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ON SYSTEMS OF ORTHOGONAL PERMUTATIONS

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We introduce the notion of complete system of orthogonal permutations in $\mathbb{R}^n$ and show that any such system generates a structure of divisions algebra in $\mathbb{R}^n$. We describe the systems of orthogonal permutations that correspond to the fields of real and complex numbers, the skew field of quaternions and, the Cayley algebra.

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Введено понятие полной системы ортогональных перестановок в $\mathbb{R}^n$ и показано, что каждая такая система порождает на $\mathbb{R}^n$ структуру алгебры с делением. Описаны системы ортогональных престановок, соответствующие полям действительных и комплексных чисел, телу кватернионов и алгебре Кели.

Let $\mathbb{R}^n$ be the n-dimensional real vector space, and let $\mathbf{e}_i(0,\dots,1,\dots,0)$, $i=1,\dots,n$, be its standard basis. A linear transformation $P:\mathbb{R}^n\to\mathbb{R}^n$ is called a *linear permutation* (or simply *permutation*) if, for any $\mathbf{a}\in\mathbb{R}^n$, $\mathbf{a}=(a_1,\dots,a_n)$, $P\mathbf{a}=(\varepsilon_1(P)a_{\pi_P(1)},\dots,\varepsilon_n(P)a_{\pi_P(n)})$, where $\pi_P\in\mathbf{S}_n$ is a permutation and $\varepsilon_i(P)\in\{+1,-1\}$. Two linear permutations P and P' are called *orthogonal* if $(P\mathbf{a},P'\mathbf{a})=0$ for every $\mathbf{a}\in\mathbb{R}^n$, $(\_,\_)$ being the standard scalar product in $\mathbb{R}^n$. A set $\mathcal{P}=(P_1,P_2,\dots,P_m)$ of (linear) permutations is called an *orthogonal system of permutations* if P_i and P_j are orthogonal for any two distinct $i,j\in\{1,\dots,m\}$. Obviously, then $m\le n$. If $m=n$, this orthogonal system of permutations is said to be *complete.*

It is clear that, given any orthogonal system of permutations $\mathcal{P}=(P_1,\dots,P_m)$ and any permutation P, one can construct a new orthogonal system $P\mathcal{P}=(PP_1,\dots,PP_m)$. That is why we shall only consider the systems such that $P_1=E$ (the identity mapping).

Given an orthogonal system of permutations $\mathcal{P}=(P_1,\dots,P_n)$, put $\varepsilon_{ij}=\varepsilon_i(P_j)$ and $p_i=\sum_{j=1}^n P_i\mathbf{e}_j$. Then $p_i=(p_{i1},\dots,p_{in})$ with $p_{ij}=\varepsilon_{ij}p_{\pi_i(j)}$, where $\varepsilon_{ij}=\varepsilon_j(P_i)$ and $\pi_i=\pi_{P_i}$. Obviously, the system $(p_1,\dots,p_n)$ determines the orthogonal system of permutations $\mathcal{P}$. Remark that π_1 is the identity permutation as $P_1=E$.

To each complete system of orthogonal permutations $\mathcal{P}=(P_1,\dots,P_n)$ we associate an $\mathbb{R}$-algebra (not necessarily associative) $A_{\mathcal{P}}$ with a basis $\mathbf{e}_i$ $(i=1,\dots,n)$

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and the multiplication given by the rule: $\mathbf{e}_i\mathbf{a} = P_i\mathbf{a}$. Remark that if $P_1 = E$, the vector $\mathbf{e}_1$ is a left unit of this algebra.

Theorem 1. *For every complete system of orthogonal permutations $\mathcal{P}$, the algebra $A_{\mathcal{P}}$ is a division algebra* (i.e., for any $\mathbf{a}, \mathbf{b} \in A_{\mathcal{P}}$, $\mathbf{a} \neq 0$, each of equations (1) $\mathbf{xa} = \mathbf{b}$ and (2) $\mathbf{ay} = \mathbf{b}$ has a unique solution).

Proof. It is well-known (and easy to see) that, as $A_{\mathcal{P}}$ is finite dimensional, it is enough to prove that one of the equations (1) or (2) has a solution for every $\mathbf{a} \neq 0$ or, the same, that the vectors $\mathbf{e}_i\mathbf{a}$ $(i = 1, \dots, n)$ form a basis of $A_{\mathcal{P}}$. But in our case the vectors $\mathbf{e}_i\mathbf{a} = P_i\mathbf{a}$ are non-zero and pairwise orthogonal. Hence, they form even an orthogonal basis of $A_{\mathcal{P}}$. □

The following theorem is well-known [1]:
If A is a finite dimensional division algebra over $\mathbb{R}$, then $\dim_{\mathbb{R}} A = 2^n$ *for* $n = 0, 1, 2, 3$.
As a corollary we obtain:

Corollary. *There exists no complete system of orthogonal permutations of n-dimensional vectors if* $n \neq 1, 2, 4, 8$.

This corollary gives an answer to a question of B. Sendov, who formulated it during his lecture at Kyiv University in November 1995.

We show how to obtain orthogonal permutations, constructed by B.Sendov, using the division algebras theory.

A division algebra A is called *alternative* if all its subalgebras generated by two elements are associative. The following finite dimensional division algebras over the field of real numbers $\mathbb{R}$ are well known (cf. [2, Ch. 5, § 6]):

0) the field of real numbers $\mathbb{R}$;

1) the field of complex numbers $\mathbb{C}$;

2) the skew field of quaternions $\mathbb{H}$. This algebra possesses a basis $\{1, i, j, k\}$ with the multiplication rule:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

3) the Cayley algebra (the algebra of octaves) $\mathbb{O}$. It is an 8-dimensional (non-associative) division algebra over the field of real numbers. The Cayley algebra consists of all formal terms $\alpha + \beta e$, where α, β are quaternions and e is a new symbol, with obvious addition and multiplication by real numbers, and the multiplication given by the rule:

$$(\alpha + \beta e)(\gamma + \delta e) = (\alpha\gamma - \bar{\gamma}\beta) + (\delta\alpha + \beta\bar{\gamma})e,$$

where $\bar{\delta}$ and $\bar{\gamma}$ are the conjugate quaternions to the quaternions, respectively, δ and γ.

In other words, it is an 8-dimensional vector space over $\mathbb{R}$ with a basis $\{1, i, j, k, e, ie, je, ke\}$ and the multiplication rule:

	1	i	j	k	e	ie	je	ke
1	1	i	j	k	e	ie	je	ke
i	i	-1	k	-j	ie	-e	-ke	je
j	j	-k	-1	i	je	ke	-e	-ie
k	k	j	-i	-1	ke	-je	ie	-e
e	e	-ie	-je	-ke	-1	i	j	k
ie	ie	e	-ke	je	-i	-1	-k	j
je	je	ke	e	-ie	-j	k	-1	-i
ke	ke	-je	ie	e	-k	-j	i	-1

The following theorem (cf. [2, Ch. 5, § 6]) is called the generalized Frobenius theorem:

Theorem 3.
1. *Only the real number field $\mathbb{R}$ and the complex number field $\mathbb{C}$ are finite dimensional real associative-commutative algebras without zero divisors.*
2. *Only the skew field $\mathbb{H}$ is a finite dimensional real associative but non-commutative algebra without zero divisors.*
3. *Only Cayley algebra is a finite dimensional real alternative but non-associative algebra without zero divisors.*

Here is the structure of orthogonal permutations which correspond to the field of complex numbers $\mathbb{C}$, the skew field of quaternions $\mathbb{H}$ and the Cayley algebra $\mathbb{O}$:

1) Complex numbers $\mathbb{C}$.

Multiplying the complex number $a_1 + a_2 i$ corresponding to the vector $\mathbf{a} = (a_1, a_2)$ by the basic elements 1 and i, we get:

$$P_1\mathbf{a} = (a_1, a_2), \quad P_2\mathbf{a} = (-a_2, a_1).$$

2) Quaternions $\mathbb{H}$.

Again, multiplying the quaternion $a_1 1 + a_2 i + a_3 j + a_4 k$ corresponding to the vector $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ by the basic elements $1, i, j, k$, we get the following permutations in $\mathbb{R}^4$:

$$P_1\mathbf{a} = (a_1, a_2, a_3, a_4), \quad P_2\mathbf{a} = (-a_2, a_1, -a_4, a_3),$$
$$P_3\mathbf{a} = (-a_3, a_4, a_1, -a_2), \quad P_4\mathbf{a} = (-a_4, -a_3, a_2, a_1).$$

3) Cayley algebra $\mathbb{O}$.

Just in the same way one obtains the following permutations in $\mathbb{R}^8$ (for $\mathbf{a} = (a_1, a_2, a_3,\ a_4,\ a_5, a_6, a_7, a_8)$):

$$P_1\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8), \quad P_2\mathbf{a} = (-a_2, a_1, -a_4, a_3, -a_6, a_5, a_8, -a_7),$$
$$P_3\mathbf{a} = (-a_3, a_4, a_1, -a_2, -a_7, -a_8, a_5, a_6), \quad P_4\mathbf{a} = (-a_4, -a_3, a_2, a_1, -a_8, a_7, -a_6, a_5),$$
$$P_5\mathbf{a} = (-a_5, a_6, a_7, a_8, a_1, -a_2, -a_3, -a_4), \quad P_6\mathbf{a} = (-a_6, -a_5, a_8, -a_7, a_2, a_1, a_4, -a_3),$$
$$P_7\mathbf{a} = (-a_7, -a_8, -a_5, a_6, a_3, -a_4, a_1, a_2), \quad P_8\mathbf{a} = (-a_8, a_7, -a_6, -a_5, a_4, a_3, -a_2, a_1).$$

REFERENCES

[1] Adams J.F., *On the non-existence of elements of Hopf invariant one*, Math. Ann. **72** (1960), 20–104.

[2] Kurosh A.G., *Lectures on General Algebra*, Nauka, Moscow, 1962. (in Russian)

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