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SOME NONCLASSICAL PROBLEM OF ELASTICITY THEORY

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To memory of Yaroslav S. Pidstryhach

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Hilbert space methods are used for the statement and solution of some problem of elasticity theory. This problem is not classical in the sense that the boundary differential operator is of the same order as the one which acts inside the domain. The existence, the unicity and the continuous dependence on data of the solution is proved. Some equivalent variational problems and dual problems are considered.

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Для постановки и решения определенной задачи механики сплошной среды в этой работе применяются методы гильбертового пространства. Задача является неклассической в том смысле, что дифференцирование в краевых условиях имеет тот же порядок, что и в системе дифференциальных уравнений, описывающих состояние среды внутри области.

Доказывается существование, единственность и непрерывная зависимость от данных решения указанной краевой задачи. Рассматриваются также эквивалентные вариационные и двойственные задачи.

This research was made to order of Ya. Pidstryhach who (in particular) directed in Lviv investigations with boundary differentiation of higher order (see [1–3]). Some our results have been announced in [4].

1. Variational statement of the problem. Let a continuous medium S fill up a bounded domain $\Omega \subset \mathbb{R}^3$ with a regular boundary Σ. Denote by $u = u(x)$ the displacement of S at $x \in \Omega \cup \Sigma$. For $x \in \Omega$ the derivative $u'(x)$ is a linear map $\mathbb{R}^3 \to \mathbb{R}^3$. The corresponding *strain tensor* $\varepsilon = \varepsilon u(x)$ is defined as usual: ε is the real part of this map: $\varepsilon = \frac{1}{2}\big(u'(x) + (u'(x))^*\big)$. For $x \in \Sigma$ the definition is more complicated. Let u_Σ be the restriction of u to Σ. Then the derivative $u'_\Sigma(x)$ is a linear map from the tangent manifold $T\Sigma_x$ to $\mathbb{R}^3$. Denote by $t(x)$ the orthoprojector $\mathbb{R}^3 \to T\Sigma_x$ and form the composite map $u'_\Sigma t(x)$. Its real part $\frac{1}{2}\big(u'_\Sigma(x)t(x) + (u'_\Sigma(x)t(x))^*\big)$ is by definition the *strain tensor* $\overset{\circ}{\varepsilon} = \overset{\circ}{\varepsilon} u(x)$ for $x \in \Sigma$.

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Use local coordinates ξ_1, ξ_2, ξ_3 such that in a small neighborhood of $x \in \Sigma$ the equation of Σ is $\xi_3 = 0$. Then $\overset{\circ}{\varepsilon} u(x)$ has the following matrix

$$\frac{1}{2}\begin{pmatrix} \frac{\partial u_1}{\partial \xi_1} & \frac{\partial u_1}{\partial \xi_2} & 0 \\ \frac{\partial u_2}{\partial \xi_1} & \frac{\partial u_2}{\partial \xi_2} & 0 \\ \frac{\partial u_3}{\partial \xi_1} & \frac{\partial u_3}{\partial \xi_2} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{\partial u_1}{\partial \xi_1} & \frac{\partial u_2}{\partial \xi_1} & \frac{\partial u_3}{\partial \xi_1} \\ \frac{\partial u_1}{\partial \xi_2} & \frac{\partial u_2}{\partial \xi_2} & \frac{\partial u_3}{\partial \xi_2} \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

We see that $\overset{\circ}{\varepsilon} u(x)$ is the *usual* strain tensor which corresponds to the field u which does not depend on ξ_3.

The stressed state of the medium S will be described by two stress tensors σ and $\overset{\circ}{\sigma}$ which are defined on Ω and Σ respectively. There is a *linear* stress-strain relation (the generalized Hooke law): $\sigma_{ij} = a_{ijkl}\varepsilon_{kl}$ in Ω, $\overset{\circ}{\sigma}_{ij} = \overset{\circ}{a}_{ijkl}\overset{\circ}{\varepsilon}_{kl}$ in Σ. Here $a_{ijkl}, \overset{\circ}{a}_{ijkl}$ are bounded mesurable functions on x satisfying usual symmetry conditions: $a_{ijkl} = a_{jikl} = a_{klij}$, $\overset{\circ}{a}_{ijkl} = \overset{\circ}{a}_{jikl} = \overset{\circ}{a}_{klij}$. Besides, ellipticity conditions are assumed:

$$\begin{aligned} \operatorname{tr}(\sigma\varepsilon) &\geq \alpha \cdot \operatorname{tr}(\varepsilon^2) \quad \text{in} \quad \Omega, \\ \operatorname{tr}(\overset{\circ}{\sigma}\overset{\circ}{\varepsilon}) &\geq \alpha \cdot \operatorname{tr}(\overset{\circ}{\varepsilon}{}^2) \quad \text{on} \quad \Sigma \end{aligned} \tag{1.2}$$

($\operatorname{tr}\beta$ denotes the sum of the diagonal elements of the matrix of β).

Associate with medium S the energetic Hilbert space $\mathcal{U} := \{u \in H^1 : \Gamma u \in G^1\}$, $(u|v)_{\mathcal{U}} := (u|v)_{H^1} + (\Gamma u|\Gamma v)_{G^1}$. Here $H^1 := (H^1(\Omega))^3$, $G^1 := (H^1(\Sigma))^3$ are the corresponding Sobolev spaces, Γ is the trace operator $H^1 \to G^1$ (see [7]; roughly speaking $\Gamma u(x)$ is the value of $u \in H^1$ at x). Now introduce the bilinear forms of virtual energy a and $\overset{\circ}{a}$:

$$\begin{aligned} (\forall u, v \in \mathcal{U}) \qquad a(u,v) &= \int_\Omega \operatorname{tr}(\sigma u \cdot \sigma v)\, d\Omega, \\ \overset{\circ}{a}(u,v) &= \int_\Sigma \operatorname{tr}(\overset{\circ}{\sigma} u \cdot \overset{\circ}{\sigma} v)\, d\Sigma. \end{aligned} \tag{1.3}$$

Observe that this forms are symmetric and continuous on $\mathcal{U}$.

We assume that an external force with volume density $f \in H = (L_2(\Omega))^3$ and an external force with surface density $g \in G = (L_2(\Sigma_N))^3$ act at S. Here Σ_N denotes some part of Σ. Suppose that on $\Sigma_\Pi := \Sigma \setminus \Sigma_N$ the displacement $h \in H$ is given. Then the *potential energy* $\mathcal{I}(v)$ of the medium S which is caused by its displacement v is:

$$\mathcal{I}(v) = \frac{1}{2}c[v] - \int_\Omega (f|v)_{\mathbb{R}^3}\, d\Omega - \int_{\Sigma_N} (g|v)_{\mathbb{R}^3}\, d\Sigma, \tag{1.4}$$

where $c[v] = c(v,v); c(u,v) = a(u,v) + \overset{\circ}{a}(u,v)$.

The quadratic functional $\mathcal{I}(v)$ is also continuous on $\mathcal{U}$.

According to the minimum principle of potential energy the medium S *attains* the equilibrium state for such a displacement u which is a solution of the following (conditional) extremum problem

$$\mathcal{I}(v) \longrightarrow \inf, \qquad v \in \mathcal{U}_h, \tag{1.5}$$

where $\mathcal{U}_h := \{v \in \mathcal{U} : v = h \text{ on } \Sigma_\Pi\}$.

Theorem 1.1. *Let* $\operatorname{mes}\Sigma_\Pi > 0$. *Then problem (1.5) has a unique solution (i.e.* $(\exists! u \in \mathcal{U}_h)(\forall v \in \mathcal{U}_h)\ (\mathcal{I}(u) \le \mathcal{I}(v))$).

Proof. It is easy to show that $\mathcal{U}_h$ is a closed affine manifold in $\mathcal{U}$. Therefore it is sufficient to prove that the form $c(u,v)$ is coercive on the subspace $\mathcal{U}_0 := \{v \in \mathcal{U} : v = 0 \text{ on } \Sigma_\Pi\}$, which is parallel to $\mathcal{U}_h$.

Let

$$\varepsilon(v)^2 := \int_\Omega \operatorname{tr}(\varepsilon v)^2\, d\Omega + \int_\Sigma \operatorname{tr}(\overset{\circ}{\varepsilon} v)^2\, d\Sigma. \tag{1.6}$$

Recall that for a symmetric operator γ we have $\operatorname{tr}\gamma^2 = 0 \Rightarrow \gamma = 0$. So $\varepsilon(v)^2 = 0$ implies $\varepsilon v = 0$, which means that the displacement v is rigid. Now for $v \in \mathcal{U}_0$ we have $\varepsilon(v)^2 = 0 \Longleftrightarrow v = 0$, because the condition $\operatorname{mes}\Sigma_\Pi > 0$ implies that the solid body is unmoved.

We need the following statement:

$$(\exists c_1 > 0)(\forall v \in \mathcal{U}_0) \quad \big(\varepsilon(v)^2 \ge c_1\big(\|v\|_H^2 + \|\Gamma v\|_G^2\big)\big). \tag{1.7}$$

Let us prove it. Suppose the contrary: there exists $v_n \in \mathcal{U}_0$ such that $\|v_n\|_H^2 + \|\Gamma v_n\|_G^2 = 1$, but $\varepsilon(v_n)^2 \to 0$ for $n \to \infty$. According to Corn's inequalities [5], we have

$$\begin{aligned}(\exists c_2 > 0)(\forall v \in \mathcal{U}) \quad &\int_\Omega (\operatorname{tr}(\varepsilon v)^2 + \|v\|_{\mathbb{R}^3}^2)\, d\Omega \ge c_2 \|v\|_{H^1}^2,\\ &\int_\Sigma (\operatorname{tr}(\overset{\circ}{\varepsilon} v)^2 + \|v\|_{\mathbb{R}^3}^2)\, d\Sigma \ge c_2 \|\Gamma v\|_{G^1}^2,\end{aligned} \tag{1.8}$$

hence

$$(\forall v \in \mathcal{U}) \quad \varepsilon(v)^2 + \|v\|_H{}^2 + \|\Gamma v\|_G{}^2 \ge c_2 \|v\|_{\mathcal{U}}^2. \tag{1.9}$$

From (1.9) we conclude that the sequence (v_n) is bounded in $\mathcal{U}_0$ and without loss of generality one can assume that $v_n \to v$ weakly in $\mathcal{U}_0$. Using the Bunyakovsky inequality we get $\underline{\lim}\, \varepsilon(v_n)^2 \ge \varepsilon(v)$ and therefore $\varepsilon(v)^2 = 0$, whence $v = 0$. The boundedness of the sequence (v_n) in $\mathcal{U}$ means that (v_n) is bounded in H^1 and (Γv_n) is bounded in G^1. The embeddings $H^1 \to H$ and $G^1 \to G$ are compact. Therefore it is possible to consider that $v_n \to v = 0$ strongly in H and $\Gamma v_n \to \Gamma v = 0$ strongly in G. This contradicts to the condition $\|v_n\|_H{}^2 + \|\Gamma v_n\|_G{}^2 = 1$, and (1.7) is proved.

Now from (1.3), (1.4) we get $c[v] \ge \alpha \varepsilon(v)^2$, $v \in \mathcal{U}$, and (1.7) implies

$$(\forall v \in \mathcal{U}) \quad c[v] \ge \beta\big(\varepsilon(v)^2 + \|v\|_H^2 + \|\Gamma v\|_G^2\big),$$

where $\beta = \min(\frac{1}{2}\alpha, \frac{1}{2}\alpha c_1)$. Using (1.9) we find $(\forall v \in \mathcal{U}_0)\ c[v] \ge \gamma \|v\|_{\mathcal{U}}^2$, where $\gamma = c_2\beta$.

2. Characterization of the solution. The solution u of problem (1.5) can be described by a variational equality and as a solution of a boundary-value problem.

Theorem 2.1. *Under the assumptions of Theorem 1.1, there exists a unique field* $u \in \mathcal{U}_h$ *satisfying the condition*

$$(\forall v \in \mathcal{U}_h) \quad c(u, v - u) = \int_\Omega (f|v-u)_{\mathbb{R}^3}\, d\Omega + \int_{\Sigma_N} (g|v-u)_{\mathbb{R}^3}\, d\Sigma. \tag{2.1}$$

This field coincides with the solution of problem (1.5).

Proof follows from equality (2.1) with the Euler equation for extremum problem (1.5). Note, that (2.1) expresses the principle of virtual displacements for the medium S.

Now we need some preparations.

Remark 2.1. Let X, Y be Banach spaces, $T \in \mathcal{B}(X;Y)$, $\operatorname{im} T = Y$, $Z := X/\ker T$ and $\widetilde{T}: Z \to Y$, $\widetilde{T}\tilde{x} := Tx$, $\tilde{x} \in Z$, $x \in \tilde{x}$. Then $\widetilde{T}$ is an algebraic and topological isomorphism $Z \to Y$. So, if we denote $|y| := \|\widetilde{T}^{-1}y\|_Z$, then $(\forall y \in Y)\ \|\widetilde{T}^{-1}\|^{-1}|y| \le \|y\|_Y \le \|\widetilde{T}\| \cdot |y|$, i.e. the norms $\|\cdot\|_Y$ and $|\cdot|$ are equivalent on Y.

Corollary 2.2. *Let (H^1, H, H^{-1}), (G^1, G, G^{-1}) be equipped Hilbert spaces, Γ the trace operator ($\Gamma \in \mathcal{B}(H^1, G^{1/2})$, $\operatorname{im}\Gamma = G^{1/2}$, $\operatorname{cl}\ker\Gamma = H$). Let us define*

$$(\forall p \in G^{1/2}) \quad |p| := \inf\{\|v\|_{H^1}; v \in H^1, \Gamma v = p\}. \tag{2.2}$$

The norms $\|\cdot\|_{G^{1/2}}$ and $|\cdot|$ are equivalent on $G^{1/2}$.

Proof. Denote by $Z := H^1/\ker\Gamma$, then $\tilde{v} \in Z \iff (\exists p \in G^{1/2})\ (\tilde{v} = \{v \in H^1; \Gamma v = p\})$. Let us define $(\forall \tilde{v} \in Z)\ (\forall p \in G^{1/2}) \tilde{\Gamma}\tilde{v} = p \iff (\exists v \in \tilde{v})(\Gamma v = p)$. So we can rewrite definition (2.2) in the following way:

$$(\forall p \in G^{1/2}) \quad |p| = \|\tilde{\Gamma}^{-1}p\|_Z.$$

Now the corollary follows from Remark 2.1.

Remark 2.3. For every $u \in H^1$ the vector field $\operatorname{div}(\sigma u)$ satisfies (in the distribution sense) the condition

$$(\forall \varphi \in C_0^\infty(\Omega)^3) \qquad -\int_\Omega (\operatorname{div}(\sigma u)|\varphi)_{\mathbb{R}^3}\, d\Sigma = a(u, \varphi). \tag{2.3}$$

Indeed, by (1.3),

$$a(u,\varphi) = \int_\Omega (\sigma u)_{ij}(\varepsilon\varphi)_{ji}\, d\Omega = \int_\Omega (\sigma u)_{ij}\varphi_{i,j}\, d\Omega = -\int_\Omega (\sigma u)_{ij,j}\varphi_i\, d\Omega.$$

Now it suffices to recall that $(\operatorname{div}\sigma)_i = \sigma_{ij,i}$. (We use the usual notation: $\varphi_{i,j} := \frac{\partial \varphi_i}{\partial x_j}$.)

Proposition 2.4. *Let $u \in H^1$ and $\sigma = \sigma u$ be the stress tensor corresponding to the displacement u. Then the formula*

$$(\forall v \in H^1) \quad (\sigma n|\Gamma v) = a(u, v) + \int_\Omega (\operatorname{div}\sigma|v)_{\mathbb{R}^3}\, d\Omega \tag{2.4}$$

defines the action of the operator σ on the unit vector $n = n(x)$ of the external normal to Σ in the point x as an element of the space $G^{-1/2}$.

Proof. According to (2.3), the right part of (2.4) depends on Γv (but not on v). Considering (2.4) as a definition of a linear functional σn, we have

$$(\forall v \in H^1) \quad |(\sigma n|\Gamma v)| \le C\big(\|u\|_{H^1}\|v\|_{H^1} + \|\operatorname{div}\sigma\|_H\|v\|_H\big),$$

where $C = \text{const}$. Hence, the functional σn is bounded in norm (2.2), therefore it is bounded in the norm $\|\cdot\|_G^{1/2}$ (see Corollary 2.2).

Remark 2.5. Let $u \in \mathcal{U}$, $\overset{\circ}{\sigma} := \overset{\circ}{\sigma} u$, then the vector field $\operatorname{div} \overset{\circ}{\sigma}$ satisfies (in the distribution sense) the condition

$$(\forall v \in \mathcal{U}) \qquad -\int_\Sigma (\operatorname{div} \overset{\circ}{\sigma} | v)_{\mathbb{R}^3} \, d\Sigma = \overset{\circ}{a}(u, v). \tag{2.5}$$

Therefore the following version of the Green-Betty formula holds for $u, v \in \mathcal{U}$:

$$c(u, v) = -\int_\Omega (\operatorname{div} \sigma u | v)_{\mathbb{R}^3} \, d\Omega + \int_\Sigma (\underline{\sigma \overset{\circ}{\sigma}} u | v)_{\mathbb{R}^3} \, d\Sigma, \tag{2.6}$$

where $u \to \underline{\sigma \overset{\circ}{\sigma}} u$ is the differential operator of the second order acting at $x \in \Sigma$ and defined by

$$\underline{\sigma \overset{\circ}{\sigma}} u := -\operatorname{div}(\overset{\circ}{\sigma} u) + (\sigma u) n. \tag{2.7}$$

Note that we understand the integral on Σ in (2.6) as a duality between $G^{1/2}$ and $G^{-1/2}$.

Remark 2.6. Formula (2.6) shows that under the displacement u of the medium S the elasticity forces with volume-density $-\operatorname{div}(\sigma u)$ in Ω and surface density $\underline{\sigma \overset{\circ}{\sigma}} u$ in Σ appear.

Theorem 2.7. *Under the assumptions of Theorem 1.1, there exists a unique function $u \in \mathcal{U}$ satisfying the conditions:*

$$-\operatorname{div}(\sigma u) = f \text{ in } \Omega, \quad \underline{\sigma \overset{\circ}{\sigma}} u = g \text{ on } \Sigma_N, \quad u = h \text{ on } \Sigma_\Pi, \tag{2.8}$$

or in coordinate notations (i=1,2,3):

$$-(\sigma u)_{ij,j} = f_i \text{ in } \Omega, \quad -(\overset{\circ}{\sigma} u)_{ij,j} + (\sigma u)_{ij} n_j = g_i \text{ on } \Sigma_N, \quad u_i = h_i \text{ on } \Sigma_\Pi. \tag{2.8$'$}$$

The solution u of boundary-value problem (2.8) coincides with the solution u of variational problem (1.5) and therefore with the solution u of variational equality (2.1).

We obtain the proof of this theorem in a usual way applying Green's formula (2.6) to variational equality (2.1) and taking into account that $\mathcal{U}$ is dense in H and $\Gamma\mathcal{U} = G^1$ is dense in G.

We call boundary-value problem (2.8) a nonclassical problem, because its boundary conditions contain differentiations of the second order exactly so as equations in the domain.

3. Dual statement of problem. In this part the unknown will be not the vector field of displacements, but the tensor field of stresses.

By $\mathfrak{H}$ and $\overset{\circ}{\mathfrak{H}}$ we denote Hilbert spaces of symmetrical tensor fields defined on Ω and Σ respectively such that

$$\mathfrak{H} = \{\tau : \tau_{ij} = \tau_{ji} \in L_2(\Omega),\ i, j = 1, 2, 3\}, \quad (\forall \sigma, \tau \in \mathfrak{H}) \quad (\sigma | \tau)_{\mathfrak{H}} = \int_\Omega \operatorname{tr}(\sigma\tau) \, d\Omega,$$

$$\overset{\circ}{\mathfrak{H}} = \{\overset{\circ}{\tau} : \overset{\circ}{\tau}_{ij} = \overset{\circ}{\tau}_{ji} \in L_2(\Sigma),\ i, j = 1, 2, 3\}, \quad (\forall \overset{\circ}{\sigma}, \overset{\circ}{\tau} \in \overset{\circ}{\mathfrak{H}}) \quad (\overset{\circ}{\sigma} | \overset{\circ}{\tau})_{\overset{\circ}{\mathfrak{H}}} = \int_\Sigma \operatorname{tr}(\overset{\circ}{\sigma}\overset{\circ}{\tau}) \, d\Sigma. \tag{3.1}$$

For $\tau \in \mathfrak{H}$, $\overset{\circ}{\tau} \in \overset{\circ}{\mathfrak{H}}$ we determine $\operatorname{div}\tau$ and $\operatorname{div}\overset{\circ}{\tau}$ in the sense of the distribution theory, namely

$$\begin{aligned} (\forall\varphi \in C_0{}^{\infty}(\Omega)^3) \quad & \int_\Omega (\operatorname{div}\tau|\varphi)_{\mathbb{R}^3}\, d\Omega = -\int_\Omega \operatorname{tr}(\tau\cdot\varepsilon\varphi)\, d\Omega, \\ (\forall p \in C_0{}^{\infty}(\Sigma)^3) \quad & \int_\Sigma (\operatorname{div}\overset{\circ}{\tau}|p)_{\mathbb{R}^3}\, d\Sigma = -\int_\Sigma \operatorname{tr}(\overset{\circ}{\tau}\cdot\overset{\circ}{\varepsilon}p)\, d\Sigma, \end{aligned} \tag{3.2}$$

where $\varepsilon\varphi = \frac{1}{2}(\varphi + \varphi'^*)$, $\overset{\circ}{\varepsilon}p = \frac{1}{2}(p't + tp'^*)$.

We understand the integrals in the left-hand of (3.2) as the action of functionals $\operatorname{div}\tau$, $\operatorname{div}\overset{\circ}{\tau}$ on φ and p respectively.

Proposition 3.1. *Let $(\tau,\overset{\circ}{\tau}) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}}$, then the formula*

$$(\forall v \in \mathcal{U}) \quad (\underline{\tau\overset{\circ}{\tau}}|\Gamma v) = \int_\Omega \big((\operatorname{div}\tau|v)_{\mathbb{R}^3} + \operatorname{tr}(\tau\cdot\varepsilon v)\big)\, d\Omega + \int_\Sigma \operatorname{tr}(\overset{\circ}{\tau}\cdot\overset{\circ}{\varepsilon}\Gamma v)\, d\Sigma \tag{3.3}$$

determines uniquely $\underline{\tau\overset{\circ}{\tau}}$ as an element of G^{-1}.

Proof. The first integral in (3.3) can be evaluated by $\mathrm{const}\cdot\|v\|_{H^1}$ and according to Corollary 2.2 it represents an element of $G^{-1/2}$. The second integral in (3.3) is dominated by $\mathrm{const}\cdot\|\Gamma v\|_{G^1}$ and therefore it represents an element of G^{-1}.

We rewrite formula (3.3) in such a way:

$$\begin{aligned} &\int_\Omega \big((\operatorname{div}\tau|v)_{\mathbb{R}^3} + \operatorname{tr}(\tau\cdot\varepsilon v)\big)\, d\Omega = \\ &= \int_\Sigma \big((\overset{\circ}{\tau}|\Gamma v)_{\mathbb{R}^3} - \operatorname{tr}(\underline{\tau\overset{\circ}{\tau}}\cdot\overset{\circ}{\varepsilon}v)\big)\, d\Sigma, \quad (\tau,\overset{\circ}{\tau}) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}}, v \in \mathcal{U}. \end{aligned} \tag{3.3$'$}$$

Proposition 3.2. *Let $f \in H$, $g \in G$,*

$$\mathcal{K}_{f,g} := \big\{(\tau,\overset{\circ}{\tau}) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}} : -\operatorname{div}\tau = f \text{ on } \Omega, \quad \underline{\tau\overset{\circ}{\tau}} = g \text{ on } \Sigma_N\big\}. \tag{3.4}$$

Then the affine manifold $\mathcal{K}_{f,g}$ is closed in the Hilbert space $\mathfrak{H}\times\overset{\circ}{\mathfrak{H}}$.

Proof. The operator $\mathfrak{H}\times\overset{\circ}{\mathfrak{H}} \ni (\tau,\overset{\circ}{\tau}) \mapsto (\operatorname{div}\tau, \underline{\tau\overset{\circ}{\tau}}) \in L_2(\Omega)^3 \times L_2(\Sigma)^3$ is defined as the adjoint operator to the operator $C_0{}^{\infty}(\Omega)^3 \times C_0{}^{\infty}(\Sigma)^3 \ni (v,p) \mapsto (\varepsilon v, \overset{\circ}{\varepsilon}p) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}}$ and therefore is closed. Hence the manifold $\mathcal{K}_{f,g}$ is closed as a shift of the kernel of a closed operator.

In order to introduce the dual forms of virtual energy, we remark that the formula of the generalized Hooke law can be inverted (this follows from symmetry and ellipticity conditions (1.2)):

$$\varepsilon_{ij} = A_{ijkl}\sigma_{kl} \quad \text{in} \quad \Omega, \quad \overset{\circ}{\varepsilon}_{ij} = \overset{\circ}{A}_{ijkl}\overset{\circ}{\sigma}_{kl} \quad \text{on} \quad \Sigma, \tag{3.5}$$

or briefly:

$$\varepsilon = A\sigma \quad \text{in} \quad \Omega, \quad \overset{\circ}{\varepsilon} = \overset{\circ}{A}\overset{\circ}{\sigma} \quad \text{on} \quad \Sigma. \tag{3.5$'$}$$

Now for $\sigma, \tau \in \mathfrak{H}$, $\overset{\circ}{\sigma}, \overset{\circ}{\tau} \in \overset{\circ}{\mathfrak{H}}$ put

$$\begin{gathered}
\mathcal{A}(\sigma,\tau) = \int_\Omega \operatorname{tr}(A\sigma\cdot\tau)\,d\Omega, \quad \overset{\circ}{\mathcal{A}}(\overset{\circ}{\sigma},\overset{\circ}{\tau}) = \int_\Sigma \operatorname{tr}(\overset{\circ}{A}\overset{\circ}{\sigma}\cdot\overset{\circ}{\tau})\,d\Sigma, \\
\mathcal{C}(\sigma,\overset{\circ}{\sigma};\tau,\overset{\circ}{\tau}) = \mathcal{A}(\sigma,\tau) + \overset{\circ}{\mathcal{A}}(\overset{\circ}{\sigma},\overset{\circ}{\tau}), \quad \mathcal{C}[\tau,\overset{\circ}{\tau}] = \mathcal{C}(\tau,\overset{\circ}{\tau};\tau,\overset{\circ}{\tau}).
\end{gathered} \tag{3.6}$$

Let $h \in G^1 = H^1(\Sigma)^3$; determine the dual functional of the potential energy $\mathcal{J}$ by the formula:

$$\mathcal{J}(\tau,\overset{\circ}{\tau}) = \frac{1}{2}\mathcal{C}[\tau,\overset{\circ}{\tau}] + \int_{\Sigma_\Pi} (\underline{\tau\overset{\circ}{\tau}}|h)_{\mathbb{R}^3}\,d\Sigma, \quad (\tau,\overset{\circ}{\tau}) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}} \tag{3.7}$$

(here $\Sigma_\Pi = \Sigma \setminus \Sigma_N$).

The problem dual to (1.5) can be formulated as follows:

$$\mathcal{J}(\tau,\overset{\circ}{\tau}) \longrightarrow \inf, \quad (\tau,\overset{\circ}{\tau}) \in \mathcal{K}_{f,g}. \tag{3.8}$$

Theorem 3.3. *Let* $\operatorname{mes}\Sigma_\Pi > 0$*, then problem (3.8) has a unique solution, i.e.* $(\exists!(\sigma,\overset{\circ}{\sigma}) \in \mathcal{K}_{f,g})(\forall(\tau,\overset{\circ}{\tau}) \in \mathcal{K}_{f,g})(\mathcal{J}(\sigma,\overset{\circ}{\sigma}) \le \mathcal{J}(\tau,\overset{\circ}{\tau}))$.

Proof. Immediately from (3.6) we see that the form $\mathcal{C}$ is symmetric and continuous on $\mathfrak{H}\times\overset{\circ}{\mathfrak{H}}$. Since the rigidity coefficients $A_{ijkl}, \overset{\circ}{A}_{ijkl}$ satisfy the conditions

$$\begin{aligned}
(\exists\alpha > 0)(\forall\tau \in \mathfrak{H}) \quad & \operatorname{tr}(A\tau\cdot\tau) \ge \alpha\operatorname{tr}(\tau^2) \quad \text{in} \quad \Omega, \\
(\forall\overset{\circ}{\tau} \in \overset{\circ}{\mathfrak{H}}) \quad & \operatorname{tr}(\overset{\circ}{A}\overset{\circ}{\tau}\cdot\overset{\circ}{\tau}) \ge \alpha\operatorname{tr}(\overset{\circ}{\tau}{}^2) \quad \text{on} \quad \Sigma,
\end{aligned} \tag{3.9}$$

the form $\mathcal{C}$ is coercive on $\mathfrak{H}\times\overset{\circ}{\mathfrak{H}}$. Now the theorem follows directly from Proposition 3.2.

Consider the connection between the solutions of problem (1.5) and the dual problem (3.8).

Theorem 3.4. *Let* $(\sigma,\overset{\circ}{\sigma})$ *be the solution of problem (3.8). Then the equations*

$$\varepsilon u = A\sigma \quad \textit{in} \quad \Omega, \quad \overset{\circ}{\varepsilon} u = \overset{\circ}{A}\overset{\circ}{\sigma} \quad \textit{on} \quad \Sigma \tag{3.10}$$

with the boundary condition

$$u = h \quad \textit{on} \quad \Sigma_\Pi \tag{3.11}$$

uniquely determine the solution u *of problem (1.5).*

Proof. Let u be the solution of problem (1.5). Put $\sigma = a\varepsilon u$ in Ω, $\overset{\circ}{\sigma} = \overset{\circ}{a}\overset{\circ}{\varepsilon} u$ on Σ , then $(\sigma,\overset{\circ}{\sigma})$ is a solution of problem (3.8). Indeed, Theorem 2.8 implies that $(\sigma,\overset{\circ}{\sigma}) \in \mathcal{K}_{f,g}$. So it is sufficient to check that the variational equality

$$(\forall(\tau,\overset{\circ}{\tau}) \in \mathcal{K}_{f,g}) \quad \mathcal{C}(\sigma,\overset{\circ}{\sigma};\tau-\sigma,\overset{\circ}{\tau}-\overset{\circ}{\sigma}) = -\int_{\Sigma_\Pi} (h|\underline{\tau\overset{\circ}{\tau}} - \underline{\sigma\overset{\circ}{\sigma}})_{\mathbb{R}^3}\,d\Sigma$$

holds.

Let $(\tau, \overset{\circ}{\tau}) \in \mathcal{K}_{f,g}$, then due to (3.10) and (3.3′)

$$\mathcal{C}(\sigma, \overset{\circ}{\sigma}; \tau - \sigma, \overset{\circ}{\tau} - \overset{\circ}{\sigma}) = \int_\Omega \operatorname{tr}(\varepsilon u \cdot (\tau - \sigma))\, d\Omega + \int_\Sigma \operatorname{tr}(\overset{\circ}{\varepsilon} u \cdot (\overset{\circ}{\tau} - \overset{\circ}{\sigma}))\, d\Sigma =$$
$$= -\int_\Omega (\operatorname{div}(\tau - \sigma)|u)_{\mathbb{R}^3}\, d\Omega - \int_\Sigma (\underline{\tau\overset{\circ}{\tau}} - \underline{\sigma\overset{\circ}{\sigma}}|\Gamma u)_{\mathbb{R}^3}\, d\Sigma = -\int_{\Sigma_\Pi} (h|\underline{\tau\overset{\circ}{\tau}} - \underline{\sigma\overset{\circ}{\sigma}})_{\mathbb{R}^3}\, d\Sigma,$$

since $\operatorname{div}(\tau - \sigma) = 0$ in Ω, $\underline{\tau\overset{\circ}{\tau}} - \underline{\sigma\overset{\circ}{\sigma}} = 0$ on Σ_N and $u = h$ on Σ_Π (see (2.8)). So, $(\sigma, \overset{\circ}{\sigma})$ is a solution of problem (3.8). The uniqueness of the solutions of problems (1.5) and (3.8) completes the proof.

4. The Lagrange multipliers method. Following the Duvaut-Lions method [5], let us consider the variational problem (1.5) as a conditional extremum problem. To this aim instead of one independent variable $v \in \mathcal{U}$ let us introduce the triple of independent variables $(v, \tau, \overset{\circ}{\tau}) \in \mathcal{U} \times \mathfrak{H} \times \overset{\circ}{\mathfrak{H}}$ satisfying the conditions (the Hooke law):

$$\tau = a\varepsilon v \quad \text{in} \quad \Omega, \quad \overset{\circ}{\tau} = \overset{\circ}{a}\overset{\circ}{\varepsilon} v \quad \text{on} \quad \Sigma \tag{4.1}$$

and take the Lagrange multipliers $q \in \mathfrak{H}$, $\overset{\circ}{q} \in \overset{\circ}{\mathfrak{H}}$.

Put

$$\widetilde{\mathcal{I}}(v, \tau, \overset{\circ}{\tau}) = \frac{1}{2}\mathcal{C}[\tau, \overset{\circ}{\tau}] - \int_\Omega (f|v)_{\mathbb{R}^3}\, d\Omega - \int_{\Sigma_N} (g|v)_{\mathbb{R}^3}\, d\Sigma \tag{4.2}$$

and note that for conditions (4.1)

$$\widetilde{\mathcal{I}}(v, \tau, \overset{\circ}{\tau}) = \mathcal{I}(v), \tag{4.2′}$$

where $\mathcal{I}$ is the functional of potential energy (1.4). Indeed, due to (1.3), (3.5), (3.6) conditions (4.1) imply the equality $c[v] = \mathcal{C}[\tau, \overset{\circ}{\tau}]$. Thus the Lagrange functional corresponding to problem (1.5) can be taken in a natural way

$$\mathcal{I}(v, \tau, \overset{\circ}{\tau}, q, \overset{\circ}{q}) = \widetilde{\mathcal{I}}(v, \tau, \overset{\circ}{\tau}) - \int_\Omega \operatorname{tr}\big(q(\tau - a\varepsilon v)\big)\, d\Omega - \int_\Sigma \operatorname{tr}\big(\overset{\circ}{q}(\overset{\circ}{\tau} - \overset{\circ}{a}\overset{\circ}{\varepsilon} v)\big)\, d\Sigma, \tag{4.3}$$

where $(v, \tau, \overset{\circ}{\tau}, q, \overset{\circ}{q}) \in \mathcal{U}_h \times \mathfrak{H} \times \overset{\circ}{\mathfrak{H}} \times \mathfrak{H} \times \overset{\circ}{\mathfrak{H}}$, $\mathcal{U}_h$ is an affine manifold; remark that there is no Lagrange's multiplier for the condition "$v = h$ on Σ_Π".

Theorem 4.1. *Let u be the solution of problem (1.5). Then*

$$\sup\inf\{\mathcal{I}(v, \tau, \overset{\circ}{\tau}, q, \overset{\circ}{q}) : e(v, \tau, \overset{\circ}{\tau}) \in \mathcal{U}_h \times \mathfrak{H} \times \overset{\circ}{\mathfrak{H}}\} = \mathcal{I}(u),$$

where sup *is taken on the set $\mathfrak{H} \times \overset{\circ}{\mathfrak{H}}$.*

Proof. Let $\mathcal{I}(q, \overset{\circ}{q}) = \inf\{\mathcal{I}(v, \tau, \overset{\circ}{\tau}, q, \overset{\circ}{q}) : (v, \tau, \overset{\circ}{\tau}) \in \mathcal{U}_h \times \mathfrak{H} \times \overset{\circ}{\mathfrak{H}}\}$, then $\mathcal{I}(q, \overset{\circ}{q}) \leq \inf\{\mathcal{I}(v, \tau, \overset{\circ}{\tau}, q, \overset{\circ}{q}) : (v, \tau, \overset{\circ}{\tau}) \in \mathfrak{Q}\}$, where $\mathfrak{Q} = \{(v, \tau, \overset{\circ}{\tau}) \in \mathcal{U}_h \times \mathfrak{H} \times \overset{\circ}{\mathfrak{H}} : (v, \tau, \overset{\circ}{\tau})$ satisfies (4.1)$\}$. But if $(v, \tau, \overset{\circ}{\tau}) \in \mathfrak{Q}$, then $\mathcal{I}(v, \tau, \overset{\circ}{\tau}, q, \overset{\circ}{q}) = \mathcal{I}(v)$ and therefore $\mathcal{I}(q, \overset{\circ}{q}) \leq \mathcal{I}(v)$ for $v \in \mathcal{U}_h$. Hence

$$\sup\{\mathcal{I}(q, \overset{\circ}{q}) : (q, \overset{\circ}{q}) \in \mathfrak{H} \times \overset{\circ}{\mathfrak{H}}\} \leq \mathcal{I}(u). \tag{4.4}$$

It is sufficient to prove the opposite inequality. Let $\mathfrak{Q}_{\tau,\overset{\circ}{\tau}} = \{(q,\overset{\circ}{q}\in\mathfrak{H}\times\overset{\circ}{\mathfrak{H}} : q = A\tau, \overset{\circ}{q} = \overset{\circ}{A}\overset{\circ}{\tau}, (\tau,\overset{\circ}{\tau}) \in \mathcal{K}_{f,g}\}$. We want to prove that

$$\mathcal{I}(q,\overset{\circ}{q}) = \begin{cases} -\frac{1}{2}\mathcal{C}[\tau,\overset{\circ}{\tau}] + \int_{\Sigma_\Pi}(\underline{\tau\overset{\circ}{\tau}}|h)_{\mathbb{R}^3}\,d\Sigma, & \text{if } (q,\overset{\circ}{q}) \in \mathfrak{Q}_{\tau,\overset{\circ}{\tau}}, \\ -\infty, & \text{in other cases.} \end{cases} \tag{4.5}$$

To this aim note that $\mathcal{I}(v,\tau,\overset{\circ}{\tau},q,\overset{\circ}{q}) = \mathcal{I}_1(\tau,\overset{\circ}{\tau},q,\overset{\circ}{q}) + \mathcal{I}_2(v,q,\overset{\circ}{q})$, where

$$\mathcal{I}_1(\tau,\overset{\circ}{\tau},q,\overset{\circ}{q}) = \frac{1}{2}\mathcal{C}[\tau,\overset{\circ}{\tau}] - \int_\Omega \operatorname{tr}(q\cdot\tau)\,d\Omega - \int_\Sigma \operatorname{tr}(\overset{\circ}{q}\cdot\overset{\circ}{\tau})\,d\Sigma,$$

$$\mathcal{I}_2(v,q,\overset{\circ}{q}) = -\int_\Omega (f|v)_{\mathbb{R}^3}\,d\Omega - \int_{\Sigma_N}(g|v)_{\mathbb{R}^3}\,d\Sigma + \int_\Omega \operatorname{tr}(q\cdot a\varepsilon v)\,d\Omega + \int_\Sigma \operatorname{tr}(\overset{\circ}{q}\cdot\overset{\circ}{a}\overset{\circ}{\varepsilon}v)\,d\Sigma.$$

Now the symmetricity of the operators $\overset{\circ}{a}$ and a and Green's formula (2.6) imply:

$$\mathcal{I}_2(v,q,\overset{\circ}{q}) = -\int_\Omega (f|v)_{\mathbb{R}^3}\,d\Omega - \int_{\Sigma_N}(g|v)_{\mathbb{R}^3}\,d\Sigma - \int_\Omega (\operatorname{div} aq|v)_{\mathbb{R}^3}\,d\Omega - \int_\Sigma (\underline{aq\cdot\overset{\circ}{a}\overset{\circ}{q}}|v)_{\mathbb{R}^3}\,d\Sigma.$$

It is easy to see that for fixed $(q,\overset{\circ}{q}) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}}$ the functional $\mathcal{I}_1$ reaches the infimum at $A\tau = q$, $\overset{\circ}{A}\overset{\circ}{\tau} = \overset{\circ}{q}$ (see (3.6)). Therefore $\inf\{\mathcal{I}_1(\tau,\overset{\circ}{\tau},q,\overset{\circ}{q}) : (\tau,\overset{\circ}{\tau}) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}}\} = -\frac{1}{2}\mathcal{C}[\tau,\overset{\circ}{\tau}]$, where $\tau = aq, \overset{\circ}{\tau} = \overset{\circ}{a}\overset{\circ}{q}$. But for such $\tau,\overset{\circ}{\tau}$ and $(q,\overset{\circ}{q}) \in \mathfrak{Q}_{\tau,\overset{\circ}{\tau}}$, $v \in \mathcal{U}_h$ we have $\mathcal{I}_2(v,q,\overset{\circ}{q}) = -\int_{\Sigma_\Pi}(\underline{\tau\overset{\circ}{\tau}}|h)_{\mathbb{R}^3}\,d\Sigma$. So, (4.5) is proved. From (4.5) we conclude that

$$\sup\{\mathcal{I}(q,\overset{\circ}{q}) : (q,\overset{\circ}{q}) \in \mathfrak{H}\times\overset{\circ}{\mathfrak{H}}\} =$$
$$= \sup\Big\{-\frac{1}{2}\mathcal{C}[\tau,\overset{\circ}{\tau}] - \int_{\Sigma_\Pi}(\underline{\tau\overset{\circ}{\tau}}|h)_{\mathbb{R}^3}\,d\Sigma : (q,\overset{\circ}{q}) \in \mathfrak{Q}_{\tau,\overset{\circ}{\tau}}\Big\} \geq -\frac{1}{2}\mathcal{C}[\sigma,\overset{\circ}{\sigma}] - \int_{\Sigma_\Pi}(\underline{\sigma\overset{\circ}{\sigma}}|h)_{\mathbb{R}^3}\,d\Sigma,$$

where $\sigma = \sigma u$, $\overset{\circ}{\sigma} = \overset{\circ}{\sigma}u$ and u is the solution of problem (1.5). However, $-\frac{1}{2}\mathcal{C}[\sigma,\overset{\circ}{\sigma}] - \int_{\Sigma_\Pi}(\underline{\sigma\overset{\circ}{\sigma}}|h)_{\mathbb{R}^3}\,d\Sigma = \mathcal{I}(u)$, so the inequality opposite to (4.4) is proved.

Remark 4.2. From Theorem 4.1 and (4.5) we get $\inf\{\frac{1}{2}\mathcal{C}[\tau,\overset{\circ}{\tau}] + \int_{\Sigma_\Pi}(\underline{\tau\overset{\circ}{\tau}}|h)_{\mathbb{R}^3}\,d\Sigma : (\tau,\overset{\circ}{\tau}) \in \mathcal{K}_{f,g}\} + \mathcal{I}(u) = 0$. Due to (3.7) this implies $\inf\{\mathcal{J}(\tau,\overset{\circ}{\tau}) : (\tau,\overset{\circ}{\tau}) \in \mathcal{K}_{f,g}\} + \inf\{\mathcal{I}(v) : v \in \mathcal{U}_h\} = 0$, which proves Theorem 3.4 once again.

5. Nonclassical problem with friction. In this problem on Σ_N the normal component g_n of the surface density of external forces is given only. Besides, on Σ_N the surface density g_t of the friction force is given. We suppose that $g_n, g_t \in L_2(\Sigma_N)$. The energy of the friction force under the displacement v we determine by formula

$$(\forall v \in H^1(\Omega)^3) \quad j(v) = \int_{\Sigma_N} g_t\|tv\|_{\mathbb{R}^3}\,d\Sigma, \tag{5.1}$$

where $t = t(x)$, $x \in \Sigma, t$ is an orthoprojector $\mathbb{R}^3 \to T\Sigma_x$ ($T\Sigma_x$ is the tangent manifold to Σ in the point x); here and in what follows we understand the value $v \in H^1(\Omega)^3$ on $\Sigma = \partial\Omega$ in the sense of the theorem of traces [7]. The functional of potential energy is given by the formula ($v \in \mathcal{U}$):

$$\mathcal{I}(v) = \frac{1}{2}c[v] + j(v) - \int_{\Omega} (f|v)_{\mathbb{R}^3}\, d\Omega - \int_{\Sigma_N} g_n (v|n)_{\mathbb{R}^3}\, d\Sigma, \tag{5.2}$$

where $f \in L_2(\Omega)^3$ is a volume density of external forces acting on S, $n = n(x)$ is a unit vector of external normal in the point $x \in \Sigma$. The medium S reaches the equilibrium state for such a displacement $u \in \mathcal{U}_h = \{v \in \mathcal{U} : v = h \text{ on } \Sigma_\Pi\}$, $\Sigma_\Pi = \Sigma \setminus \Sigma_N$, which gives the minimum to the functional $\mathcal{I}$. So, it is necessary to research the problem

$$\mathcal{I}(v) \longrightarrow \inf, \qquad v \in \mathcal{U}_h. \tag{5.3}$$

Theorem 5.1. *Problem (5.3) has a unique solution.*

Proof. The functional j is not differentiable, but it is convex on $H^1(\Omega)^3$. So the functional $v \mapsto \mathcal{I}(v) - j(v)$ is differentiable, strictly convex and coercive on $\mathcal{U}_0$ (see the proof of Theorem 1.1). Therefore Theorem 1.1 [6, part 1] can be applied.

The solution u of problem (5.3) can be characterized by a variational inequality.

Theorem 5.2. *The displacement field $u \in \mathcal{U}_h$ is a solution of variational problem (5.3) iff*

$$c(u, v-u) + j(v) - j(u) \geq \int_{\Omega} (f|v-u)_{\mathbb{R}^3}\, d\Omega + \int_{\Sigma_N} g_n (v-u|n)_{\mathbb{R}^3}\, d\Sigma. \tag{5.4}$$

The proof immediately follows from Theorem 1.6 [6, part 1].

Now we will formulate the boundary-value problem whose solution coincides with a solution of variational problem (5.3) and hence with a solution of variational inequality (5.4).To simplify the notations let us denote:

$$\sigma = \sigma u, \quad \overset{\circ}{\sigma} = \overset{\circ}{\sigma} u, \quad \underline{\sigma\overset{\circ}{\sigma}} = -\operatorname{div} \overset{\circ}{\sigma} + \sigma n. \tag{5.5}$$

Observe, that $u \mapsto \underline{\sigma\overset{\circ}{\sigma}}$ is a differential operator of the second order, and $\underline{\sigma\overset{\circ}{\sigma}}$ is a surface density of stresses on Σ. As before we understand the differentiation in the sense of distribution theory.

Theorem 5.3. *The field $u \in \mathcal{U}$ is a solution of variational problem (5.3) (and hence of variational inequality (5.4)) iff*

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega, \tag{5.6}$$

$$(\underline{\sigma\overset{\circ}{\sigma}}|n)_{\mathbb{R}^3} = g_n \quad \text{on } \Sigma_N, \tag{5.7}$$

$$\|t\underline{\sigma\overset{\circ}{\sigma}}\|_{\mathbb{R}^3} < g_t \Longrightarrow tu = 0 \quad \text{on } \Sigma_N, \tag{5.8}$$

$$\|t\underline{\sigma\overset{\circ}{\sigma}}\|_{\mathbb{R}^3} = g_t \Longrightarrow \exists \lambda \geq 0 \quad tu = -\lambda t\underline{\sigma\overset{\circ}{\sigma}} \quad \text{on } \Sigma_N, \tag{5.9}$$

$$u = h \quad \text{on } \Sigma_\Pi. \tag{5.10}$$

The proof of Theorem 5.3 needs some preparation. Remark that according to implication (5.8) the tangent component tu of a displacement u is equal to zero until the tangent component $t\underline{\sigma\overset{\circ}{\sigma}}$ of a stress is less (by norm) than some critical value g_t. As soon as this critical value has been reached, the vectors tu and $t\underline{\sigma\overset{\circ}{\sigma}}$ become collinear and have opposite direction.

Lemma 5.4. *If*

$$\|t\underline{\sigma\overset{\circ}{\sigma}}\|_{\mathbb{R}^3} \le g_t \quad on \quad \Sigma_N \tag{5.11}$$

and

$$(t\underline{\sigma\overset{\circ}{\sigma}}|tu)_{\mathbb{R}^3} + g_t\|tu\|_{\mathbb{R}^3} = 0 \quad on \quad \Sigma_N \tag{5.12}$$

then implications (5.8), (5.9) hold.

Proof. Let $z = t\underline{\sigma\overset{\circ}{\sigma}}$. Suppose, that $\|z\| < g_t$, and, hence $g_t > 0$. If $tu = 0$, then according to (5.12) $g_t\|tu\| \le z\|tu\| < g_t\|tu\|$, i.e. $g_t < g_t$, and therefore $tu = 0$. If $\|z\| = g_t$, then according to (5.12) $(z|tu) + \|z\| \cdot \|tu\| = 0$, and therefore the vectors z and tu are collinear. Putting $tu = -\lambda z$ in (5.12) we get $(z| - \lambda z) + g_t\|tu\| = 0$ and hence $\lambda \ge 0$.

Lemma 5.5. *Let implications (5.8), (5.9) hold, then* $\forall v \in \mathcal{U}$

$$(t\underline{\sigma\overset{\circ}{\sigma}}|tu - tv)_{\mathbb{R}^3} + g_t(\|tv\|_{\mathbb{R}^3} - \|tu\|_{\mathbb{R}^3}) \ge 0 \text{ on } \Sigma_N. \tag{5.13}$$

Proof. If $\|t\underline{\sigma\overset{\circ}{\sigma}}\| \le g_t$ and $tu = 0$, then (5.13) follows from the Bunyakovsky inequality. If $\|t\underline{\sigma\overset{\circ}{\sigma}}\| = g_t$ and $\lambda > 0$, then $\|tu\| = \lambda g_t$ and left-hand of (5.13) is equal to $(-\frac{1}{\lambda}tu|tv - tu) + \frac{1}{\lambda}\|tu\|(\|tv\| - \|tu\|) = \frac{1}{\lambda}(\|tu\| \cdot \|tv\| - (tu|tv\|)) \ge 0$.

Remark 5.6. Condition (5.12) rewriting in a form

$$\Big(t\underline{\sigma\overset{\circ}{\sigma}}|\frac{tu}{\|tu\|}\Big) = -g_t \tag{5.12'}$$

shows what is the orthoprojection of the tangent component of a stress in the direction of the tangent component of a displacement.

Proof of Theorem 5.3. We will show that the solution u of variational problem (5.3) is a solution of boundary-value problem (5.6)–(5.10). To this aim note that if the field u gives the minimum to the functional (5.2) on the manifold $\mathcal{U}_h$, then u satisfies the variational inequality (5.4). Putting in (5.4) $v = u \pm \varphi$, $\varphi \in C_0^\infty(\Omega)^3$ and observing that in this case $j(v) = j(u)$, we obtain $a(u, \varphi) \ge \pm \int_\Omega (f|g)_{\mathbb{R}^3}\, d\Omega$, what implies equality (5.6) immediately. Now applying Green's formula (2.6) we get from (5.4) $\int_{\Sigma_N} (\underline{\sigma\overset{\circ}{\sigma}}|v - u)_{\mathbb{R}^3}\, d\Sigma + j(v) - j(u) \ge \int_{\Sigma_N} g_n(v - u)\, d\Sigma$. Using (5.1) and the formula $tz = z - (z|n)_{\mathbb{R}^3} \cdot n$, $(z \in \mathbb{R}^3)$, we get

$$\int_{\Sigma_N} \big((t\underline{\sigma\overset{\circ}{\sigma}}|t(v-u))_{\mathbb{R}^3} + g_t(\|tu\| - \|tv\|)\big)\, d\Sigma + \int_{\Sigma_N} (\underline{\sigma\overset{\circ}{\sigma}} - g_n \cdot n|n)_{\mathbb{R}^3}(v - u|n)\, d\Sigma \ge 0. \tag{5.14}$$

In particular, if $tu = tv$ on Σ_N,

$$\int_{\Sigma_N} (\underline{\sigma\overset{\circ}{\sigma}} - g_n \cdot n|n)(v - u|n)\, d\Sigma \ge 0. \tag{5.15}$$

If v runs on $H^1(\Omega)^3$ so, that $tu = tv$ on Σ_N, then $(v|h)_{\mathbb{R}^3}$ covers the space $H^{1/2}(\Sigma)$, in particular, the set of functions φ with $\operatorname{supp}\varphi \subset \Sigma_N$ and (5.15) implies that the boundary condition (5.7) holds. Now (5.14) means

$$(\forall v \in \mathcal{U}_h) \int_{\Sigma_N} (t\underline{\sigma\overset{\circ}{\sigma}}|tv - tu) + g_t(\|tv\| - \|tu\|)\, d\Sigma \ge 0. \tag{5.16}$$

Let $\Psi = \{\psi \in H^{1/2}(\Sigma)^3 : \operatorname{supp}\psi \subset \Sigma_N\}$. Applying (5.16) to the function $v \in \mathcal{U}_h$ such that $tv = t\psi$ on Σ_N, and taking into account that $\|t\psi\| \le \|\psi\|$, $g_t > 0$, $t^2 = t = t^*$ we obtain

$$\int_{\Sigma_N} \big((t\underline{\sigma\overset{\circ}{\sigma}}|\psi) + g_t\|\psi\|\big)\, d\Sigma \ge \int_{\Sigma_N} \big((t\underline{\sigma\overset{\circ}{\sigma}}|tu) + g_t\|tu\|\big)\, d\Sigma.$$

Changing ψ into $\pm\lambda\psi$ we get: $(\forall \lambda \ge 0)$

$$\lambda \int_{\Sigma_N} \big(\pm(t\underline{\sigma\overset{\circ}{\sigma}}|\psi) + g_t\|\psi\|\big)\, d\Sigma \ge \int_{\Sigma_N} \big((t\underline{\sigma\overset{\circ}{\sigma}}|tu) + g_t\|tu\|\big)\, d\Sigma.$$

This implies

$$(\forall \psi \in \Psi) \qquad \int_{\Sigma_N} (t\underline{\sigma\overset{\circ}{\sigma}}|\psi)\, d\Sigma \le \int_{\Sigma_N} g_t\|\psi\|\, d\Sigma, \tag{5.17}$$

and

$$\int_{\Sigma_N} \left((t\underline{\sigma}\overset{\circ}{\sigma}|tu) + g_t\|tu\| \right) d\Sigma \le 0. \tag{5.18}$$

Inequality (5.17) implies that the functional $g_t\psi \mapsto \int_{\Sigma_N} (t\underline{\sigma}\overset{\circ}{\sigma}|\psi)\, d\Sigma = \int_{\Sigma_N} \left(\frac{1}{g_t} t\underline{\sigma}\overset{\circ}{\sigma}|g_t\psi \right) d\Sigma$ is continuous on the set $g_t\Psi$ which we consider as a subspace of $L_1(\Sigma_N)^3$, and its norm is equal or less than 1. Since $g_t\Psi$ is dense in $L_1(\Sigma_N)^3$, vraimax$\{\frac{1}{g_t}\|t\underline{\sigma}\overset{\circ}{\sigma}\| \le 1\}$ (vraimax is taken on the set Σ_N), i.e. (5.11) holds.

Therefore $(t\underline{\sigma}\overset{\circ}{\sigma}|tu) + g_t\|tu\| \ge 0$ and (5.18) imply that $(t\underline{\sigma}\overset{\circ}{\sigma}|tu) + g_t\|tu\| = 0$, i.e. (5.12) holds, so one can apply Lemma 5.4.

Finally we will show that the solution u of boundary-value problem (5.6)–(5.10) is a solution of variational problem (5.3). To this aim observe that equation (5.6) and the Green's formula imply:

$$\int_\Omega (f|v-u)_{\mathbb{R}^3}\, d\Omega = -\int_\Omega (\operatorname{div}\sigma|v-u)_{\mathbb{R}^3} d\Omega = c(u, v-u) - \int_\Sigma (\underline{\sigma}\overset{\circ}{\sigma}|v-u)_{\mathbb{R}^3}\, d\Sigma.$$

But if $u, v \in \mathcal{U}_h$, then $\int_\Sigma (\underline{\sigma}\overset{\circ}{\sigma}|v-u)_{\mathbb{R}^3}\, d\Sigma = \int_{\Sigma_N} (\underline{\sigma}\overset{\circ}{\sigma}|v-u)_{\mathbb{R}^3}\, d\Sigma$.

Using $tz = z - (z|n)n$ and the boundary condition (5.7) we get:

$$c(u, v-u) + j(v) - j(u) =$$

$$= \int_\Omega (f|v-u)\, d\Omega + \int_{\Sigma_N} g_n(n|v-u) d\Sigma + \int_{\Sigma_N} ((t\underline{\sigma}\overset{\circ}{\sigma}|v-u) + g_t(\|tv\| - \|tu\|))\, d\Sigma.$$

Applying Lemma 5.5 we conclude that variational inequality (5.4) holds. Now it is sufficient to use Theorem 5.2.

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