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SLOW GROWTH OF POWER SERIES CONVERGENT IN THE UNIT DISK

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The conditions on Taylor coefficients of an analytic in the unit disk function are found in order that the logarithm of maximal term is a slowly increasing function. A problem of slow increasing of maximum modulus is investigated.

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Найдены условия на тейлоровские коэффициенты аналитической в единичном круге функции, при которых логарифм максимального члена является медленно возрастающей функцией. Исследуются проблемы медленного возрастания логарифма максимума модуля.

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the disk $\{z:|z|<1\}$, $M(r)=\max\{|f(z)|:|z|=r\}$, $\mu(r)=\max\{|a_n|r^n : n\ge 0\}$ the maximal term, and $\nu(r)=\max\{n:|a_n|r^n=\mu(r)\}$ the central index, $0\le r<1$. A positive continuous and increasing to $+\infty$ on $[0,1)$ function l is called slowly increasing iff $l((x+1)/2)\sim l(x)$ as $x\uparrow 1$. We will investigate the problem when the functions $\ln\mu(r)$ and $\ln M(r)$ are slowly increasing.

Theorem 1. *Let $\varkappa(n,m)=(\ln|a_m|-\ln|a_n|)/(n-m)$, $n>m$. The function $\ln\mu(r)$ is slowly increasing iff there exists an increasing sequence (n_k) of positive integers such that*

$$\varkappa(n_{k+1},n_k)>\varkappa(n_k,n_{k-1}),\quad k\ge 2, \tag{1}$$

$$\varkappa(n_{k+1},n_k)\le\varkappa(n,n_k),\quad n_k<n<n_{k+1}, \tag{2}$$

and

$$\frac{\ln|a_{n_{k+1}}|}{n_{k+1}|\varkappa(n_{k+1},n_k)|}\to+\infty,\quad k\to+\infty. \tag{3}$$

Proof. Using the equality $\ln\mu(r)-\ln\mu(r_0)=\int_{r_0}^{r}\nu(t)/t\,dt$, $0\le r_0\le r<1$, it is easy to see that $\ln\mu(r)$ is slowly increasing iff

$$\frac{\nu(r)(1-r)}{\ln\mu(r)}\to 0,\quad r\uparrow 1. \tag{4}$$

On the plane XOY we consider the point set $P_n=(n,-\ln|a_n|)$, and besides, if $|a_n|=0$ then we assume that $-\ln|a_n|=+\infty$. Since $\overline{\lim}_{n\to\infty}\ln|a_n|=+\infty$ and $\underline{\lim}_{n\to\infty}-\ln|a_n|/n=0$, then

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an infinite number points P_n lies in the half-plane $\{(x,y): \ y<0\}$, their convex hull from below is contained in this half-plane (provided $|a_0|=1$ for simplicity), and the angle coefficients of its edge increase to 0. Let $y=g(x)$, $x\geq 0$, be the equation of this polygon. Put $c_n=\exp\{-g(n)\}$ and consider an analytic function $f_0(z)=\sum_{n=0}^{\infty} c_n z^n$ in $\{z: \ |z|<1\}$. This function is called the Newton majorant for f. It is known that $\ln\mu(r,f)=\ln\mu(r,f_0)$ and $\nu(r,f)=\nu(r,f_0)$.

Further, let (n_k) be an arbitrary increasing sequence of positive integers and $F(z)=\sum_{k=1}^{\infty} a_{n_k} z^{n_k}$. If (n_k) satisfies conditions (1) and (2) then the functions f and F have the same Newton majorant. Indeed, let $y=g_1(x)=\inf\{y:(x,y)\in\Gamma\}$, where Γ is the convex hull of the point set $(n_k,-\ln|a_{n_k}|)$. Then from (1) it follows that all points $(n_k,-\ln|a_{n_k}|)$ are vertices of the polygon $y=g_1(x)$. The equation of a segment joining the points $(n_k,-\ln|a_{n_k}|)$ and $(n_{k+1},-\ln|a_{n_{k+1}}|)$ has the form $\frac{y+\ln|a_{n_k}|}{x-n_k}=\varkappa(n_{k+1},n_k)$, and (2) implies that all points $(n,-\ln|a_n|)$, $n_k<n<n_{k+1}$, does not lie under of this segment. Thus, F and f have the same Newton majorant. Hence, $\ln\mu(r,f)=\ln\mu(r,F)$ and $\nu(r,f)=\nu(r,F)$.

Now, if

$$\varkappa(n_k,n_{k-1})\leq \ln r<\varkappa(n_{k+1},n_k) \tag{5}$$

then $\nu(r,F)=n_k$, $\ln\mu(r,F)=\ln|a_{n_k}|+n_k\ln r$ and, in view of (3) and of the increase of $\varkappa(n_k,n_{k-1})$ to 0, we obtain

$$\begin{gathered}\frac{\ln\mu(r,F)}{(1-r)\nu(r,F)}=\frac{1}{1-r}\Big(\frac{1}{n_k}\ln|a_{n_k}|+\ln r\Big)\geq\frac{\frac{1}{n_k}\ln|a_{n_k}|+\varkappa(n_k,n_{k-1})}{1-\exp\{\varkappa(n_k,n_{k-1})\}}=\\ =\frac{\frac{1}{n_k}\ln|a_{n_k}|+\varkappa(n_k,n_{k-1})}{-(1+o(1))\varkappa(n_k,n_{k-1})}=(1+o(1))\left\{\frac{\ln|a_{n_k}|}{n_k|\varkappa(n_k,n_{k-1})|}-1\right\}\to+\infty,\quad k\to\infty\end{gathered} \tag{6}$$

that is, in view of (4), the function $\ln\mu(r,F)=\ln\mu(r,f)$ is slowly increasing.

On the contrary, let $(n_k,-\ln|a_{n_k}|)$ be the vertices of the polygon $y=g(x)$. It is clear that for such sequence (n_k) inequalities (1) and (2) hold, and for the values $\ln r$ from (5), in view of (4), we have

$$\frac{1}{1-r}\left(\frac{1}{n_k}\ln|a_{n_k}|+\ln r\right)=\frac{\ln\mu(r,f)}{(1-r)\nu(r,f)}\to+\infty,\quad k\to+\infty.$$

If we put here $\ln r=\varkappa(n_k,n_{k-1})$ then, as in the proof of (6), we obtain (3). Theorem 1 is proved.

Corollary. *Let $|a_n/a_{n+1}|\uparrow 1$, $n\to\infty$. The function $\ln\mu(r)$ is slowly increasing, iff $\ln\Big|\frac{a_n}{a_{n+1}}\Big|=o\left(\frac{1}{n}\ln|a_n|\right)$, $n\to\infty$.*

The question arises whether conditions (1)–(3) are also necessary and sufficient for the slow increase of the function $\ln M(r)$. The answer is affirmative when $\ln\frac{1}{1-r}=o(\ln\mu(r))$, $r\uparrow 1$, because $\mu(r)\leq M(r)\leq\mu\left(\frac{1+r}{2}\right)\frac{2}{1-r}$. If this condition does not hold then the question on the slow increase of the function $\ln M(r)$ remains open.

Conjecture 1. *There exists an analytic in the unit disk function such that $\ln M(r)$ is slowly increasing and $\ln\mu(r)$ is not slowly increasing.*

Conjecture 2. *There does not exist an analytic in the unit disk function such that $a_n\neq O(1)$, $n\to\infty$, $\ln\mu(r)$ is slowly increasing and $\ln M(r)$ is not slowly increasing.*

Conjecture 3. *There exists an analytic function in the unit disk such that $a_n=O(1)$, $n\to\infty$, and $\ln M(r)$ is not slowly increasing.*

We succeeded to prove only Conjecture 1. For its proof we use the following

Theorem 2. *Let $A>1$ be an arbitrary number, a function l is positive continuously differentiable on $[1,+\infty)$ increasing to $+\infty$ and $l'(x)\downarrow 0$, $l(x/l(x))\sim l(x)$, $l(1/l'(x))\sim l(x)$ as $x\to+\infty$. Then there exists an analytic in the unit disk function such that*

$$(1+o(1))l\Big(\frac{1}{1-r}\Big)\leq\ln\mu(r)\leq A(1+o(1))l\Big(\frac{1}{1-r}\Big),\quad r\to 1. \tag{7}$$

and $\ln\mu(r)$ *is not slowly increasing.*

Proof. Since $l(x)\uparrow+\infty$ and $x/l(x)\to+\infty$ as $x\to+\infty$, we can choose an increasing sequence (n_k) of positive integers such that for $k\to\infty$

$$n_{k-1}=o(n_k),\quad l(n_{k-1})=o(l(n_k)),\quad \frac{n_{k-1}}{l(n_{k-1})}=o\left(\frac{n_k}{l(n_k)}\right). \tag{8}$$

Put $a_0=1$, $a_n=\exp\{l(n)\}$ for $n\ge1$, $n\ne n_k$, and $a_{n_k}=\exp\{Al(n_k)\}$, $k\ge1$. Function (1) with such coefficients is analytic in $\{z:\ |z|<1\}$ and

$$\mu(r,f)=\max\{\max_{n\ge0,\,n\ne n_k}|a_n|r^n,\max_{k\ge1}|a_{n_k}|r^{n_k}\}=\max\{\mu(r,f_1),\mu(r,f_2)\}, \tag{9}$$

where

$$f_1(z)=1+\sum_{n=1,\,n\ne n_k}^{\infty}e^{l(n)}z^n,\quad f_2(z)=\sum_{k=1}^{\infty}e^{Al(n_k)}z^{n_k}.$$

Since the function $l(x)+x\ln r$ reaches its maximum in the point $x=x(r)$ such that $l'(x(r))=-\ln r$ and $|\nu(r,f_1)-x(r)|<2$ then, in view of the relation $xl'(x)=o(l(x))$, $x\to+\infty$, we have

$$\begin{aligned}\ln\mu(r,f_1)&=l(\nu(r,f_1))+\nu(r,f_1)\ln r=l(x(r)+O(1))-(x(r)+O(1))l'(x(r))=\\&=l(x(r))-x(r)l'(x(r))+o(l(x(r)))+O(l'(x(r)))=(1+o(1))l(x(r)),\quad r\uparrow1.\end{aligned}$$

But $x(r)=\varphi(|\ln r|)$ where φ is the function inverse to l'. Therefore, using the condition $l(1/l'(x))=(1+o(1))l(x)$, $x\to+\infty$, we obtain

$$l(x(r))=l(\varphi(|\ln r|))=(1+o(1))l\left(\frac{1}{l'(\varphi(|\ln r|))}\right)=(1+o(1))l\left(\frac{1}{|\ln r|}\right),\quad r\uparrow1.$$

Thus,

$$\ln\mu(r,f_1)=(1+o(1))l\Big(\frac{1}{|\ln r|}\Big)=(1+o(1))l\Big(\frac{1}{1-r}\Big),\quad r\uparrow1. \tag{10}$$

From the proof of this relation we see that $\ln\mu(r,f_2)\le(1+o(1))Al\left(\frac{1}{1-r}\right)$, $r\uparrow1$. Therefore, (9) easily implies (7).

Now, put $\varkappa_k=-A\frac{l(n_{k+1})-l(n_k)}{n_{k+1}-n_k}$. Since l is concave, we have $\varkappa_k\uparrow0$ $(k\to\infty)$ and, therefore, if $\varkappa_{k-1}\le\ln r<\varkappa_k$ then $\nu(r,f_2)=n_k$ and $\ln\mu(r,f_2)=Al(n_k)+n_k\ln r$. Thus, in view of (8),

$$\begin{aligned}\ln\mu(\exp\{\varkappa_{k-1}\},f_2)&=Al(n_k)+\varkappa_{k-1}n_k=A\frac{n_kl(n_{k-1})-n_{k-1}l(n_k)}{n_k-n_{k-1}}=\\&=(1+o(1))A\frac{n_kl(n_{k-1})}{n_k}=(1+o(1))Al(n_{k-1}),\quad k\to+\infty,\end{aligned}$$

and from (10) and (8) it follows that

$$\ln\mu(\exp\{\varkappa_{k-1}\},f_1)=(1+o(1))l\Big(\frac{1}{|\varkappa_{k-1}|}\Big)=(1+o(1))l\Big(\frac{n_k}{l(n_k)}\Big)=(1+o(1))l(n_k),\quad k\to+\infty.$$

But $l(n_{k-1})=o(l(n_k))$, $k\to\infty$. Therefore, in view of (9), we obtain

$$\ln\mu(\exp\{\varkappa_{k-1}\},f)=(1+o(1))l(n_k),\quad k\to+\infty. \tag{11}$$

Further, (9) implies

$$\frac{\varkappa_{k-1}}{\varkappa_k}=(1+o(1))\frac{n_{k+1}}{l(n_{k+1})}\frac{l(n_k)}{n_k}\to+\infty,\quad k\to\infty.$$

Therefore, $\varkappa_{k-1} < \frac{A-1}{A+1}\varkappa_{k-1} < \varkappa_k$ for $k \ge k_0$ and

$$\ln\mu\Big(\exp\Big\{\frac{A-1}{A+1}\varkappa_{k-1}\Big\}, f_2\Big) = Al(n_k) + \frac{A-1}{A+1}\varkappa_{k-1}n_k =$$
$$= A\Big(l(n_k) - (1+o(1))\frac{A-1}{A+1}l(n_k)\Big) = (1+o(1))\frac{2A}{A+1}l(n_k), \quad k\to+\infty,$$

and (10) implies

$$\ln\mu\Big(\exp\Big\{\frac{A-1}{A+1}\varkappa_{k-1}\Big\}, f_1\Big) = (1+o(1))l\left(\frac{A+1}{(A-1)|\varkappa_{k-1}|}\right) = (1+o(1))l(n_k), \quad k\to+\infty.$$

Therefore, from (9) we have

$$\ln\mu\Big(\exp\Big\{\frac{A-1}{A+1}\varkappa_{k-1}\Big\}, f\Big) = (1+o(1))\frac{2A}{A+1}l(n_k), \quad k\to+\infty. \tag{12}$$

From (11) and (12) it follows that

$$\lim_{k\to\infty}\frac{\ln\mu\Big(\exp\{\frac{A-1}{A+1}\varkappa_{k-1}\}, f\Big)}{\ln\mu(\exp\{\varkappa_{k-1}\}, f)} = \frac{2A}{A-1} > 1. \tag{13}$$

If we put $t_k = 1-\exp\{\varkappa_{k-1}\}$ then $t_k\downarrow 0$, $k\to\infty$, and

$$\exp\Big\{\frac{A-1}{A+1}\varkappa_{k-1}\Big\} = (1-t_k)^{(A-1)/(A+1)} = 1 - \frac{A-1}{A+1}t_k(1+o(1)), \quad k\to\infty.$$

Therefore, one can write (13) in the form

$$\lim_{k\to\infty}\frac{\ln\mu\Big(1-\frac{A-1}{A+1}t_k(1+o(1))\Big)}{\ln\mu(1-t_k)} = \frac{2A}{A-1} > 1,$$

whence it follows that $\ln\mu(r,f)$ is not slowly increasing. Theorem 2 is proved.

If we choose in Theorem 2 $l(r) = \ln r$ then we obtain an analytic in the unit disk function such that $\ln\mu(r)$ is not slowly increasing and $\ln\mu(r) = O\big(\ln\frac{1}{1-r}\big)$, $r\uparrow 1$.

We will show that $\ln M(r)$ for such function is slowly increasing. For simplicity we choose $n_k = \exp_3 k$, $A=2$. Obviously, conditions (8) hold and

$$M(r) = 1 + \sum_{n=1,\, n\ne n_k}^{+\infty} nr^n + \sum_{k=1}^{+\infty} n_k^2 r^{n_k} =$$
$$= \int_0^{+\infty} te^{\sigma t}\,dt - \int_0^{+\infty} te^{\sigma t}\,dn(t) + \int_0^{+\infty} t^2e^{\sigma t}\,dn(t) + o(1), \quad r\uparrow 1,$$

where $\sigma = \ln r < 0$, $n(t)$ is the counting function of the sequence (n_k) and, thus, $n(t) = (1+o(1))\ln_3 t$, $t\to+\infty$. Since $\int_0^{+\infty} t\exp\{\sigma t\}\,dt = \sigma^{-2}$ and

$$\int_0^{+\infty} te^{\sigma t}\,dn(t) \le \int_0^{+\infty} t^2e^{\sigma t}\,dn(t) \le (-\sigma+o(1))\int_0^{+\infty} t^2e^{\sigma t}\ln_3^+ t\,dt \le \frac{\ln_3(-1/\sigma)}{\sigma^2}K$$

for $\sigma\uparrow 0$, then

$$\frac{1+o(1)}{(1-r)^2} \le M(r) \le \frac{(K+o(1))\ln_3(\frac{1}{1-r})}{(1-r)^2}, \quad r\uparrow 1,$$

where $K>0$ is some constant. Hence, we easily obtain that $\ln M(r) = (2+o(1))\ln\frac{1}{1-r}$, $r\uparrow 1$, that is $\ln M(r)$ is slowly increasing. Conjecture 1 is proved.

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