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PARABOLIC VARIATIONAL INEQUALITIES WITH DEGENERATION

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We study parabolic variational inequalities with degeneration at the initial moment of time. Conditions are obtained for existence and uniqueness of solutions of these inequalities in the class of functions whose order of degenaration at the initial moment of time depends only on that of the inequalities.

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Рассмотрены параболические вариационные неравенства, которые вырождаются в начальный момент времени. Получены условия существования и единственности решений этих неравенств в классе функций, порядок вырождения которых в начальный момент времени зависит от порядка вырождения неравенств.

A Cauchy problem and initial-boundary value problems for parabolic equations and systems that degenerate in the set where initial data are posed, are sufficiently well known at the present time. For instance, such problems had been considered in [1]–[6]. Variational inequalities without degeneration are considered in [7]–[12].

Here we investigate some parabolic variational inequality without initial conditions in a bounded cylinder degenerating to an elliptic variational inequality at the initial moment of time $t = 0$. Conditions for existence and uniqueness of solution of this inequality are obtained.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega \subset C^1$, $Q = \Omega \times (0, T)$, $T < \infty$; $Q_{t_1,t_2} = \Omega \times (t_1, t_2)$, $t_1 < t_2$, $t_1, t_2 \in [0, T]$; $H^1_q(Q)$ the closure of the space $C^\infty(\overline{Q})$ with respect to the norm

$$||u|| = \left(\int_Q \Big(\sum_{i=1}^n |u_{x_i}|^2 + q(t)|u|^2\Big)\, dx\, dt\right)^{\frac{1}{2}},$$

where $q(t) \geqslant 0$, $t \in (0, T)$. The measure of the set where $q(t) = 0$ equals to zero.

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We consider functions $\varphi=\varphi(t), \psi=\psi(t)$ such that $\varphi,\psi\in C^1([0,T])$, $\varphi(t)>0$, $\psi(t)>0$ for $t\in(0,T]$, $\varphi(0)=0$, $\varphi'(t)>0$. The function $\varphi'(t)$ increases on $(0,T]$ and $\lim_{t\to+0}\frac{\varphi\psi'}{\varphi'\psi}=\varphi_0<\infty$.

Let V be a closed subspace such that $\overset{\circ}{H}{}^1(\Omega)\subset V\subset H^1(\Omega)$; K a convex closed subset of V which includes zero; $W=\{w(x,t)|w\in L^2_{\text{loc}}((0,T];V), w_t\in L^2_{\text{loc}}((0,T];V^*)\}$.

Now we consider the variational inequality

$$\int_{Q_{t_1,t_2}}\Big[\varphi(t)v_t(v-u)+\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}(v_{x_j}-u_{x_j})+$$
$$+\sum_{i=1}^n b_i(x,t)u_{x_i}(v-u)+c(x,t)u(v-u)+g(x,t)|u|^{p-2}u(v-u)+$$
$$+\frac{1}{2}\varphi'(t)(v-u)^2+\frac{\varphi\psi'}{2\psi}(v-u)^2-f(x,t)(v-u)\Big]\psi(t)\,dx\,dt\geqslant$$
$$\geqslant\frac{1}{2}\int_\Omega\varphi(t_2)\psi(t_2)|v(x,t_2)-u(x,t_2)|^2\,dx-\frac{1}{2}\int_\Omega\varphi(t_1)\psi(t_1)|v(x,t_1)-u(x,t_1)|^2\,dx. \tag{1}$$

Definition. If a function $u\in L^\infty_{\text{loc}}((0,T];L^2(\Omega))\cap W$, $u\psi^{\frac{1}{p}}\in L^p((0,T);L^p(\Omega))$, $u\psi^{\frac{1}{2}}\in H^1_{\varphi'}(Q)$, $u\in K$ for a. e. $t\in(0,T]$ satisfies variational inequality (1) for all functions $v\in W$, $v\in K$ for a. e. $t\in(0,T]$ and for all $t_1,t_2\in(0,T]$, $t_1<t_2$ then $u(x,t)$ is called a solution of this inequality.

Let a_{ij}, b_i, $(i,j=\overline{1,n})$, c, g in inequality (1) satisfy the following conditions:
(A): $a_{ij}\in L^\infty(Q)$, $a_{ij}(x,t)=a_{ji}(x,t), (i,j=\overline{1,n})$, $\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j\geqslant a_0\sum_{i=1}^n\xi_i^2$, $a_0>0$ for a. e. $(x,t)\in Q$ and for all $\xi\in\mathbb{R}^n$;
(B): $b_i\in L^\infty(Q)$, $(i=\overline{1,n})$; $b_0(t)=\sup_{Q_{0,t}}\sum_{i=1}^n\frac{b_i^2(x,\tau)}{\varphi'(\tau)}$, $b_1=\inf_{[0,T]}b_0(t)$;
(C): $c\in L^\infty(Q)$, $c_0=\inf_{Q_{0,t}}\frac{c(x,\tau)}{\varphi'(\tau)}, c_1=\sup_{[0,T]}c_0(t)$;
(G): $g\in L^\infty(Q)$, $g(x,t)\geqslant g_0>0$ for a. e. $(x,t)\in Q$, $1<p\leqslant 2$. Let $\alpha_0=4\alpha_0c_1-b_1/2\alpha_0$. If $\alpha_0-1-\varphi_0>0$ then $\varkappa_0>0$ such that $\alpha_0-1-\varphi_0-\varkappa_0>0$, if $\alpha_0-1-\varphi_0\leqslant 0$ then $\varkappa_0>0$ is an arbitrary constant.

Theorem 1. *If the coefficients of inequality (1) satisfy conditions (A), (B), (C), (G) then inequality (1) has at most one solution which satisfies the condition*

$$\lim_{t\to+0}\int_\Omega\varphi^{\alpha_0-\varphi_0-\varkappa_0}(t)u^2(x,t)\,dx=0.$$

Proof. Let $u^{(1)},u^{(2)}$ be solutions of inequality (1). Since $u^{(1)},u^{(2)}\in W$, we have $u^{(i)}\in C((0,T];L^2(\Omega))$, $(i=1,2)$ and the following integrals are defined

$$\int_{Q_{t_1,t_2}}u^{(i)}(x,t)u_t^{(i)}(x,t)\,dx\,dt,\quad(i=1,2).$$

Consider the operator A such that

$$\langle Au, v\rangle(t) = \int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}v_{x_j} + \sum_{i=1}^n b_i(x,t)u_{x_i}v + c(x,t)uv + g(x,t)|u|^{p-2}uv\Big]\,dx,$$

for a. e. $t \in (0,T]$, where $u, v \in W$.

The operator A is bounded. In addition, we can find $t_0 \in (0,T]$ such that

$$\langle Au - Av, u - v\rangle(t) = \int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(u_{x_i} - v_{x_i})(u_{x_j} - v_{x_j}) +$$
$$+\sum_{i=1}^n \frac{b_i(x,t)}{\sqrt{\varphi'(t)}}(u_{x_i} - v_{x_i})(u-v)\sqrt{\varphi'(t)} + \frac{c(x,t)}{\varphi'(t)}(u-v)^2\varphi'(t) +$$
$$+g(x,t)(|u|^{p-2}u - |v|^{p-2}v)(u-v)\Big]\,dx \geqslant$$
$$\geqslant \int_\Omega \Big[\Big(a_0 - \frac{\delta_0 b_0(t)}{2}\Big)\sum_{i=1}^n (u_{x_i} - v_{x_i})^2 + \Big(c_0(t) - \frac{1}{2\delta_0}\Big)(u-v)^2\varphi'\Big]\,dx =$$
$$= \frac{4\alpha_0 c_0(t) - b_0(t)}{2a_0}\int_\Omega |u(x,t) - v(x,t)|^2\varphi'(t)\,dx \geqslant$$
$$\geqslant \frac{\alpha_0 - \varkappa_0}{2}\int_\Omega (u(x,t) - v(x,t))^2\varphi'(t)\,dx \qquad (2)$$

for all $t \in (0, t_0]$.

It is easy to see that if functions $u_1, u_2 \in L^2_{\mathrm{loc}}((0,T]; V) \cap C((0,T]; L^2(\Omega))$ satisfy the inequalities

$$\int_{Q_{t_1,t_2}} (\varphi v_t - f_i)(v - u_i)\psi\,dx\,dt \geqslant \frac{1}{2}\int_\Omega \varphi(t_2)\psi(t_2)|v(x,t_2) - u_i(x,t_2)|^2\,dx -$$
$$-\frac{1}{2}\int_\Omega \varphi(t_1)\psi(t_1)|v(x,t_1) - u_i(x,t_1)|^2\,dx - \frac{1}{2}\int_{Q_{t_1,t_2}} \varphi'\psi(v - u_i)^2\,dx\,dt -$$
$$-\frac{1}{2}\int_{Q_{t_1,t_2}} \varphi\psi'(v-u_i)^2\,dx\,dt \ (i = 1,2), \qquad (3)$$

then the following estimate is true

$$\int_{Q_{t_1,t_2}} (f_1 - f_2)(u_1 - u_2)\psi\,dx\,dt \geqslant \frac{1}{2}\int_\Omega \varphi(t_2)\psi(t_2)|u_1(x,t_2) - u_2(x,t_2)|^2\,dx -$$
$$-\frac{1}{2}\int_\Omega \varphi(t_2)\psi(t_1)|u_1(x,t_1) - u_2(x,t_1)|^2\,dx - \frac{1}{2}\int_{Q_{t_1,t_2}} \varphi'\psi|u_1 - u_2|^2\,dx\,dt -$$
$$-\frac{1}{2}\int_{Q_{t_1,t_2}} \varphi\psi'(u_1 - u_2)^2\,dx\,dt. \qquad (4)$$

Let in (4) $f_i = f - Au^{(i)}$, $u_i = u^{(i)}$, $(i = 1,2)$. We may do this, because inequalities (3), (4) are true for these functions (the latter holds because $u^{(1)}(x,t)$, $u^{(2)}(x,t)$ are solutions of inequality (1)). Now we obtain

$$\int_{t_1}^{t_2} \langle Au^{(1)} - Au^{(2)}, u^{(1)} - u^{(2)}\rangle\psi\,dt + \frac{1}{2}\int_{t_1}^{t_2} \frac{d}{dt}\Big(\int_\Omega \varphi\psi(t)|u_1 - u_2|^2\,dx\Big)\,dt -$$

$$-\frac{1}{2}\int_{Q_{t_1,t_2}} \varphi'\psi|u_1-u_2|^2\,dx\,dt - \frac{1}{2}\int_{Q_{t_1,t_2}} \varphi\psi'(u_1-u_2)^2\,dx\,dt \leqslant 0.$$

Let $y(t) = \int_\Omega \varphi(t)\psi(t)|u^{(1)}(x,t) - u^{(2)}(x,t)|^2\,dx$. Then taking into account (2) we obtain

$$\int_{t_1}^{t_2} \Big[(\alpha_0 - \varkappa_0)\frac{\varphi'}{\varphi}y + y' - \frac{\varphi'}{\varphi}y - \frac{\psi'}{\psi}y\Big]\,dt \leqslant 0$$

for all $t_1, t_2 \in (0, t_0]$, $t_1 < t_2$.

Hence $y'(t) + \frac{\varphi'}{\varphi}\Big(\alpha_0 - \varkappa_0 - 1 - \frac{\varphi\psi'}{\varphi'\psi}\Big)y(t) \leqslant 0$ for a. e. $t \in (0, t_0]$. Therefore, $\varphi y' + (\alpha_0 - 1 - \varphi_0 - \varkappa_0)\varphi' y \leqslant 0$. Hence it follows $y(t_2)\varphi^{\alpha_0-1-\varphi_0-\varkappa_0}(t_2) \leqslant y(t_1)\varphi^{\alpha_0-1-\varphi_0-\varkappa_0}(t_1) \to 0$ if $t_1 \to 0$. Then $y(t_2) \leqslant 0, t_2 \in (0, t_0]$ i. e. $y(t) = 0$ on the $(0, t_0]$. Hence $u^{(1)}(x,t) = u^{(2)}(x,t)$ for a. e. $(x,t) \in Q_{0,t_0}$. Uniqueness of solution (1) in $Q_{t_0,T}$ can be obtained as in [9].

Theorem 2. *Suppose that all conditions of Theorem 1 hold, the functions $t \to a_{ij}(x,t)$, $t \to b_i(x,t)$, $(i, j = \overline{1,n})$, $t \to c(x,t)$, $t \to g(x,t)$, $t \to f(x,t)$ continuous on $(0,T]$ for a. e. $x \in \Omega$; the function $\varphi'(t)t^\rho(\varphi^{2+\varphi_0-\alpha_0+\varkappa_1}(t))^{-1}$ does not increase on $(0,T]$ for some $\varkappa_1 > \varkappa_0 > 0$, $\rho(0 < \rho < 1)$. Let these constants and functions satisfy the following assumptions*
a) If $\alpha_0 - 1 - \varphi_0 > 0$ then

$$\int_Q \frac{f^2\psi}{\varphi'}\,dx\,dt < \infty.$$

b) If $\alpha_0 - 1 - \varphi_0 \leqslant 0$ then

$$F_2 = \int_Q \frac{f^2\psi t^\rho\,dx\,dt}{\varphi^{2+\varkappa_1+\varphi_0-\alpha_0}} < \infty.$$

Then there exists a solution of inequality (1).

Proof. Consider the problem in $Q_{t_0,T}$:

$$\varphi u_t + A(t)u + \frac{1}{\varepsilon}B(u) = f_{t_0}(x,t), \tag{5}$$

$$u(x,t_0) = 0, \quad t_0 \in (0,T], \tag{6}$$

where $\varepsilon > 0$, $B(u) = J(u - P_K(u))$, J is the duality operator between V and V^*, P_K is the projection operator onto K,

$$f_{t_0}(x,t) = \begin{cases} f(x,t), & (x,t) \in Q_{t_0,T}, \\ 0, & (x,t) \in Q_{0,t_0}. \end{cases}$$

In accordance with [9], p. 384, the operator B is bounded and monotonous. Theorem 1.2 ([9], p. 173) states that there exists a solution of problem (5), (6) in $Q_{t_0,T}$ such that $u \in L^2((t_0,T);V)$, $u_t \in L^2((t_0,T);V^*)$.

If we choose now $t_0 = T/2, T/3, \dots, T/k, \dots$, we obtain a sequence of solutions of problem (5), (6) $\{u^{k,\varepsilon}(x,t)\}$. We extend every function $u^{k,\varepsilon}(x,t)$ by zero onto $Q_{0,\frac{T}{k}}$.Then for all k i $\tau \in (0,T]$ we obtain the equality

$$\int_{Q_{0,\tau}} \Big[\varphi u_t^{k,\varepsilon} u^{k,\varepsilon} + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}^{k,\varepsilon} u_{x_j}^{k,\varepsilon} + \sum_{i=1}^{n} b_i(x,t) u_{x_i}^{k,\varepsilon} u^{k,\varepsilon} +$$
$$+ c(x,t)|u^{k,\varepsilon}|^2 + g(x,t)|u^{k,\varepsilon}|^p - f_{\frac{T}{k}}(x,t)u^{k,\varepsilon}\Big]\psi\,dx\,dt + \frac{1}{\varepsilon}\int_0^{\tau} \langle B(u^{k,\varepsilon}), u^{k,\varepsilon}\rangle \psi\,dt = 0. \tag{7}$$

It easy to obtain from this the estimate

$$\frac{1}{2}\int_{\Omega} \varphi(\tau)|u^{k,\varepsilon}(x,\tau)|^2\psi(\tau)\,dx + \int_{T/k}^{\tau} dt \int_{\Omega} \Big[\Big(a_0 - \frac{\delta_0 b_0(t)}{2}\Big)\sum_{i=1}^{n} |u_{x_i}^{k,\varepsilon}|^2 +$$
$$+ \Big(c_0(t) - \frac{1}{2\delta_0} - \frac{\varkappa}{2} - \frac{1}{2} - \frac{\varphi_0}{2} - o(t)\Big)|u^{k,\varepsilon}|^2\varphi'\Big]\psi\,dx +$$
$$+ \frac{1}{\varepsilon}\int_{T/k}^{\tau} \langle B(u^{k,\varepsilon}), u^{k,\varepsilon}\rangle\psi\,dt + \int_{Q_{T/k,\tau}} g_0|u^{k,\varepsilon}|^p\psi\,dx\,dt \leqslant \frac{1}{2\varkappa}F_1 \tag{8}$$
$$F_1 = \int_{T/k}^{\tau} dt \int_{\Omega} \frac{f^2(x,t)}{\varphi'(t)}\psi\,dx = \frac{1}{2\varkappa}F_1,$$

where $\varkappa > 0$. Now we can find $t_0 \in (0,T]$ such that the sign of the expression $\big(c_0(t) - \frac{1}{2\delta_0} - \frac{\varkappa}{2} - \frac{1}{2} - \frac{\varphi_0}{2} - o(t)\big)$ will be determined by that of the expression $\Big(c_0(t) - \frac{1}{2\delta_0} - \frac{\varkappa}{2} - \frac{1}{2} - \frac{\varphi_0}{2}\Big)$ on $(0,t_0]$. Putting in (8) $\delta_1 = \frac{2a_0}{b_0(t)}$ we obtain

$$\int_{\Omega} \varphi\psi(\tau)|u^{k,\varepsilon}|^2\,dx + \int_{Q_{0,\tau}} (2c_0(t) - \frac{b_0(t)}{2a_0} - 1 - \varphi_0 - \varkappa)\varphi'\psi|u^{k,\varepsilon}|^2\,dx \leqslant \frac{1}{\varkappa}F_1. \tag{9}$$

a) Let $\alpha_0 - 1 - \varphi_0 > 0$. Then from (8), choosing $\varkappa_1 = \varkappa_0 + \varkappa, \varkappa_1 > \varkappa_0$ such that $\alpha_0 - 1 - \varphi_0 - \varkappa_1 > 0$, we obtain the estimates

$$\int_{\Omega} \varphi(\tau)\psi(\tau)|u^{k,\varepsilon}(x,\tau)|^2\,dx \leqslant \mu_1 F_1,$$
$$\int_{Q_{0,\tau}} \Big[\sum_{i=1}^{n} |u_{x_i}^{k,\varepsilon}(x,t)|^2 + \varphi'(t)|u^{k,\varepsilon}(x,t)|^2 + |u^{k,\varepsilon}(x,t)|^p\Big]\psi(t)\,dx\,dt \leqslant \mu_1 F_1,$$
$$\int_0^{\tau} \langle B(u^{k,\varepsilon}), u^{k,\varepsilon}\rangle\psi(t)\,dt \leqslant \mu_1 \varepsilon F_1, \tag{10}$$

where $\tau \in (0,t_0]$, μ_1 is independent of k i ε.

б) Let $\alpha_0 - 1 - \varphi_0 = -\gamma$, $\gamma \geqslant 0$. If we define $y_{k,\varepsilon}(\tau)$ as follows

$$y_{k,\varepsilon}(\tau) = \int_{\frac{T}{k}}^{\tau} dt \int_{\Omega} |u^{k,\varepsilon}(x,t)|^2\varphi'\psi\,dx.$$

and take $\varkappa_1 > \varkappa_0$ then from (9) we will have

$$\frac{\varphi(\tau)}{\varphi'(\tau)} y'_{k,\varepsilon}(\tau) - (\gamma + \varkappa_1) y_{k,\varepsilon}(\tau) \leqslant C_1 F_1, \quad \left(\frac{y_{k,\varepsilon}}{\varphi^{\gamma+\varkappa_1}}\right)' \leqslant \frac{C_1 \varphi' \tau^\rho}{\tau^\rho \varphi^{\gamma+\varkappa_1+1}} F_1 \leqslant \frac{C_1}{\tau^\rho} F_2.$$

Therefore, $y_{k,\varepsilon} \leqslant C_2 F_2 t^{1-\rho} \varphi^{\varkappa_1+\gamma}$. Then from (8) we obtain estimates (10) where the constants F_1, μ_1 will be replaced by F_2, μ_2 and μ_2 is independent of k, ε. If we consider (8) we will be able to show that estimates (10) are true for $\tau \in [t_0, T]$. Now we consider (9). In case b) we obtain

$$\int_\Omega \varphi^{\alpha_0 - \varphi_0 - \varkappa_0} \psi |u^{k,\varepsilon}|^2 \, dt \leqslant C_3 \left(t^{1-\rho} \varphi^{\varkappa}(t) + \frac{\varphi^{\varkappa+1}(t)}{\varphi'(t) t^\rho} \right) F_2 = \beta(t) F_2, \qquad (11)$$

where $\varkappa = \varkappa_1 - \varkappa_0$. We may assume that $\varphi^{\varkappa+1}(t)/\varphi'(t)t^\rho \to 0$ if $t \to +0$. Hence $\beta(t) \to 0$ in estimate (11) if $t \to +0$. In addition, $\beta(t)$ is bounded on (0,T].

Now we consider case a). Estimates (10) give existence of a subsequence of the sequence $\{u^{k,\varepsilon}(x,t)\}$ (we denote it again by $\{u^{k,\varepsilon}(x,t)\}$) such that

$$\sqrt{\varphi\psi} u^{k,\varepsilon}(x,t) \to \sqrt{\varphi\psi} u^\varepsilon(x,t) \quad * -\text{weakly in } L^\infty((0,T); L^2(\Omega));$$
$$\sqrt{\psi} u^{k,\varepsilon}(x,t) \to \sqrt{\psi} u^\varepsilon(x,t) \quad \text{weakly in } H^1_{\varphi'}(Q);$$
$$\psi^{\frac{1}{p}} u^{k,\varepsilon}(x,t) \to \psi^{\frac{1}{p}} u^\varepsilon(x,t) \quad \text{weakly in } L^p((0,T); L^p(\Omega)).$$

Then the function $u^\varepsilon(x,t)$ is a solution of the equation

$$\varphi u_t + A(t) u + \frac{1}{\varepsilon} B(u) = f(x,t), \qquad (12)$$

and $\sqrt{\varphi\psi} u^\varepsilon \in L^\infty((0,T); L^2(\Omega))$, $\psi^{\frac{1}{p}} u^\varepsilon \in L^p((0,T); L^p(\Omega))$, $u^\varepsilon \in L^2_{\text{loc}}((0,T]; V)$, $u^\varepsilon_t \in L^2_{\text{loc}}((0,T]; V^*)$, $\sqrt{\psi} u^\varepsilon \in H^1_{\varphi'}(Q)$ and for $u^\varepsilon(x,t)$ estimates (10) hold.

Let $v \in W$, $v \in K$ for a. e. $t \in (0,T]$. The operator B is monotonous and $B(v) = 0$. The function u^ε is a solution of (12). Then

$$\int_{Q_{t_1,t_2}} \Big[\varphi(t) v_t (v - u^\varepsilon) + \sum_{i,j=1}^n a_{ij}(x,t) u^\varepsilon_{x_i} (v_{x_j} - u^\varepsilon_{x_j}) +$$
$$+ \sum_{i=1}^n b_i(x,t) u^\varepsilon_{x_i} (v - u^\varepsilon) + c(x,t) u^\varepsilon (v - u^\varepsilon) + \frac{1}{2} \varphi'(t) (v - u^\varepsilon)^2 +$$
$$+ g(x,t) |u^\varepsilon|^{p-2} u^\varepsilon (v - u^\varepsilon) + \frac{\varphi\psi'}{2\psi} (v - u^\varepsilon)^2 - f(x,t)(v - u^\varepsilon) \Big] \psi \, dx \, dt =$$
$$= \frac{1}{\varepsilon} \int_{t_1}^{t_2} \langle B(v) - B(u^\varepsilon), v - u^\varepsilon \rangle \psi(t) \, dt + \frac{1}{2} \int_\Omega \varphi(t_2) \psi(t_2) |v(x,t_2) - u^\varepsilon(x,t_2)|^2 \, dx -$$
$$- \frac{1}{2} \int_\Omega \varphi(t_1) \psi(t_1) |v(x,t_1) - u^\varepsilon(x,t_1)|^2 \, dx \geqslant \frac{1}{2} \int_\Omega \varphi(t_2) \psi(t_2) |v(x,t_2) - u^\varepsilon(x,t_2)|^2 \, dx -$$
$$- \frac{1}{2} \int_\Omega \varphi(t_1) \psi(t_1) |v(x,t_1) - u^\varepsilon(x,t_1)|^2 \, dx \qquad (13)$$

for all $t_1, t_2 \in (0, T]$, $t_1 < t_2$.

Let $[T_1, T_2] \subset (0, T]$. Now we obtain a compact in $C([T_1, T_2]; L^2(\Omega))$ subsequence $\{\sqrt{\varphi\psi} u^{\varepsilon_m}(x,t)\} \subset \{\sqrt{\varphi\psi} u^{\varepsilon}(x,t)\}$.

The following estimate is a consequence of the Fatou lemma and estimate (10_2):

$$\int_{\frac{T_1}{2}}^{T_2} \liminf ||u^\varepsilon||^2 \psi\, dt \leqslant \liminf \int_{Q_{\frac{T_1}{2},T_2}} \left[\sum_{i=1}^n |u^\varepsilon_{x_i}|^2 + \varphi' |u^\varepsilon|^2\right] \psi\, dx\, dt \leqslant \mu_1 F_1.$$

Hence $\liminf ||u^\varepsilon(x,t)||^2 < \infty$ for a. e. $t \in [T_1/2, T_2]$. Then there is $\hat{T} \in [T_1/2, T_2]$ and a subsequence $\{u^{\varepsilon_m}(x,t)\} \subset \{u^\varepsilon(x,t)\}$ such that $\liminf ||u^\varepsilon(x,\hat{T})||^2 = \lim_{m\to\infty} ||u^{\varepsilon_m}(x,\hat{T})||^2$.

Let $\hat{T} = T_1$. Then

$$||u^{\varepsilon_m}(x,T_1)||^2 \leqslant \mu_3 \tag{14}$$

for all m. We multiply (10) by $u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t)$ and integrate by t on the segment $[T_1, T_1+\delta]$. The operator B is monotonous. Besides, we have estimates (10), (14). Therefore,

$$\int_\Omega \varphi\psi(T_1+\delta)|u^{\varepsilon_m}(x,T_1+\delta) - u^{\varepsilon_m}(x,T_1)|^2\, dx \leqslant$$
$$\leqslant \frac{2}{\varepsilon_m}\int_{T_1}^{T_1+\delta} \langle B(u^{\varepsilon_m}(x,T_1)), u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t)\rangle \psi\, dt +$$
$$+2\int_{Q_{T_1,T_1+\delta}} \Bigg[\sum_{i,j=1}^n a_{ij} u^{\varepsilon_m}_{x_i}[u^{\varepsilon_m}_{x_j}(x,T_1) - u^{\varepsilon_m}_{x_i}(x,t)] + \sum_{i=1}^n b_i u^{\varepsilon_m}_{x_i}[u^{\varepsilon_m}(x,T_1) -$$
$$-u^{\varepsilon_m}(x,t)] + c(x,t)u^{\varepsilon_m}(x,t)[u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t)] +$$
$$+g(x,t)|u^{\varepsilon_m}(x,t)|^{p-2}u^{\varepsilon_m}(x,t)(u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t)) + \frac{\varphi\psi'}{2\psi}(u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t))^2 +$$
$$+\frac{1}{2}\varphi'[u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t)]^2 - f(x,t)[u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t)]\Bigg]\psi\, dx\, dt \leqslant$$
$$\leqslant J + C_6 \int_{Q_{T_1,T_1+\delta}} \Bigg[\sum_{i=1}^n |u^{\varepsilon_m}_{x_i}|^2 + \varphi'|u^{\varepsilon_m}|^2 + \sum_{i=1}^n |u^{\varepsilon_m}_{x_i}(x,T_1)|^2 +$$
$$+(\varphi_0 + o(t))\varphi'(t)(|u^{\varepsilon_m}|^2 + |u^{\varepsilon_m}(x,T_1)|^2) +$$
$$+\varphi'(t)|u^{\varepsilon_m}(x,T_1)|^2 + \frac{f^2(x,t)}{\varphi'(t)}\Bigg]\psi\, dx\, dt \leqslant J + C_7 F_1 \delta,$$

where

$$J = \frac{2}{\varepsilon_m}\int_{T_1}^{T_1+\delta} \langle B(u^{\varepsilon_m}(x,T_1)), u^{\varepsilon_m}(x,T_1) - u^{\varepsilon_m}(x,t)\rangle \psi(t)\, dt.$$

If $0 < \lim_{m\to\infty} ||u^{x,\varepsilon_m}(T_1)|| < \infty$ then we can show that $J \leqslant C_8\delta$ and thus

$$\int_\Omega \varphi(T_1+\delta)\psi(T_1+\delta)|u^{\varepsilon_m}(x,T_1+\delta) - u^{\varepsilon_m}(x,T_1)|^2\, dx \leqslant C_9\delta, \tag{15}$$

where C_9 is independent of ε_m.

If $\lim\limits_{m\to\infty} ||u^{\varepsilon_m}(x,T_1)|| = 0$ then we can also obtain estimate (15).

If $\alpha_0 - 1 - \varphi_0 \leqslant 0$ then estimate (15) is obtained in the same way.

Now we may take in (4)

$$t_1 = T_1, \quad t_2 = t, \quad u_1(x,t) = u^{\varepsilon_m}(x,t), \quad u_2(x,t) = u^{\varepsilon_m}(x,t+\delta),$$
$$f_1(x,t) = f(x,t) - A(t)u^{\varepsilon_m}(x,t), \quad f_2(x,t) = f(x,t+\delta) - A(t+\delta)u^{\varepsilon_m}(x,t+\delta).$$

Then we obtain

$$\int_\Omega \varphi\psi(t)|u^{\varepsilon_m}(x,t+\delta) - u^{\varepsilon_m}(x,t)|^2\,dx \leqslant \int_\Omega \varphi\psi(T_1)|u^{\varepsilon_m}(x,T_1+\delta)-$$
$$-u^{\varepsilon_m}(x,T_1)|^2\,dx + 2\int_{Q_{T_1,t}} [f(x,t) - f(x,t+\delta)][u^{\varepsilon_m}(x,t) - u^{\varepsilon_m}(x,t+\delta)]\psi\,dx\,dt-$$
$$-2\int_{T_1}^t \langle A(\tau+\delta)u^{\varepsilon_m}(x,\tau+\delta) - A(\tau)u^{\varepsilon_m}(x,\tau), u^{\varepsilon_m}(x,\tau+\delta) - u^{\varepsilon_m}(x,\tau)\rangle\psi(\tau)\,d\tau+$$
$$+\int_{Q_{T_1,t}} \varphi'\psi|u^{\varepsilon_m}(x,t+\delta) - u^{\varepsilon_m}(x,t)|^2\,dx\,dt+$$
$$+\int_{Q_{T_1,t}} \varphi\psi'|u^{\varepsilon_m}(x,t+\delta) - u^{\varepsilon_m}(x,t)|^2\,dx\,dt = J_1 + J_2 - J_3 + J_4 + J_5. \quad (16)$$

Now we consider every summand at the right side of (16).

Using estimate (15) we obtain

$$J_1 = \int_\Omega \varphi\psi(T_1)|u^{\varepsilon_m}(x,T_1+\delta) - u^{\varepsilon_m}(x,T_1)|^2\,dx \leqslant$$
$$\leqslant C_{10}\int_\Omega \varphi\psi(T_1+\delta)|u^{\varepsilon_m}(x,T_1+\delta) - u^{\varepsilon_m}(x,T_1)|^2\,dx \leqslant C_{11}\delta,$$
$$J_2 = 2\int_{Q_{T_1,t}} [f(x,t) - f(x,t+\delta)][u^{\varepsilon_m}(x,t) - u^{\varepsilon_m}(x,t+\delta)]\psi(t)\,dx\,dt \leqslant$$
$$\leqslant \int_{Q_{T_1,t}} \frac{|f(x,t+\delta) - f(x,t)|^2\psi(t)}{\varphi'(t)}\,dx\,dt + J_4.$$

The function f is continuous with respect to t. Hence for all $\varepsilon > 0$ there is $\delta_1 > 0$ such that for $\delta (0 < \delta < \delta_1)$ the first summand in the estimate of J_2 will be less than ε.

Using (2), we obtain

$$J_3 = 2\int_{T_1}^t \langle A(\tau+\delta)u^{\varepsilon_m}(x,\tau+\delta) - A(\tau)u^{\varepsilon_m}(x,\tau), u^{\varepsilon_m}(x,\tau+\delta) - u^{\varepsilon_m}(x,\tau)\rangle\psi(\tau)\,d\tau \geqslant$$
$$\geqslant 2\int_{Q_{T_1,t}} \Big[\sum_{i,j=1}^n (a_{ij}(x,\tau+\delta) - a_{ij}(x,\tau))u^{\varepsilon_m}_{x_i}(x,\tau+\delta)(u^{\varepsilon_m}_{x_j}(x,\tau+\delta) - u^{\varepsilon_m}_{x_j}(x,\tau))+$$
$$+\sum_{i=1}^n (b_i(x,\tau+\delta) - b_i(x,\tau))u^{\varepsilon_m}_{x_i}(x,\tau+\delta)(u^{\varepsilon_m}(x,\tau+\delta) - u^{\varepsilon_m}(x,\tau))+$$
$$+(c(x,\tau+\delta) - c(x,\tau))u^{\varepsilon_m}(x,\tau+\delta)(u^{\varepsilon_m}(x,\tau+\delta) - u^{\varepsilon_m}(x,\tau)) + (g(x,\tau+\delta) - g(x,\tau))\times$$
$$\times|u^{\varepsilon_m}(x,\tau+\delta)|^{p-2}u^{\varepsilon_m}_2(x,\tau+\delta)(u^{\varepsilon_m}(x,\tau+\delta) - u^{\varepsilon_m}(x,\tau))\Big]\psi\,dx + \alpha_0 J_4.$$

The functions a_{ij}, b_i, $(i,j=\overline{1,n})$, c, g are continuous. Therefore $-J_3 \leqslant \varepsilon-(\alpha_0-\varkappa)J_4$ for $\delta<\delta_2$,

$$J_5=\int_{Q_{T_1,t}}\varphi\psi'(t)|u^{\varepsilon_m}(x,t+\delta)-u^{\varepsilon_m}(x,t)|^2\,dx\,dt\leqslant C_{11}J_4.$$

The integral J_4 is a function of t. Let $\delta<\varepsilon$. Then (16) gives us

$$\frac{\varphi}{\varphi'}J_4'\leqslant C_{12}\varepsilon+C_{13}J_4,\ \ J_4'\leqslant\frac{\varphi'(t)}{\varphi(t)}C_{13}\varepsilon+C_{13}\frac{\varphi'(t)}{\varphi(t)}J_4\leqslant C_{14}\varepsilon+C_{15}J_4.$$

Hence

$$J_4(t)\leqslant C_{14}T\varepsilon+C_{15}\int_{T_1}^{t}J_4(t)\,dt,\quad J_4(t)\leqslant C_{14}T\varepsilon e^{C_{15}(t-T_1)}\leqslant C_{16}\varepsilon.$$

Then we conclude that for all $\varepsilon>0$ there is δ_3 $(0<\delta_3<\varepsilon)$ such that for every $t\in[T_1,T_2]$ and every δ $(0<\delta<\delta_3)$

$$\int_\Omega\varphi(t)\psi(t)|u^{\varepsilon_m}(x,t+\delta)-u^{\varepsilon_m}(x,t)|^2\,dx<\varepsilon.$$

Hence the sequence $\{u^{\varepsilon_m}(x,t)\}$ is compact in $C([T_1,T_2];L^2(\Omega))$. Since $\{u^{\varepsilon_m}(x,t)\}$ satisfies conditions (10), we can choose a subsequence of this sequence (we denote it again by $\{u^{\varepsilon_m}(x,t)\}$) such that

$$\begin{gathered}\sqrt{\varphi\psi}u^{\varepsilon_m}(x,t)\to\sqrt{\varphi\psi}\tilde u(x,t)\quad *-\text{weakly in }L^\infty((0,T];L^2(\Omega));\\ \sqrt{\psi}u^{\varepsilon_m}(x,t)\to\sqrt{\psi}\tilde u(x,t)\quad\text{weakly in }H^1_{\varphi'}(Q);\\ u^{\varepsilon_m}(x,t)\psi^{\frac1p}\to\tilde u(x,t)\psi^{\frac1p}\quad\text{weakly in }L^p((0,T);L^p(\Omega));\\ \sqrt{\varphi\psi}u^{\varepsilon_m}(x,t)\to\sqrt{\varphi\psi}\tilde u(x,t)\quad\text{uniformly in }C([T_1,T_2];L^2(\Omega))\end{gathered}$$

if $\varepsilon_m\to0$. We consider now the segments $[T/2,T]$, $[T/3,T]$, … , $[T/k,T]$, … . We can choose a diagonal subsequence $\{u^{m,m}(x,t)\}$ such that

$$\begin{gathered}\sqrt{\varphi\psi}u^{m,m}(x,t)\to\sqrt{\varphi\psi}u(x,t)\quad *-\text{weakly in }L^\infty((0,T);L^2(\Omega));\\ \sqrt{\psi}u^{m,m}(x,t)\to\sqrt{\psi}u(x,t)\quad\text{weakly in }H^1_{\varphi'}(Q);\\ u^{m,m}(x,t)\psi^{\frac1p}\to u(x,t)\psi^{\frac1p}\quad\text{weakly in }L^p((0,T);L^p(\Omega));\\ \sqrt{\varphi\psi}u^{m,m}(x,t)\to\sqrt{\varphi\psi}u(x,t)\quad\text{uniformly in }C([T_1,T];L^2(\Omega))\end{gathered}$$

if $m\to\infty$ for all $T_1\in(0,T)$. We also see that the functions $\{u^{m,m}(x,t)\}$ satisfy inequality (13) for all $t_1,t_2\in(0,T]$, $t_1<t_2$; $v\in W$, $v\in K$ for a. e. $t\in(0,T]$. Inequality (10_3) implies that $B(u)=0$. Hence $u\in K$ for a. e. $t\in(0,T]$. Therefore as well as in [9, p. 407] we can obtain that the function $u(x,t)$ is a solution of inequality (1).

Remark. If the conditions of Theorem 2 are satisfied then inequality (1) has a unique solution. We will have proved this if we show that

$$\lim_{t\to+0}\int_\Omega\varphi^{\alpha_0-\varphi_0-\varkappa_0}(t)u^2(x,t)\,dx=0.$$

If $\alpha_0-1-\varphi_0>0$ then this is a consequence of estimate (10_1).

If $\alpha_0-1-\varphi_0\geqslant0$ then this is a consequence of estimate (11).

Example. Let $K = V = H^1(\Omega)$. Then a solution of the variational inequality (1) is a solution of the problem:

$$\varphi u_t + Au = f \text{ for a. e. } (x,t) \in Q, \tag{18}$$

$$\frac{\partial u}{\partial \nu_A} = 0 \text{ for a. e. } (x,t) \in \partial\Omega \times [0,T], \tag{19}$$

$$\lim_{t\to+0} \int_\Omega \varphi^{\alpha_0}(t) u^2(x,t)\, dx = 0, \tag{20}$$

where $\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i} \nu_j$ is the derivative with respect to conormal and $\nu = (\nu_1, \dots, \nu_n)$ is an external normal to the surface $\partial\Omega$.

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