' \cap \mathcal{Q}^1$;

if Δ^2 is a 2-simplex of $\mathcal{Q}$, then

(v) the closure of each component of $S' \cap \mathrm{Int}\,\Delta^2$ meets $\partial\Delta^2$ in at most two points;

(vi) the closure of each component of $S' \cap \mathrm{Int}\,\Delta^2$ is either a point, or an arc, or a simple closed curve, and moreover, these arcs and simple closed curves have the finite sum of their lengths.

Proof. Assume that $4\varepsilon < \operatorname{diam} S$. In the first four steps we construct a triangulation $\mathcal{Q}$ of $\mathbb{R}^3$, a 2-sphere S_0 and a homeomorphism $F_0{:}\, S \to S_0$ satisfying the conditions (i)–(iv) provided that S', ε, and F in Lemma 4.1 are replaced by $S_0, \varepsilon/2$, and F_0, respectively.

1. Take $\delta, 0 < \delta < \varepsilon/4$, such that no continuum in S of diameter less than δ has two complementary domains in S each of diameter more than $\varepsilon/4$. Let $\mathcal{Q}_1$ be a triangulation of $\mathbb{R}^3$ of mesh less than δ such that $S \cap \mathcal{Q}_1^0 = \varnothing$. We study the structure S in the vicinity of $S \cap \mathcal{Q}_1^1$.

Let $\Delta_1^1, \dots, \Delta_m^1$ be the 1-simplices of $\mathcal{Q}_1$ each meeting S. For each $\Delta_i^1, i = 1, \dots, m$, let $C_i^3(r)$ be a regular circular closed solid cylinder with the axis on Δ_i^1 of radius r which does not contain the end points of Δ_i^1 such that the bases of $C_i^3(r)$ miss S^2 and the interior of $C_i^3(r)$ covers $S^2 \cap \Delta_i^1$. Assume also that the $C_i^3(r)$'s are mutually disjoint and each of diameter less than δ. The part of $\partial C_i^3(r)$ off the bases of $C_i^3(r)$ is the side of $C_i^3(r)$ and is denoted by $C_i^2(r)$.

By Corollary 3.1, there exists t, $t < r < \delta$, such that each non-degenerate component of $C_i^2(r) \cap S, i = 1, 2, \dots, m$, is a simple closed curve of diameter less than δ, and let $\Gamma_n, n = 1, 2, \dots$, be all such curves. Denote by d_n the smaller of two disks bounded by $\Gamma_n = \partial d_n$ on S; clearly diam $d_n < \varepsilon/4$. We suppose the notation is adjusted so that $d_1, d_2, \dots$ is the collection of maximal d_n's no one of them contained in another of them.

So we obtain at most a countable collection of mutually disjoint closed disks $\bar{d}_n, n = 1, 2, \dots$, in S such that

(1) the union $\bigcup_n \bar{d}_n$ contains all nondegenerate components of $S \cap \left(\bigcup_{i=1}^m C_i^2(t)\right)$;
(2) $S \cap \left(\bigcup_{i=1}^m \operatorname{Int} C_i^3(t)\right) \subset \bigcup_n \bar{d}_n$;
(3) $\bigcup_n \partial d_n \subset \bigcup_{i=1}^m \operatorname{Int} C_i^2(t)$;
(4) the sum of lengths of ∂d_n is finite;
(5) $\operatorname{diam} \partial d_n < \delta$, and hence $\operatorname{diam} d_n < \varepsilon/4$;
(6) $\lim_{n\to\infty} \operatorname{diam} d_n = 0$.

2. Each simple closed curve ∂d_n lies in some $C_i^2(t), i = 1, \dots, m$. It follows from Theorem 2.7 that there is a minimal disk d_{n0} of least area spanning ∂d_n and lying in $C_i^3(t)$. Since $\partial d_n \subset \operatorname{Int} C_i^2(t)$, we have $\operatorname{Int} d_{n0} \cap \partial C_i^3(t) = \varnothing$. By Theorem 2.8, none of two such minimal disks intersects the other if both contained in the same cylinder $C_i^3(t)$ and, of course, they do not intersect if contained in different $C_i^3(t)$'s.

For each $d_n, n = 1, 2, \dots$, let H_n be a homeomorphism which takes $\bar{d}_n$ onto $\bar{d}_{n0}$ and is fixed on ∂d_n. We define a continuous map $H_0: S \to E^3$ by setting

$$H_0(x) = \begin{cases} x, & \text{if } x \in S \setminus (\bigcup_n \bar{d}_n), \\ H_n(x), & \text{if } x \in \bar{d}_n. \end{cases}$$

By the convexity property, a compact minimal surface in $\mathbb{R}^3$ is always contained in the convex hull of its boundary. So $|H_n(x) - x| < \operatorname{diam} d_n < \varepsilon/4$ if $x \in d_n$ and therefore $|H_0(x) - x| < \max_n \operatorname{diam} d_n < \varepsilon/4$ if $x \in S$. Since no two minimal disks $d_{n0} = H_n(d_n)$ intersect each other, we conclude that $S_0' = (S \setminus (\bigcup_n \bar{d}_n)) \cup (\bigcup_n \bar{d}_{n0})$ is a 2-sphere and $H_0: S \to S_0'$ is the homeomorphism.

We also have $L(S_0') \le L(S)$ because d_{n0}'s are the minimal disks of least areas. So S_0' satisfies the condition (ii) for S' of Lemma 4.1.

3. In Step 2 we have replaced S with the 2-sphere S_0' by making use the collection of minimal disks d_{no}. Since $\mathcal{Q}_1^1 \cap S_0' = \mathcal{Q}_1^1 \cap (\bigcup_n d_{n0})$, we shall deal with $\mathcal{Q}_1^1 \cap (\bigcup_n d_{n0})$ (that is, the set of intersections of lines with the minimal surfaces). In this step we find a triangulation $\mathcal{Q}$ of $\mathbb{R}^3$ so that $\mathcal{Q}^0 \cap S_0' = \varnothing$ and $\mathcal{Q}^1 \cap S_0'$ is a finite set.

By Step 2, $(\bigcup_{i=1}^m \operatorname{Int} C_i^3(t)) \cap S_0' = \bigcup_n d_{n0}$. Since a minimal surface admits a C^2-parametrization (and even a conformal one) [11, p.59], the Lebesgue area $L(d_{n0})$ is equal to the two-dimensional Hausdorff measure $\Lambda^2(d_{n0})$ [12, p.92]. It follows from this that $D_0 = \bigcup_n d_{n0}$ is of finite Hausdorff measure and $\Lambda^2(D_0) = \sum_n L(d_{n0}) < \infty$. Note that the inequalities $\Lambda^2(d_n) < \infty$ may be not valid for the d_n's.

For any plane π, let P_π is the map of orthogonal projection of $\mathbb{R}^3$ onto π. For any subset M of $\mathbb{R}^3$ and $y \in \pi$, let $N(\pi, M, y)$ be the number (maybe 0 or ∞) of points $x \in M$ such that $P_\pi(x) = y$. Since P_π is a Lipschitzian map with the constant 1, we have ([13 , p.194])

$$\int_\pi N(\pi, D_0, y)\, d\Lambda^2(y) \le \Lambda^2(D_0) < \infty.$$

Then for almost every $y \in \pi$, the line through y and orthogonal to π meets D_0 in at most a finite number of points.

Let π' be a plane which is parallel to no Δ_i^1 of $\mathcal{Q}_1^1, i = 1, 2, \dots, m$ (see Step 1). Then there is a sufficiently small vector v parallel to π' and a translation Φ of E^3 in the direction of v and distance $|v|$ such that the triangulation $\mathcal{Q} = \Phi(\mathcal{Q}_1)$ will satisfy the conditions

(1) $S_0' \cap \mathcal{Q}^0 = \varnothing; S_0' \cap \mathcal{Q}^1$ is a finite set;

(2) no point of $\mathcal{Q}^0$ is contained in $\bigcup_{i=1}^m C_i^3(t)$;

(3) each $C_i^3(t), i = 1, \dots, m$ meets both Δ_i^1 and $\Phi(\Delta_i^1)$) and does not meet the other Δ_i's and $\Phi(\Delta_i^1)$'s.

So, by (1), S_0' satisfies the condition (iii) for S' of Lemma 4.1. It follows from (3) that $S_0' \cap \mathcal{Q}^1 \subset \bigcup_n d_{n0} = D_0$.

4. In this step we eliminate nonpiercing points of $S_0' \cap \mathcal{Q}^1$ (that is, such points at which 1-simplices of $\mathcal{Q}^1$ do not pierce S_0'). More general cases of eliminating nonpiercing points were presented in [10, Section 5]; but my attempts in checking up the alternation of the surface area for those cases were unavailing.

Suppose that the 1-simplex Δ^1 of $\mathcal{Q}$ does not pierce S_0' at some point a of $\Delta^1 \cap S_0'$ and let d be the one of the minimal disks d_{n0} which contains a. We introduce orthogonal coordinates (x^1, x^2, x^3) with the origin at a so that the (x^1, x^2)-plane is tangent to d at a and the x^1-axis contains Δ^1. By [11, p.59], d admits the C^2-representation $x^3 = f(x^1, x^2)$ in some neighborhood $U^{1,2}$ (in (x^1, x^2)-plane) of a. Since Δ^1 does not pierce S_0' at a, there is a neigborhood $U^{1,3}$ (in (x^1, x^3)-plane) of a such that $d \cap U^{1,3}$ is a graph of $f(x^1, 0)$ provided that x^1 is sufficiently close to 0 and $d \cap U^{1,3}$ abuts on the x^1-axis from one side. Therefore if $U(a)$ is a neighborhood of a in $\mathbb{R}^3$ that contains no point of $S_0' \cap \mathcal{Q}^1$ other than a, we may construct a piecewise linear homeomorphism $H_a: \mathbb{R}^3 \to \mathbb{R}^3$ which is as close to identity as desired and such that (1) H_a is fixed outside $U(a)$; (2) the Lebesgue area $L(S_0')$ alters under H_a by no more than a prescribed small quantity; and (3) $H_a(S_0') \cap \Delta^1 \cap U(a) = \varnothing$. The homeomorphism H_a may be obtained in just the same way as the map G_Δ in Step 7 (see below).

We assume that H_a is so near to the identity that $|H_a(x) - x| < \varepsilon'$ for $x \in S_0'$ and $L(H_a(S_0')) \leq L(S_0') + \varepsilon''$, where sufficiently small ε' and ε'' will be specified below. Clearly the number of nonpiercing points of $H_a(S_0') \cap \mathcal{Q}^1$ is less than that of $S_0' \cap \mathcal{Q}^1$. We eliminate the other nonpiercing points of $H_a(S_0') \cap \mathcal{Q}^1$ by modifying $H_a(S_0')$ like we did with S_0' to obtain $H_a(S_0')$. By applying the similar constructions finitely many times we obtain the resulting homeomorphism $H: \mathbb{R}^3 \to \mathbb{R}^3$ such that $H(S_0') \cap \mathcal{Q}^1$ contains no nonpiercing point. Moreover, if k is a total number of nonpiercing points of $S_0' \cap \mathcal{Q}^1$, we may construct H so that $|H(x) - x| < k\varepsilon'$ for $x \in S_0'$ and $L(H(S_0')) \leq L(S_0') + k\varepsilon''$.

Let now $\varepsilon', \varepsilon''$ be such that $k\varepsilon' < \varepsilon/4$ and $k\varepsilon'' < \varepsilon/2$. For H_0 of Step 2, let $G_0 = H \circ H_0$ and $S_0 = H(S_0') = G_0(S)$. Then $|G_0(x) - x| < \varepsilon/2$ for $x \in S$ and

$$L(S_0) \leq L(S) + \varepsilon/2. \tag{4.1}$$

We also have that $S_0 \cap \mathcal{Q}^0 = \varnothing; S_0 \cap \mathcal{Q}^1$ is a finite set, and S_0 is pierced by an arc of $\mathcal{Q}^1$ at each point of $\mathcal{Q}^1 \cap S_0$.

5. So if $a' \in S_0 \cap \mathcal{Q}^1$, then some 1-simplex $\Delta^1(a')$ of $\mathcal{Q}^1$ pierces S_0 at the point a' which lies in some minimal disk d_{n0} in S_0 (Step 3). Then by [7, Lemma 2] there is a neighborhood $V(a')$ of a' in d_{n0} such that every plane containing $\Delta^1(a')$ intersects $V(a')$ along a finite number of smooth arcs having a' as a unique pairwise common point.

6. Suppose Δ^2 is a 2-simplex of $\mathcal{Q}$ meeting S_0. Let v_1, v_2, and v_3 be the vertices of Δ^2 and let b be its barycenter. Let T and T^* be two 3-simplices of $\mathcal{Q}$ with Δ^2 as the 2-face in common. Denote by I a segment of length α (α will be specified below) which is perpendicular to Δ^2 and has one end point at b and the other in $\operatorname{Int} T$. Let x_t be the point in I at a distance t from b, K_t be the cone with the vertex x_t based on $\partial\Delta^2$, and T_t be the 3-simplex with the vertices x_t, v_1, v_2, and v_3. Denote by T_t^* the 3-simplex which is symmetric to T_t with respect to Δ^2 and assume that $T_\alpha^* \subset T^*$. In this step we show that there is x_t in I as near to b as desired and such that the set $S_0 \cap \operatorname{Int} K_t$ satisfies the conditions (v) and (vi) for $S' \cap \operatorname{Int} \Delta^2$ of Lemma 4.1.

According to the argument of [2, p.463], for any I there exists a nondegenerate segment I' in I such that no component of $S_0 \cap (\bigcup \operatorname{Int} K_t)$, where the union is extended over all x_t of I', contains three limit points on $\partial\Delta$. Therefore if $x_t \in I'$, then the components of $S_0 \cap \operatorname{Int} K_t$ satisfy the condition (v) for $S' \cap \operatorname{Int} \Delta^2$ of Lemma 4.1.

Let $\Delta^2(u), 0 < u \leq 1$, be the triangle in $\operatorname{Int} \Delta^2$ with the vertices $v_1(u) = (1-u)b + u \cdot v_i$, $i = 1, 2, 3$, and let $\Delta^2(0) = b$. We choose $\alpha \leq \min \rho(b, [v_j, v_k]), 1 \leq j \neq k \leq 3$ (here $\rho(A, B)$ denotes the distance between sets A and B). We denote by $T_\alpha(u)$ the set of those x of T_α for which the orthogonal projections on Δ^2 lie in $\Delta^2(u)$. For each x of $T_\alpha(u)$ and for a unique K_t, $0 \leq t \leq \alpha$, containing x, let $f_u(x) = t$. Then $K_t(u) = K_t \cap T_\alpha(u)$ is the t-level of f_u and $|f_u(x) - f_u(x')| < \frac{\sqrt{2}}{1-u}|x - x'|$ for x and x' of $T_\alpha(u)$.

We now fix u of $(0; 1)$ and extend f_u to obtain a function $\tilde{f}_u : \mathbb{R}^3 \to \mathbb{R}$ satisfying the Lipschitz condition with constant $\sqrt{2}/(1-u)$ (see [14]). Let $\tilde{f}_{u0}$ be the restriction of $\tilde{f}_u$ to S_0 and let $E_t(\tilde{f}_{u0})$ be a t-level of $\tilde{f}_{u0}$. Since $K_t(u) \cap S_0$ is a subset of $E_t(\tilde{f}_{u0})$, it follows from Theorem 3.1 that for almost all t of $[0; \alpha]$, $K_t(u) \cap S_0$ contains arcs and simple closed curves (as the nondegenerate components) which have the finite sum of their lengths and also one-point components. Therefore there is a set I^c of complete measure in I such that if $t \in I^c$, then for all u of $(0; 1)$, $K_t(u) \cap S_0$ contains arcs and simple closed curves which have the finite sum of their lengths and also one-point components. And finally, since $K_t \cap S_0$ near $\partial\Delta^2$ is the union of a finite number of smooth curves (Step 5) and at most a countable collection of continua in S_0 may contain branching points, we may suppose that $S_0 \cap \operatorname{Int} K_t$ satisfies the conditions (v) and (vi) for $S' \cap \operatorname{Int} \Delta^2$ of Lemma 4.1 if $t \in I^c$. Let $\tau \in I^c$ (we specify τ below).

7. We complete the proof of Lemma 4.1. First we triangulate $T_\alpha \cup T_\alpha^*$ of $T \cup T^*$ by means of tetrahedra which have x_τ as their common vertex and the 2-faces of the polyhedron $\partial(T_\alpha \cup T_\alpha^*)$ (opposite to x_τ) as their bases. Let $\mathcal{Q}(x_\tau)$ be any triangulation of $\mathbb{R}^3$ which agrees with the triangulation already defined for $T_\alpha \cup T_\alpha^*$ and agrees with $\mathcal{Q}$ outside $T \cup T^*$.

We now define a piecewise linear autohomeomorphism G_Δ of $\mathbb{R}^3$. We start G_Δ to be fixed for all vertices of $\mathcal{Q}^0(x_\tau)$ but one x_τ for which let $G_\Delta(x_\tau) = b$. We extend G_Δ linearly to each element of $\mathcal{Q}(x_\tau)$ and thus obtain a piecewise linear homeomorphism $G_\Delta : \mathbb{R}^3 \to \mathbb{R}^3$. Note that the construction of the homeomorphism H_a of Step 4 would be carried out by the same way as G_Δ already described.

Since τ is chosen as in Step 6, the components of $G_\Delta(S_0) \cap \operatorname{Int} \Delta^2$ satisfy the conditions (v) and (vi) for $S' \cap \operatorname{Int} \Delta^2$ of Lemma 4.1. If x_τ of I is sufficiently close to b, G_Δ is very near to the identity. Moreover, if S^* is any 2-sphere in $\mathbb{R}^3$ of finite Lebesgue area $L(S^*)$, then by taking $\tau = |x_\tau - b|$ sufficiently close to zero we may require $|L(G_\Delta(S^*)) - L(S^*)|$ to be less than a prescribed small number (see [3, p.232]). Therefore for any $\beta > 0$ there is $\eta > 0$ such that if $\tau < \eta$, then $|G_\Delta(x) - x| < \beta$ for $x \in S_0$ and $L(G_\Delta(S_0)) < L(S_0) + \beta$.

Further, we define β-homeomorphisms in neighborhoods of the other 2-simplices of $\mathcal{Q}$ like we defined G_Δ in the neighborhood of Δ^2. Let $G\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a so obtained resulting homeomorphism (modifying S_0) and let $S' = G(S_0) = G(G_0(S))$. If l is a number of the 2-simplices of $\mathcal{Q}$ meeting S_0, we may and do require that $|G(x)-x| < l\beta$ for $x \in S_0$ and $L(G(S_0)) < L(S_0)+l\beta$. It follows from this and from (4.1) that $|G(G_0(x))-x| < \varepsilon/2+l\beta$ for $x \in S$ and $L(G(G_0(S))) < L(S)+\varepsilon/2+l\beta$. If we choose $\beta < \varepsilon/(2l)$ and put $F = G \circ G_0$, then all the conditions of Lemma 4.1 will be satisfied.

5. The equality $A(S) = L(S)$**.** Throughout this section we mean the area of a surface in the sense of Lebesgue, unless the contrary is allowed. Suppose S is a 2-sphere of finite area $L(S)$ and let $\varepsilon > 0$. To prove Theorem 1.1 we construct a polyhedral 2-sphere Ω and a homeomorphism $H\colon S \to \Omega$ such that $|H(x) - x| < \varepsilon$ for $x \in S$ and $L(\Omega) < L(S) + \varepsilon$.

In this section we make use the proof of Bing's approximation theorem [2]. First we describe briefly (after [2, p.464–467], see also [10, p.155–159]) a collection of some disks in S and then in Steps 1–6 construct the polyhedral 2-sphere Ω. We suppose the diameter of S is large compared to ε and let δ, $0 < \delta < \varepsilon/2$, be such that no continuum in S of diameter less than δ has two complementary domains each of diameter more than $\varepsilon/2$.

Let $\mathcal{Q}$ be a triangulation of $\mathbb{R}^3$ of mesh less than δ. We assume that S satisfies the conditions (iii)–(vi) for S' of Lemma 4.1. If Δ_1^2 is a 2-simplex of $\mathcal{Q}$ and X is a component of $S \cap \operatorname{Int}\Delta_1^2$ with two limit points a_1 and b_1 on $\partial\Delta_1^2$, then $\operatorname{diam} X < \delta$ and by Lemma 4.1, X is an arc of finite length which spans Δ_1^2. There are at most a finite collection of components of $S \cap \operatorname{Int}\Delta_1^2$ which are the arcs of finite lengths with a_1 and b_1 as their end points and let X and X' be two components of the collection such that the simple closed curve $\bar{X} \cup \bar{X}'$ bounds in S a disk K_1 which contains all the other components with the same end points a_1 and b_1 and such that $\operatorname{diam} K_1 < \varepsilon/2$. We speak of a_1 and b_1 as the end points of K_1. If there is only one component X of $S \cap \operatorname{Int}\Delta_1^2$ with the end points a_1 and b_1, we set $K_1 = \bar{X}$.

Other K's are defined inductively. Suppose K_β, $\beta = 1, \dots, i$, and the corresponding 2-simplices Δ_β^2 of $\mathcal{Q}$ have been defined. To obtain K_{i+1} we choose a 2-simplex Δ_{i+1}^2 of $\mathcal{Q}$ (not necessarily different from the preceding Δ_β^2's) and a finite collection of components (arcs X of finite lengths) X_α, $\alpha = 1, \dots, j$, of $S \cap \operatorname{Int}\Delta_{i+1}^2$ having two end points in common on $\partial\Delta_{i+1}^2$ and such that no X_α meets K_β's. If there is only one such X_1, we put $K_{i+1} = \bar{X}$. Otherwise suppose the notation is adjusted so that the simple closed curve $\bar{X}_1 \cup \bar{X}_2$ bounds (in S) a disk K which contains all the other X_α's of $S \cap \operatorname{Int}\Delta_{i+1}^2$ with the same end points and such that $\operatorname{diam} K < \varepsilon/2$. Then we put $K_{i+1} = K$. Thus we can define a finite collection of the K's.

Let K_i, $i = 1, \dots, k$, be the collection of all such K's that are a not contained in a larger K. For each K_i, let $M(K_i) = \partial K_i$ if K_i is a disk and $M(K_i) = K_i$ if K_i is an arc. Then for each K_i with the end points a_i and b_i there exists a 2-simplex Δ_i^2 of $\mathcal{Q}$ such that $M(K_i) \subset \Delta_i^2$ and $M(K_i) \cap \partial\Delta_i^2 = a_i \cup b_i$. The intersection of any two of the K_i's is at most one or two their common end points. If $K_i \cap K_{i'} = a_i \cup b_i$, both end points a_i and b_i lie in one and the same common edge of two different 2-simplices of $\mathcal{Q}$ one of them containing $M(K_i)$ and the other containing $M(K_{i'})$.

Note that the restriction $L(S) < \infty$ on S made it possible to obtain less complicated K_i's than the K_i's which had been obtained in [2, p.465] under no restriction on S.

Let $D_j,\ j = 1, \dots , d$, be a finite collection of components of $S \setminus (\bigcup_{i=1}^{k} K_i)$. Further, each D_j is enlarged (see [2, p.465]) by adding some of K_i's and small D_j's in order to obtain a simply connected domain E_j such that $D_j \subset E_j$ and $\partial E_j \subset \partial D_j$.

We present after [2, p.465–467] some properties of $E_j,\ j = 1, \dots , d$. For each E_j there exists a subcollection $K^j_1, \dots , K^j_{m(j)}, K^j_{m(j)+1} = K^j_1$ of the collection of K_i's such that

(1) each $K^j_i,\ i = 1, \dots , m(j)$, has non end points on ∂E_j;

(2) the K^j_i's are indexed so that $K^j_i \cap K^j_{i+1}$ is either a common end point (if $m(j) > 2$), or two end points (if $m(j) = 2$);

(3) $K^j_i \cap K^j_{i'} = \varnothing$ if $1 < |i - i'| < m(j) - 1$;

(4) if $B(E_j) = \bigcup_{i=1}^{m(j)} K^j_i$, then $S \setminus B(E_j)$ is a union of two components of which E_j is the smaller one; and hence $\partial E_j \subset \bigcup_{i=1}^{m(j)} M(K^j_i) \subset B(E_j)$;

(5) there is a component E'_j of $E_j \setminus \mathcal{Q}^2$ such that $\partial E_j \subset \partial E'_j$; and therefore ∂E_j lies in the boundary of some 3-simplex of $\mathcal{Q}$.

It follows from this that each E_j is a disk (the E_j's of [2] are not disks) such that ∂E_j has the finite length and lies on a boundary of some 3-simplex of $\mathcal{Q}$. So $\operatorname{diam} D_j \le \operatorname{diam} E_j < \varepsilon/2$.

We shall enumerate the following steps in the proof.

1. In this step we describe a cell-like decomposition of S. Two different E_j's either do not intersect or otherwise one contains the other. We keep those of E_j's that are not contained in a larger E_j and use the same notation $E_1, \dots , E_d$ for them (as for all the E_j's). Also we keep those of K_i's that are contained in the union $\bigcup_{j=1}^{d} B(E_j)$ and use the same notation $K_1, \dots , K_k$ for them (as for all the K_i's).

For each $K_i,\ i = 1, \dots , k$, let A_i be an arc in $M(K_i)$ from one end point of K_i to the other. Then $S \setminus (\bigcup_{i=1}^{k} A_i)$ is the union of a finite number of open disks $P_j,\ j = 1, \dots d$, such that $E_j \subset P_j$. We also have $\bigcup_{j=1}^{d} \partial P_j = \bigcup_{i=1}^{k} A_i$. Since both A_i and $M(K_i)$ lie in a 2-face of $\mathcal{Q}$, ∂P_j and ∂E_j lie in the boundary ∂T_j of a tetrahedron T_j of $\mathcal{Q}$. Hence $\operatorname{diam} P_j < \varepsilon/2$. Clearly all the A_i's are of finite lengths and so are all the ∂P_j's.

By Theorem 6 of [2], if two simple closed curves ∂P_j and $\partial P_{j'}$ lie in the boundary of the same tetrahedron $T_j = T_{j'}$, none of ∂P_j and $\partial P_{j'}$ intersect both components of the complement of the other with respect to ∂T_j.

In the following steps each disk P_j is replaced by a polyhedral one; next, the union of all the polyhedral disks thus defined gives the required polyhedral 2-sphere. The estimations of the areas of surfaces, which are employed in the proof, are obtained by applying of Dehn's lemma; the cut-and-paste technique are also used.

We need some preliminaries. Suppose F is a continuous map of a 2-simplex σ into $\mathbb{R}^3$. Then $G = F(\sigma)$ is a singular disk, $\Gamma = F(\partial\sigma)$ is its boundary, and F is a parametrization of G. We also say that Γ bounds G. If Γ is a simple closed curve, we require in addition $F\colon \partial\sigma \to \Gamma$ to be a homeomorphism. If $F\colon \sigma \to G$ is piecewise linear, we speak of the singular polyhedral disk G.

The singularity set of G (denoted by $N(G)$) is the closure of the set of all x of $\mathbb{R}^3$ such that $F^{-1}(x)$ contains at least two points. If $N(G) = \varnothing$, then the corresponding parametrization $F\colon \sigma \to \mathbb{R}^3$ is an embedding and hence G is a disk.

An area $L(G)$ of the singular disk G is the Lebesgue area $L(F)$. If G is a disk, then the area $L(G)$ is the area of any homeomorphism of σ onto G.

2. Suppose P is one of the P_j's defined in the preceding step. Then the boundary ∂P, being of finite length, lies in the boundary ∂T of some tetrahedron T of $\mathcal{Q}$. In this step we show that for each $\alpha > 0$ there is $\beta > 0$ such that for any polygon (that is a simple closed polygonal curve) Γ in ∂T which is at a Frechet distance (see [1, p.15]) less then β from ∂P, there exists a singular polyhedral disk G' in T bounded by Γ such that

(1) $G' \setminus \Gamma \subset \mathrm{Int}T$;

(2) $N(G') \cap \Gamma = \varnothing$;

(3) $L(G') \leq L(P) + \alpha$.

It follows from (2) that G' will satisfy the condition of Denh's lemma [15].

We use $d_{\mathcal{F}}$ for the Frechet distance between two curves.

We choose $\alpha_1 > 0$ (it will be specified below) and enclose ∂P in a polyhedral annulus C in ∂T of area less than α_1 such that ∂P separates in C two components of ∂C.

First we suppose that Γ is a polygon within C separating two components of ∂C of which one is denoted by Γ_1.

For any simple closed curve λ in $\mathbb{R}^3$, let $m(\lambda)$ be the greatest lower bound of the areas of all singular disks bounded by λ. But in case λ is a polygon, the greatest lower bound of the areas of singular polyhedral disks bounded by λ is equal to $m(\lambda)$ (see [16, p.83]). Therefore there exists a singular polyhedral disk G''' which has the boundary Γ and a parametrization $F_1\colon \sigma \to G'''$ (where σ is a 2-simplex) such that

$$L(G''') = L(F_1) < m(\Gamma) + \alpha_1 < m(\Gamma_1) + 2\alpha_1 \leq m(\partial P) + 3\alpha_1 \leq L(P) + 3\alpha_1. \quad (5.1)$$

We define a continuous piecewise linear map p of $\mathbb{R}^3$ onto T by setting $p(x) = y$, where y is the point of T such that $\rho(x, T) = |x - y|$. Obviously $p \circ F_1$ is a parametrization of a singular disk $G'' = p(G''')$ bounded by Γ, $G'' \subset \bar{T}$, and $L(G'') \leq L(G''')$.

Let $q\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the homothety with a center b (the barycenter of T) and a coefficient t, $0 < t < 1$, and let $K(b, \Gamma)$ be the cone from b over Γ. Then $C_1 = K(b, \Gamma) - K(b, q(\Gamma))$ is the polyhedral annulus with the boundary $\partial C_1 = \Gamma \cup q(\Gamma)$ such that the singular polyhedral disk $G' = q(G'') \cup C_1$ bounded by Γ satisfies the conditions (1) and (2). The corresponding parametrization can be obtained by extending the map $q \circ p \circ F_1\colon \sigma \to q(G'')$ to a piecewise linear map of a larger 2-simplex σ_1 containing σ in its interior so that the extended map takes an annulus $\sigma_1 \setminus \sigma$ homeomorphically onto C_1. Clearly an area $L(G')$ depends on t. Therefore taking into account (5.1) we can choose t sufficiently near 1 to obtain $L(G') < L(P) + 4\alpha_1$. So if $\alpha_1 < \alpha/4$ (and in case Γ lies within C and separates two components of ∂C), there is a singular polyhedral disk G' bounded by Γ and satisfying the conditions (1)–(3).

However for an annulus C (of area less than $\alpha/4$) there is $\beta > 0$ such that any polygon Γ lying in ∂T at a Frechet distance less than β from ∂P separates in C two boundary components of ∂C. Consequently each such Γ bounds a singular polyhedral disk G' satisfying the conditions (1)–(3).

3. In this step we replace the collection of ∂P_j, $j = 1, \dots, d$ (Step 1), with a collection of polygons. By applying Step 2 for $\alpha = \varepsilon/(2d)$ and for each ∂P_j, we find $\beta_j > 0$ such that any polygon Γ_j in ∂T_j for which $d_{\mathcal{F}}(\Gamma_j, \partial P_j) < \beta_j$ bounds a singular polyhedral disk G'_j satisfying the conditions

(1) $G'_j \setminus \Gamma_j \subset \mathrm{Int}\, T_j$;

(2) $N(G'_j) \cap \Gamma_j = \varnothing$;

(3) $L(G'_j) \leq L(P_j) + \varepsilon/(2d)$.

Let $\beta_0 = \min \beta_j,\ j = 1, \dots, d$. We define the Γ_j's more concretely in the following way. By Step 1, $\bigcup_{j=1}^{d} \partial P_j = \bigcup_{i=1}^{k} A_i$ and each A_i spans some 2-simplex Δ_i^2 of $\mathcal{Q}$. For each A_i, let A_i' be a polygonal spanning arc of Δ_i^2 that has the same end points as A_i, $d_{\mathcal{F}}(A_i, A_i') < \beta_0$, and such that no two A_i''s intersect except possibly at their end points. If ∂P_j is a union of some A_i's, the union of the corresponding A_i''s forms a polygon which we use as Γ_j mentioned above and also keep the same notation. From the construction it follows that Γ_j lies in the boundary of the same 3-simplex T_j of $\mathcal{Q}$ as ∂P_j and $d_{\mathcal{F}}(\Gamma_j, \partial P_j) < \beta_0$. Moreover if two different Γ_j and $\Gamma_{j'}$ lie in ∂T_j, then, by Theorem 6 of [2], none of Γ_j and $\Gamma_{j'}$ intersect both components of the complement of the other with respect to ∂T_j.

Let H be a homeomorphism of $\bigcup_{i=1}^{k} A_i$ onto $\bigcup_{i=1}^{k} A_i'$ that leaves the end points of the A_i's fixed and takes each A_i onto A_i'. Since $A_i \bigcup A_i' \subset \Delta_i^2$, $|H(x) - x| < \delta$ for $x \in \bigcup_{i=1}^{k} A_i$.

4. Let T be a 3-simplex of $\mathcal{Q}$ such that ∂T contains some of the Γ_j's defined in the preceding step. We may adjust the notation so that Γ_r, $r = 1, \dots, l$, $l < d$, lie in ∂T, where d is the total number of the Γ_j's. In this section we show that for $\gamma > 0$ and for each Γ_r, $r = 1, \dots, l$, there exists a polyhedral disk $\tilde{G}_r$ bounded by Γ_r such that

(1) $\mathrm{Int}\tilde{G}_r \subset \mathrm{Int}T$;

(2) $L(\tilde{G}_r) < L(P_r) + \varepsilon/(2d) + \gamma$;

(3) there is a neighborhood of Γ_r in $\tilde{G}_r$ which does not meet the other open disks $\tilde{G}_{r'} \setminus \Gamma_{r'}$, $r' \neq r$, $1 \leq r' \leq l$.

Since $d_{\mathcal{F}}(\Gamma_r, \partial P_r) < \beta_0$, there is (by Step 3) a singular polyhedral disk G_r' bounded by Γ_r such that $G_r' \setminus \Gamma_r \subset \mathrm{Int}\, T$, $N(G_r') \cap \Gamma_r = \varnothing$, and $L(G_r') \leq L(P_r) + \varepsilon/(2d)$. Let $x \in \partial T \setminus (\bigcup_{r=1}^{l} \Gamma_r)$ and let $J(\Gamma_r)$ be the component of $\partial T \setminus \Gamma_r$ not containing x. Two different $J(\Gamma_r)$'s intersect only if one contains the other (Step 3). Hence we may assume that the Γ_r's are indexed so that $J(\Gamma_r) \subset J(\Gamma_{r+i})$ if $J(\Gamma_r) \cap J(\Gamma_{r+i}) \neq \varnothing$.

For each $J(\Gamma_r)$ let Γ_r' be a polygon in $J(\Gamma_r)$ and let C_r be an annulus in $J(\Gamma_r)$ bounded by Γ_r and Γ_r'. We choose Γ_r' sufficiently close in the Frechet sense to Γ_r so that the area of C_r is less than $\gamma/2$ and there is a singular polyhedral disk G_r'' bounded by Γ_r' such that G_r'' satisfies the conditions of Dehn's lemma, $G_r'' \setminus \Gamma_r' \subset \mathrm{Int}\, T$, and $L(G_r'') < L(G_r') + \gamma/4$. By the version of Dehn's lemma of [4, p.31], there is a polyhedral disk G_r''' with the boundary Γ_r' such that $L(G_r''') < L(G_r'') + \gamma/8$ and $G_r''' \setminus \Gamma_r' \subset \mathrm{Int}\, T$. By adding C_r to G_r''' we obtain a polyhedral disk G_r^{iv} bounded by Γ_r. From the construction it follows that

$$L(G_r^{iv}) < L(P_r) + \varepsilon/(2d) + 7\gamma/8.$$

But G_r^{iv} has not satisfied the conditions (1) and (3) yet.

We now construct the required polyhedral disks $\tilde{G}_r$, $r = 1, \dots, l$. Let $\mathcal{Q}(\partial T)$ be a triangulation of ∂T such that each Γ_r lies in the 1-skeleton of $\mathcal{Q}(\partial T)$, no Γ_r lies on the boundary of any 2-simplex of $\mathcal{Q}(\partial T)$, and each 1-simplex of $\mathcal{Q}(\partial T)$ having both its vertices in Γ_r lies in Γ_r. We denote by $\mathcal{Q}(C_r)$ a triangulation of C_r such that each element of $\mathcal{Q}(C_r)$ is an element of $\mathcal{Q}(\partial T)$. Let $\mathcal{Q}(C_r^{iv})$ be a triangulation of C_r^{iv} which coincides with $\mathcal{Q}(C_r)$ in C_r and with some triangulation of G_r''' in G_r'''.

Let b be the barycenter of T. For each r, $r = 1, \dots, l$, we choose t_r, $0 < t_r < 1$, and define a map q_r of G_r^{iv} in the following way. We put q_r to be fixed on vertices of $\mathcal{Q}(G_r^{iv})$ lying in Γ_r. If v is a vertex of $\mathcal{Q}(G_r^{iv})$ lying in $\mathrm{Int}\, G_r^{iv}$, we denote by $q_r(v)$

the point on the segment from b to v such that $|b-q_r(v)|/|b-v|=t_r$. Then we extend q_r linearly to each element of $\mathcal{Q}(G_r^{iv})$ to obtain the homeomorphism q_r of G_r^{iv} onto the polyhedral disk $\tilde{G}_r=q_r(G_r^{iv})$ bounded by Γ_r.

Clearly the $\tilde{G}_r$'s satisfy the condition (1). The $\tilde{G}_r$'s will also satisfy the conditions (2) and then (3) if we first choose the t_r's sufficiently close to 1 and thereupon such that $1>t_1>\dots t_l>0$.

5. In this step we replace the $\tilde{G}_r$'s by a collection of pairwise disjoint polyhedral disks in T. Namely, for each $\tilde{G}_r,\ r=1,\dots,l<d$, we define a polyhedral disk G_r bounded by Γ_r such that

(1) $\operatorname{Int} G_r\subset \operatorname{Int} T$;

(2) $\operatorname{Int} G_r\cap \operatorname{Int} G_{r'}=\varnothing$ if $r'\neq r,\ 1\leq r'\leq l$;

(3) $\sum_{r=1}^{l} L(G_r)<\sum_{r=1}^{l} L(P_r)+l\varepsilon/(2d)+l\gamma$.

Having made slight shifts of the vertices of the $\tilde{G}_r$'s (except for the vertices in Γ_r) we may assume that the $\tilde{G}_r$'s satisfy not only the conditions (1)–(3) of Step 4 but also that the intersection of the interiors of any two of them is the union of finitely many of mutually disjoint polygons.

Let a polygon Γ be one of the component of $\operatorname{Int}\tilde{G}_1\cap\operatorname{Int}\tilde{G}_2$. By cutting $\tilde{G}_1$ and $\tilde{G}_2$ along Γ and pasting split sheets in appropriate way we obtain two possibly singular polyhedral disks $\tilde{G}_1'$ and $\tilde{G}_2'$ bounded by Γ_1 and Γ_2 respectively such that the collection of components of $N(\tilde{G}_1')$, $N(\tilde{G}_2')$, and $\operatorname{Int}\tilde{G}_1'\cap\operatorname{Int}\tilde{G}_2'$ is a proper subcollection of the components of $\operatorname{Int}\tilde{G}_1\cap\operatorname{Int}\tilde{G}_2$. The iteration of the similar cut-and-paste procedure along the other components (polygons) of $N(\tilde{G}_1')$, $N(\tilde{G}_2')$, and $\operatorname{Int}\tilde{G}_1'\cap\operatorname{Int}\tilde{G}_2'$ leads to two polyhedral disks $\tilde{G}_1''$ and $\tilde{G}_2''$ (bounded by Γ_1 and Γ_2) with disjoint interiors.

After that we proceed to remove the pairwise intersections of the interiors of three polyhedral disks $\tilde{G}_1$, $\tilde{G}_2$, and $\tilde{G}_3$. To that end it is sufficient to remove the components of $(\operatorname{Int}\tilde{G}_1''\cup\operatorname{Int}\tilde{G}_2'')\cap\operatorname{Int}\tilde{G}_3$ in just the same way as we have removed the components of $\operatorname{Int}\tilde{G}_1\cap\operatorname{Int}\tilde{G}_2$. As a result we obtain three polyhedral disks with the boundaries Γ_1, Γ_2 and Γ_3 and mutually disjoint interiors. The similar argument successively applied to the other $\tilde{G}_r$'s permits us to obtain a collection of polyhedral disks $G_1,\dots,G_l$ with the boundaries $\Gamma_1,\dots,\Gamma_l$ and mutually disjoint interiors. Moreover, all these and the above transformations we can do by moving the parts of the $\tilde{G}_r$'s by so little that the G_r's satisfy also the condition (3).

6. To complete the proof of Theorem 1.1 we define polyhedral disks in the other 3-simplices of the triangulation $\mathcal{Q}$ just as we have defined the G_r's in T in the preceding two steps. Thus we construct the collection of polyhedral disks $G_j,\ j=1,\dots,d$, such that the subcollection those of them contained in each tetrahedron T of $\mathcal{Q}$ has the similar properties (1)–(3), which the G_r's have in Step 5.

In Step 3 the homeomorphism H of $\bigcup_{j=1}^{d}\partial P_j$ onto $\bigcup_{j=1}^{d}\Gamma_j$ was defined. Here we extend H to take P_j onto G_j for each $j=1,\dots,d$. Thus we obtain the homeomorphism H of S onto a polyhedral 2-sphere $\Omega=\bigcup_{j=1}^{d} G_j$. Clearly $|H(x)-x|<\varepsilon/2+\delta<\varepsilon$ for $x\in S$ and $L(\Omega)=\sum_{j=1}^{d}L(G_j)\leq\sum_{j=1}^{d}L(P_j)+\varepsilon/2+d\gamma=L(S)+\varepsilon/2+d\gamma$. If $\gamma>0$ is sufficiently small, we have that $L(\Omega)<L(S)+\varepsilon$. This proves Theorem 1.1.

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