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PARTIAL COHOMOLOGIES AND CANONIC ROOTS IN SEMIGROUPS

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A result of the author (MR 90a:20135) on an isomorphism between partial and classical cohomologies of a semigroup with a canonic root is generalized. As an application, the cohomology is calculated for the case when a semigroup is defined by a single relation of the form $aP = Q$ or $Pa = Q$, where the words P and Q don't contain the letter a.

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Обобщается результат автора (РЖМат, 1989, 6A317) об изоморфизме частичных и классических когомологий полугрупп с каноническим корнем. В качестве приложения вычислены когомологии для полугрупп с одним определяющим соотношением вида $aP = Q$ или $Pa = Q$, где слова P и Q не содержат буквы a.

Partial cohomologies are used, in particular, for calculating the classic semigroup cohomology of Eilenberg-McLane (further we shall call it EM-cohomology). Examples of such using were shown in [7] and [8] as a consequence of results obtained there. In this article we generalize Theorem 3.2 from [7] which allows us to describe EM-cohomology for a more wide class of semigroups.

1. Prelminaries

In this section necessary properties of cotriple [2, 3] and partial [7, 8] cohomologies are given for the convenience of reading.

In the sequel the notation $S = \langle X|R\rangle$ means that the semigroup S is generated by the set X with the defining relation set R. We write $S = \langle X\rangle$ if the type of R is not important at this moment.

Let S be a semigroup. A subset $X \subset S$ is called a root of S if $S = \langle X|R\rangle$, where all relations from R are of the form $xy = z$ for some $x, y, z \in X$. E. g., for every presentation

$$S = \langle a_1, \dots , a_m|P_1 = Q_1, \dots , P_n = Q_n\rangle$$

the subset consisting of all generators $a_1, \dots , a_m$ and all subwords of the words P_i and Q_i $(1 \le i \le n)$ is a root (it is easy to see that every root can be obtained in such a way).

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Now we pass to the definition of partial cohomologies.

Let X be a root of a semigroup S, A a (left) S-module. We denote by X_n the set of all n-tuples $(x_1, \dots, x_n)$ such that $x_i x_{i+1} \cdot \ldots \cdot x_j \in X$ for all i, j, $1 \le i \le j \le n$. A map $f: X_n \to A$ is called a partial n-dimensional cochain of X, or an X-cochain, with values in A. The n-dimensional X-cochains form an Abelian group $C^n(S, X, A)$. We set $C^0(S, X, A) = A$ and if $X_n = \varnothing$ then $C^n(S, X, A) = 0$.

A coboundary operator $\partial^n: C^n(S, X, A) \to C^{n+1}(S, X, A)$ is given in the usual way: $\partial^0 a(x) = xa - a$ and for $n \ge 1$,

$$\partial^n f(x_1, ..., x_{n+1}) = x_1 f(x_2, ..., x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, ..., x_i x_{i+1}, ..., x_{n+1}) + \\ +(-1)^{n+1} f(x_1, ..., x_n). \qquad (1)$$

It can be checked straightforward that $\partial^{n+1}\partial^n = 0$.

The quotient group $\operatorname{Ker} \partial^n / \operatorname{Im} \partial^{n-1}$ is called the n-dimensional X-cohomology group and is denoted by $H^n(S, X, A)$. The X-cocycles and X-coboundaries are defined and denoted analogously.

The embedding $X \to S$ induces a homomorphism $\theta^n_X: H^n(S, A) \longrightarrow H^n(S, X, A)$. It was shown in [7] that θ^n_X turned out to be an isomorphism for $n < 2$ and a monomorphism for $n = 2$. Besides, the map $\Theta: Z^1(S, A) \longrightarrow Z^1(S, X, A)$, induced by $X \to S$ is a monomorphism too.

Generally speaking, partial cohomology is not a derived functor. However, it allows a presentation as a cotriple one which allows us to obtain an additional information about homomorphisms θ^n_X.

Previously we cite some information about cotriple (co)homology [2, 3]. We follow [3] as to definitions and notations, but as against [3] we choose the contravariant version, which is necessary for construction of cohomology.

Further on $\mathcal{I}_{\mathsf{C}}$ denotes the identity functor of a given category CC and $\iota_{\mathcal{G}}$ denotes the identity natural transformation of a functor $\mathcal{G}$ (we shall delete the subscript at ι if it is clear what functor is considered).

Let C be a category, $(\mathcal{G}, \varepsilon, \delta)$ be a cotriple on C, i. e. $\mathcal{G}: \mathsf{C} \longrightarrow \mathsf{C}$ is an endofunctor, $\varepsilon: \mathcal{G} \longrightarrow \mathcal{I}_{\mathsf{C}}$ and $\delta: \mathcal{G} \longrightarrow \mathcal{G}^2$ are natural transformations satisfying the equations

$$\delta \cdot \varepsilon\mathcal{G} = \delta \cdot \mathcal{G}\varepsilon = \iota_{\mathcal{G}^2}, \qquad \delta \cdot \delta\mathcal{G} = \delta \cdot \mathcal{G}\delta.$$

Then it is possible to construct an augmented simplicial object $\mathcal{G}^* X$ (called a standard resolution of X with respect to $\mathcal{G}$) for every object $X \in \mathsf{C}$:

$$X \longleftarrow \mathcal{G}X \leftleftarrows \mathcal{G}^2 X \Lleftarrow \mathcal{G}^3 X \ldots .$$

Here n arrows from $\mathcal{G}^n X$ to $\mathcal{G}^{n-1} X$ denote the morphisms corresponding to the natural transformations $\varepsilon_i^{(n-1)} = \mathcal{G}^i \varepsilon \mathcal{G}^{n-i-1}: \mathcal{G}^n \longrightarrow \mathcal{G}^{n-1}$, $0 \le i \le n-1$.

If A is an Abelian category, $\mathcal{K}: C \longrightarrow \mathsf{A}$ a contravariant functor, we can construct a chain complex

$$0 \longrightarrow \mathcal{K}\mathcal{G}X \xrightarrow{d^1} \mathcal{K}\mathcal{G}^2 X \xrightarrow{d^2} \ldots ,$$

where $d^n = \sum_{i \le n-1} (-1)^i \mathcal{K}\varepsilon_i^{(n)}(X)$. Its cohomology is denoted by $H^n(X, \mathcal{K})_{\mathcal{G}}$ and called the cotriple one, or the Barr-Beck cohomology.

Cotriple theory deals with chain complexes of functors in which natural transformations are used as boundary operators. For example, instead of (2) one must consider the complex

$$0 \longrightarrow \mathcal{K}\mathcal{G} \longrightarrow \mathcal{K}\mathcal{G}^2 \longrightarrow \dots$$

with the natural transformations of the form $\sum_{i\le n-1}(-1)^i\mathcal{K}\varepsilon_i^{(n)}$. In general, let $\mathcal{L}_n\colon \mathsf{C} \to \mathsf{A}$ $(n \ge -1)$ be contravariant functors into an Abelian category A and $\delta^n\colon \mathcal{L}_n \longrightarrow \mathcal{L}_{n+1}$ be natural transformations such that $\delta^{n+1}\delta^n = 0$. Then the sequence

$$0 \longrightarrow \mathcal{L}_{-1} \xrightarrow{\delta^{-1}} \mathcal{L}_0 \xrightarrow{\delta^0} \mathcal{L}_1 \xrightarrow{\delta^1} \dots \tag{3}$$

is a chain complex of functors. The cohomology of complex (3) are denoted by $H^n(\mathcal{L}_{*}\text{\_})$.

Complex (3) is called $\mathcal{G}$-representable, if there are natural transformations $\tau^n\colon \mathcal{L}_n\mathcal{G} \to \mathcal{L}_n$ such that $\tau^n \cdot \mathcal{L}_n\varepsilon = \iota_{\mathcal{L}_n}$. Complex (3) is called $\mathcal{G}$-contractible if the complex $\{\mathcal{L}_n\mathcal{G}\}_{n\ge-1}$ has a (natural) contracting homotopy $\sigma^n\colon \mathcal{L}_n\mathcal{G} \longrightarrow \mathcal{L}_{n-1}\mathcal{G}$ (the latter means that $\sigma^0\delta^{-1}\mathcal{G} = \iota$ and $\delta^{n-1}\mathcal{G}\sigma^n + \sigma^{n+1}\delta^n\mathcal{G} = \iota$).

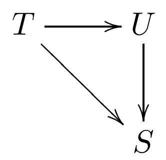
Further on we need a result about the comparison of cohomologies which is a special case of Proposition 11.2 from [3]:

Theorem 1.1. *Suppose that complex (3) is $\mathcal{G}$-representable and its cohomology is*

$$H^n(\mathcal{L}_n\mathcal{G}X) = \begin{cases} \mathcal{L}_*\mathcal{G}X, & \textit{if } n = 0, \\ 0, & \textit{if } n > 0 \end{cases} \tag{4}$$

for every $X \in C$. Then $H^n(\mathcal{L}_*\text{\_}) \cong H^n(\text{\_}, \mathcal{L}_{-1})_{\mathcal{G}}$ for all $n \ge 0$.

A category PSem was constructed in [8] to present partial cohomology by cotriple one. Its objects were all pairs (S, X) where $X \subseteq S$ and a morphism $\alpha\colon (S, X) \longrightarrow (T, Y)$ was defined as a semigroup homomorphism $\alpha\colon S \longrightarrow T$ with $\alpha(X) \subseteq Y$. An object (S, X) of PSem will be denoted by S too if it doesn't lead to a confusion. For a fixed object S from PSem the notation $\mathsf{PSem} \downarrow S$ is used for the comma-category: its objects are morphisms $T \longrightarrow S$ of PSem and its morphisms are commutative diagrams of the form



Let $X \subseteq S$, X be the set of symbols of the form x which are in 1-1 correspondence with elements $x \in X$, F_X be a free semigroup generated by X and $\tilde{X} = \{x_1 \dots x_n \,|\, (x_1, \dots, x_n) \in X_n\}$ (X_n has been defined above). In particular, $X \subseteq \tilde{X}$. It is easy seen that the mapping $\mathcal{G}\colon (S, X) \longrightarrow (F_X, \tilde{X})$ gives an endofunctor of PSem. We define natural transformations $\delta\colon \mathcal{G} \longrightarrow \mathcal{G}^2$ and $\varepsilon\colon \mathcal{G} \longrightarrow \mathcal{I}_{\mathsf{PSem}}$ by the formulae: $\delta(S)\colon x \longrightarrow x$, $\varepsilon(S)\colon x \longrightarrow x$ $(x \in X)$. Then $(\mathcal{G}, \varepsilon, \delta)$ is a cotriple in PSem which induces the cotriple $(\overline{\mathcal{G}}, \overline{\varepsilon}, \overline{\delta})$ in $\mathsf{PSem} \downarrow S$.

Let $(S, X) \in \mathsf{PSem}$. For every S-module A and every object $\alpha\colon (T, Y) \longrightarrow (S, X)$ from $\mathsf{PSem} \downarrow S$, we denote by $\operatorname{Der}((T, Y) \to (S, X), A)$ (or $\operatorname{Der}(T, Y, A)$ for brevity) the set of maps $f\colon Y \longrightarrow A$ such that $f(xy) = \alpha(x)f(y) + f(x)$ as soon as $xy \in X$; we shall also write $xf(y)$ instead of $\alpha(x)f(y)$. Since this set turns naturally into an Abelian group, thereby we obtain a contravariant functor $\operatorname{Der}(\text{\_}, A)$ from

$\mathsf{PSem} \downarrow S$ to the category of Abelian groups. This enables us to construct a cotriple complex of functors $\mathcal{K}_* = \{\mathcal{K}_n = \mathrm{Der}(\overline{\mathcal{G}}^{n+1}{}_{-}, A)\}_{n \geq -1}$, where $\mathcal{K}_{-1} = \mathrm{Der}(\_, A)$. The cohomology of this complex (more exactly, the values of the cohomology functors of the complex) will be denoted by $H^n(T, Y, A)_{\mathcal{G}}$.

On the other hand, since S-module A turns into T-module by stepping back along the homomorphism $T \longrightarrow S$, a contravariant functor $C^n(\_, A)$ is defined. This functor compares the Abelian group of the partial cochains $C^n(T, Y, A)$ to the object $T \longrightarrow S$. By the boundary homomorphisms ∂^n we obtain the augmented complex of functors $\mathcal{L}_* = \{\mathcal{L}_n = C^{n+1}(\_, A)\}_{n \geq -1}$, where $\mathcal{L}_{-1} = \mathrm{Der}(\_, A)$. The boundary natural transformations of this complex will be denoted by ∂^n too.

It was shown in [8] (Theorem 2.1) that $H^n(T, Y, A)_{\mathcal{G}} \cong H^{n+1}(T, Y, A)$ for $n > 0$.

In particular, in the case $T = S$ we obtain the required presentation of the partial cohomology by the Barr-Beck one. Besides, for $X = S$ we have obtained an analogous result for EM-cohomology: $H^{n+1}(T, A) \cong H^n(T, T, A)_{\mathcal{G}}$ in the category $\mathsf{PSem} \downarrow (S, S)$ for every S-module A.

Moreover, let B be a $\overline{\mathcal{G}}$-closed subcategory of $\mathsf{PSem} \downarrow S$ (i. e. a full subcategory closed with respect to action of $\overline{\mathcal{G}}$ on its objects) and $H^n(T, Y, A)^{\mathsf{B}}_{\mathcal{G}}$ be the groups of cotriple cohomology which are constructed in B. Then $H^{n+1}(T, Y, A) \cong H^n(T, Y, A)^{\mathsf{B}}_{\mathcal{G}}$ if B contains the objects $(S, X) \longrightarrow (S, X)$ (the identity morphism) and $(T, Y) \longrightarrow (S, X)$ (Theorem 2.2 [8]).

2. Comparison with EM-cohomology

Let S be a semigroup, X be its root. A decomposition $x = x_1 \dots x_n$ $(x_i \in X)$ of an element $x \in S \setminus X$ is called reduced if $x_i x_{i+1} \dots x_j \notin X$ for each i, j, $1 \leq i < j \leq n$. We mean that a reduced decomposition of an element $x \in X$ is its decomposition into product of one multiplier. A root X is said to be canonic if each element $x \in S$ has a unique reduced decomposition.

For example, the set of all element of S is a canonic root.

A root X is called a J-root if $xy = x$, $yz = z$ implies $xz \in X$ for all $x, y, z \in X$.

Further on we shall need a result from [5]. Theorem 1 of that article being applied to our case is formulated as follows:

Lemma 2.1. *Let X be a root of a semigroup $U = \langle X \rangle$ satisfying the following condition: $uv, vw \in X$ implies $uvw \in X$ for all $u, v, w \in U$. Then X is canonic.*

As above we consider the category $\mathsf{PSem} \downarrow (S, X)$ and set for each its object $(T, Y) \to (S, X)$ (the latter one will be also denoted by $T \to S$) $\mathcal{K}_n(T \to S) = \mathrm{Der}(\overline{\mathcal{G}}^{n+1}(T \to S), A)$, $\mathcal{M}_{-1}(T \to S) = \mathrm{Der}(T, A)$ and $\mathcal{M}_n(T \to S) = C^{n+1}(T, A)$, the group of EM-cochains. Here S-module A is considered as a T-module with the evident action of T.

Since $H^n(\mathcal{M}_* \overline{\mathcal{G}}(T \to S)) = H^{n+1}(F_Y, A)$ and F_Y is a free semigroup, condition (4) carries out evidently (see, e. g., [4]).

In what follows we have to refer to Lemma 3.2 from [8]:

Lemma 2.2. *Let B be a $\overline{\mathcal{G}}$-closed subcategory of $\mathsf{PSem} \downarrow (S, X)$. If there is a collection of maps $\{\rho_{T \to S}: T \longrightarrow F_Y\}_{(T \to S) \in \mathsf{B}}$ such that:*

1) $\varepsilon(T)\rho_{T \to S}$ is the identity map;

2) for each morphism $\varphi\colon (T,Y) \longrightarrow (U,Z)$ from B the diagram

$$\begin{array}{ccc} T & \xrightarrow{\ \varphi\ } & U \\ {\scriptstyle \rho_{T\to S}}\downarrow & & \downarrow{\scriptstyle \rho_{U\to S}} \\ F_Y & \xrightarrow{\ \overline{\mathcal{G}}_\varphi\ } & F_U \end{array}$$

is commutative, then the complex $\mathcal{M}$ is $\overline{\mathcal{G}}$-representable in the category B.

Let X be a canonic J-root of a semigroup S. Denote by B the full subcategory in $\mathsf{PSem} \downarrow (S,X)$ whose objects are the morphisms $\varphi\colon (T,Y) \to (S,X)$, where Y is a canonic J-root in T.

Lemma 2.3. *The subcategory B is $\overline{\mathcal{G}}$-closed.*

Proof. Show that for every object $\varphi(T,Y) \to (S,X)$ from B the object $\overline{\mathcal{G}}\varphi(F_Y,\tilde{Y}) \to (S,X)$ is in B too. Evidently, $\tilde{Y}$ is a root. Moreover, F_Y is a free semigroup and the equalities $xy = x,\ yz = z$ don't hold in it; therefore $\tilde{Y}$ is a J-root.

Let $ab, bc \in \tilde{Y}$, where

$$a = x_1 \dots x_m,\ b = y_1 \dots y_n,\ c = z_1 \dots z_p.$$

To prove that $abc \in \tilde{Y}$ it is enough to check that $x_i \dots x_m y_1 \dots y_n z_1 \dots z_j \in Y$ for $1 \le i \le m$, $1 \le j \le n$. Denote

$$u = x_i \dots x_m,\ v = y_1 \dots y_n,\ w = z_1 \dots z_j.$$

It follows from $ab, bc \in \tilde{Y}$ that $u, v, w, uv, vw \in Y$. If $uvw \notin Y$, there are two reduced decompositions: $(uv)w = u(vw)$. Hence $uv = u$, $vw = w$, from where $uvw \in Y$ and $abc \in \tilde{Y}$. Now Lemma 2.1 implies that $\tilde{Y}$ is canonic. □

Theorem 2.1. *If X is a canonic J-root of S then θ^n_X are isomorphisms for all $n \ge 0$.*

Proof. Let B be the category defined above, $\varphi{:}(T,Y) \to (S,X)$ be its object, $t = y_1 \dots y_r$ $(y_i \in Y)$ be a reduced decomposition of an element $t \in T$. Set $\rho_{T\to S} t = y_1 \dots y_r$. Then $\varepsilon(T)\rho_{T\to S} t = y_1 \dots y_r = t$. Besides, if

$$\begin{array}{ccc} T & \xrightarrow{\ \mu\ } & U \\ & {\scriptstyle \varphi}\searrow & \downarrow{\scriptstyle \psi} \\ & & S \end{array}$$

is a morphism from B, then $\mu y_1 \dots \mu y_r$ is a reduced decomposition of μt; hence $\rho_{U\to S}\mu t = \mu y_1 \dots \mu y_r = (\overline{\mathcal{G}}\mu)\rho_{T\to S} t$, i.e. conditions 1) and 2) of Lemma 2.2 hold and the complex $\mathcal{M}$ is $\overline{\mathcal{G}}$-representable.

Evidently, $\mathcal{K}_{-1}(T \to S) = Z^1(T,Y,A)$ and $\mathcal{M}_{-1}(T \to S) = Z^1(T,A)$. The embedding $X \to S$ induces a homomorphism $\Theta{:}Z^1(T,A) \to Z^1(T,Y,A)$. The injectivity of Θ follows from the equality $f(s_1 \dots s_r) = \sum_{i=0}^{r-1} s_1 \dots s_i f(s_{i+1})$, which holds for every 1-dimensional partial cocycle f, if $f(s_1 \dots s_i)$ and $f(s_i)$ are defined at $1 \le i \le r$. The surjectivity of Θ is proved in [7] (Proposition 2.1). It follows from here that the functors $\mathcal{K}_{-1}$ and $\mathcal{M}_{-1}$ are isomorphic. Now Theorem 1.1 implies an isomorphism of the cohomology functors of complexes $\mathcal{K}$ and $\mathcal{M}$. Since Theorem 2.2 [8] is valid for a $\overline{\mathcal{G}}$-closed subcategory (see the remark in the end of the preceding section), $H^n(S,A) \cong H^n(S,X,A)$. According to Proposition 11.1 from [3] this isomorphism coincides with θ^n_X, because θ^n_X is induced by the map $C^n(S,A) \to C^n(S,X,A)$. □

3. Application: calculating EM-cohomology

Theorem 2.1 enables us to use a partial cohomology for calculating EM-cohomology of semigroups in the case that succeeds in finding a "good" root in a given semigroup. For instance, if $S = T * U$ is the free product of semigroups T and U, then $X = T \cup U$ is its canonic J-root and $X_n = T_n \cup U_n$. Thus, employing Theorem 2.1 we get

$$H^n(S, A) \cong H^n(S, X, A) \cong H^n(T, A) \bigoplus H^n(U, A)$$

for every S-module A. Below we consider less trivial examples.

Let $S = \langle a, b_1, b_2, \dots | aP = Q\rangle$ be such a semigroup that the defining words P and Q don't contain the letter a. According to [10] (see also [9]) the subsemigroup $F = \langle b_1, b_2, \dots\rangle$ is free. Denote $X = F \cup \{a\}$. Evidently, X is a root: one can take the multiplication table of the semigroup F and the equality $P \cdot a = Q$ as defining relations. It is easy to see that $X_2 = (F \times F) \cup (\{a\} \times PF^1)$ and $X_3 = (F \times F \times F) \cup (\{a\} \times PF^1 \times F)$. We are going now to calculate the 2-dimensional X-cohomology of S.

Lemma 3.1. *$H^2(S, X, A) = 0$ for every X-module A.*

Proof. Let $f \in Z^2(S, X, A)$. Since F is free, one can assume that the restriction of f on F equals zero: $f(x, y) = 0$ for all $x, y \in F$. Applying the equality $\partial f(a, P, x) = 0$ we obtain $f(a, Px) = f(a, P)$ for $x \in F$.

Set $\varphi(x) \equiv 0$, $\varphi(a) = f(a, P)$. Then

$$\partial\varphi(a, Px) = \varphi(a) = f(a, P) = f(a, Px),$$

i. e. $f = \partial\varphi$. □

Lemma 3.1 and the injectivity of θ^2_X imply $H^2(S, A) = 0$ for every S-module A. So the following assertion has been proved:

Theorem 3.1. *$H^n(S, A) = 0$ for every S-module A and for all $n \geq 2$.*

As another example we consider the semigroup $T = \langle a, b_1, b_2, \dots | Pa = Q\rangle$, antiisomorphic to S. As for S, the subset $X = F \cup \{a\}$ is a root; in addition $X_2 = (F \times F) \cup (F^1P \times \{a\})$ and $X_3 = (F \times F \times F) \cup (F \times F^1P \times \{a\})$.

We recall the definition of the Fox' derivative [6] adapted to semigroups in [8].

Let $F = \langle b_1, b_2, \dots | \varnothing\rangle$ be a free semigroup, $x = x_1 b_i x_2 b_i \dots x_{n-1} b_i x_n \in F$, where the words $x_k \in F$ don't contain the letter b_i. We define a derivative of the word x with respect to b_i to be an element of the semigroup algebra $\mathbb{Z}F^1$:

$$\frac{\partial x}{\partial b_i} = \sum_{k=1}^{n-1} x_1 b_i \dots x_{k-1} b_i x_k;$$

moreover, if $x_1 = \varnothing$ then the first summand in $\frac{\partial x}{\partial b_i}$ equals 1; if the letter b_i doesn't occur in x then we set $\frac{\partial x}{\partial b_i} = 0$.

Lemma 3.2. *For every X-module A*

a) $H^2(T, X, A) \cong A/B$, where $B = PA + \sum_i \left(\frac{\partial P}{\partial b_i} - \frac{\partial Q}{\partial b_i}\right) A$;

b) $H^3(T, X, A) = 0$.

Proof. a) Let $f \in Z^2(T,X,A)$. As above one can set

$$f(x,y) = 0 \quad (x,y \in F). \tag{5}$$

The equality $\partial f(x,P,a) = 0$ implies

$$f(xP,a) = xf(P,a). \tag{6}$$

Conversely, if an X-cochain $f \in C^2(T,X,A)$ satisfies (5) and (6) then $f \in Z^2(T,X,A)$. Hence, instead of $Z^2(T,X,A)$ it is enough to treat with the subgroup Z_0 of X-cocycles yielding to equalities (5), (6). Each of these X-cocycles is defined by its value $f(P,a)$, i. e. $Z_0 \cong A$.

Now we find the coboundaries from Z_0. The condition $f = \partial\varphi$ is equivalent to the system of equations

$$x\varphi(y) - \varphi(xy) + \varphi(x) = 0 \ (x,y \in F), \quad P\varphi(a) - \varphi(Q) + \varphi(P) = f(P,a).$$

It follows from the first equation that

$$\varphi(x) = \sum_i \frac{\partial x}{\partial b_i}\varphi(b_i),$$

and from the second one

$$f(P,a) = \sum_i \left(\frac{\partial P}{\partial b_i} - \frac{\partial Q}{\partial b_i}\right)\varphi(b_i) + P\varphi(a).$$

As one can choose the values $\varphi(b_i)$ and $\varphi(a)$ arbitrarily, we have:

$$f \sim 0 \quad \Longleftrightarrow \quad f(P,a) \in PA + \sum_i \left(\frac{\partial P}{\partial b_i} - \frac{\partial Q}{\partial b_i}\right)A.$$

b) Let $f \in Z^3(T,X,A)$. We suppose again that $f(x,y,z) = 0 \ (x,y,z \in F)$. Define the X-cochain φ by the following way:

$$\varphi(x,y) = \varphi(P,a) = 0, \quad \varphi(xP,a) = -f(x,P,a).$$

It is easy to see that $f = \partial\varphi$. □

Lemma 3.3. *X is a canonic J-root of T.*

Proof. As T has no right cycles in Adyan's sense [1], it is right cancellative. Therefore, the equality $yz = z$ is impossible in it and X turns out to be a J-root.

Every element of T can be written in the form

$$u = x_1ax_2\ldots ax_n, \quad \text{where } n \geq 1, x_i \in F \setminus PF^1 \ (i < n), \tag{7}$$

where the words x_1 and x_n for $n > 1$ can be empty. We show that $u \notin F$ if $n > 1$. Indeed, in the opposite case there exists a sequence $u \longrightarrow u^{(1)} \longrightarrow \ldots \longrightarrow u^{(r)}$ consisting of words in the alphabet $\{a, b_1, b_2, \ldots\}$ such that every word is obtained from the preceding one by applying the equality $Pa = Q$ and besides the last word $u^{(r)}$ doesn't contain the letter a. Since on some step the first of the letters a in the word u must disappear, so the word x_1 must be transformed into a word of the form yP, $(y \in F)$ before this step; moreover, this transformation is realized independently on the other ones. Therefore $x_1 = yP$ in the semigroup T and so in F. However the semigroup F is free and we have got a contradiction with $x_1 \in F \setminus PF^1$.

Let $u, v \in T$ and $uv \in F$. Show that $u \in F$ in this case. Indeed, if it doesn't hold, then after writing u and v as (7) and substituting in uv the expressions Pa (which can arise on the border of the words u and v) by Q we obtain a reduced decomposition $uv = x_1 a \dots$. Since $x_1 \notin PF^1$, it turns out that $uv \notin F$, according to the above.

We can use now Lemma 2.1 to prove the canonicity. If $uv, vw \in X$, then evidently $uv, vw \in F$. It implies that $u \in F$ and $u(vw) \in F$, as we have shown. □

Applying Theorem 2.1, we conclude from here:

Theorem 3.2. *For every T-module A*

a) $H^2(T, A) \cong A/B$, where $B = PA + \sum_i \left(\frac{\partial P}{\partial b_i} - \frac{\partial Q}{\partial b_i}\right) A$;

b) $H^n(T, A) = 0$ for all $n \geq 3$.

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