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DIRICHLET SERIES OF BOUNDED l-M-INDEX

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The class of absolutely convergent in $(-\infty, A)$, $A \in (-\infty, +\infty]$, Dirichlet series of bounded linear l-M-index is introduced. The growth of functions from this class is investigated.

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Вводится класс абсолютно сходящихся на $(-\infty, A)$, $A \in (-\infty, +\infty]$, рядов Дирихле ограниченного линейного l-M-индекса. Исследуется возрастание функций из этого класса.

1°. Let l be a positive continuous function on $[0, +\infty]$. An entire function f is said to be of bounded l-M-index [1–2] if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and all $r \in [0, +\infty)$

$$\frac{M_{f^{(n)}}(r)}{n!l^n(r)} \le \max\left\{\frac{M_{f^{(k)}}(r)}{k!l^k(r)} : 0 \le k \le N\right\},$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$. Using results from [3], in [4] with some additional conditions on l it is established that an entire function f is of bounded l-M-index iff

$$\overline{\lim_{r\to+\infty}} \frac{\ln M_f(r)}{L(r)} < +\infty, \quad L(r) = \int_0^r l(t)dt.$$

An entire (absolutely convergent in $\mathbb{C}$) Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

with exponents $0 = \lambda_0 < \lambda_n \uparrow +\infty$, $n \to \infty$, is a direct generalization of power development of an entire function. The results from [3] are generalized in [5] for

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Dirichlet series (1) with an arbitrary abscissa $A \in (-\infty, +\infty]$ of absolute convergence. Therefore, a problem of analogues of theorems from [4] for such Dirichlet series is natural.

Let now l be a positive continuous function on $(-\infty,\, A)$ such that for every $\sigma \in (-\infty,\, A)$

$$\int_{\sigma}^{A} l(t)dt = +\infty, \tag{2}$$

and for Dirichlet series (1) with an arbitrary abscissa $A \in (-\infty, +\infty]$ of absolute convergence let $M(\sigma, F) = \sup\{|F(\sigma + it)| :\ t \in \mathbb{R}\}$, $\sigma < A$. The function F is said to be of bounded linear l-M-index if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and all $\sigma \in (-\infty,\, A)$

$$\frac{M(\sigma, F^{(n)})}{n! l^n(\sigma)} \le \max\left\{\frac{M(\sigma, F^{(k)})}{k! l^k(\sigma)} :\ 0 \le k \le N\right\}. \tag{3}$$

The least such integer N is called the linear l-M-index of series (1).

2°. It is known that the function $\ln M(\sigma, F)$ is convex on $(-\infty,\, A)$ and thus it is continuous and has a continuous nondecreasing derivative out, perhaps, an enumerable set of points. In this points there exist one-sided derivatives and, besides, the left-sided derivative does not exceed the right-sided derivative. Hence it follows that the function $M(\sigma, F)$ is continuously differentiable on $(\infty,\, A)$, except an enumerable set of points in which here exist one-sided derivatives and, besides, the left-sided derivative does not exceed the right-sided derivative.

Lemma 1. *For every Dirichlet series (1) with an arbitrary abscissa of absolute convergence $A \in (-\infty, +\infty]$ and for all $\sigma < A$ the inequality $M'(\sigma, F) \le M(\sigma, F')$ holds (in the points, in which the derivative not exists, one can put an arbitrary one-sided derivative).*

Proof. In virtue of the remarks given above, it is sufficient to prove this inequality only for the right-side derivative $M'(\sigma, F)$. We remark that if a function f is absolutely continuous on $[a, b]$ then

$$|f(b)| - |f(a)| \le |f(b) - f(a)| = \left|\int_a^b f'(t)\ dt\right| \le \int_a^b |f'(t)|\ dt.$$

Therefore, for every $\sigma < \sigma_1 < A$ we have

$$|F(\sigma_1 + it)| \le |F(\sigma + it)| + \int_{\sigma}^{\sigma_1} |F'(y + it)|\ dy \le M(\sigma, F) + \int_{\sigma}^{\sigma_1} M(y, F')\ dy,$$

and thus

$$M(\sigma_1, F) \le M(\sigma, F) + \int_{\sigma}^{\sigma_1} M(y, F')\ dy.$$

Hence, it follows

$$\frac{M(\sigma_1, F) - M(\sigma, F)}{\sigma_1 - \sigma} \le \frac{1}{\sigma_1 - \sigma}\int_{\sigma}^{\sigma_1} M(y, F')\ dy.$$

Tending $\sigma_1 \to \sigma$, we obtain the required inequality.

Lemma 2. *Let functions $f_1, \dots, f_n$ be absolutely continuous on $[a, b]$ and*

$$f(x) \stackrel{\text{def}}{=\!=} \max\{f_k(x):\ 1 \le k \le n\}, \quad x \in [a, b].$$

Then the function f is absolutely continuous on $[a, b]$ and almost everywhere on $[a, b]$

$$(f'(x))^+ \le \max\{(f_k'(x))^+:\ 1 \le k \le n\}, \tag{4}$$

where $(x)^+ = \max\{x, 0\}$.

Proof. Let $\alpha, \beta \in [a, b]$. We show that if $f(\beta) \ge f(\alpha)$ that

$$f(\beta) - f(\alpha) \le \max\{f_k(\beta) - f_k(\alpha):\ 1 \le k \le n\}. \tag{5}$$

Indeed, let $f(\beta) = f_{m_1}(\beta)$, $f(\alpha) = f_{m_2}(\alpha)$. Then $f_{m_1}(\alpha) \le f_{m_2}(\alpha)$ and thus

$$f(\beta) - f(\alpha) = f_{m_1}(\beta) - f_{m_2}(\alpha) \le f_{m_1}(\beta) - f_{m_1}(\alpha).$$

Hence (5) follows and for all $\alpha, \beta \in [a, b]$ we obtain

$$|f(\beta) - f(\alpha)| \le \max\{|f_k(\beta) - f_k(\alpha)|:\ 1 \le k \le n\}$$

and

$$|f(\beta) - f(\alpha)| \le \sum_{k=1}^{n} |f_k(\beta) - f_k(\alpha)|, \tag{6}$$

i. e. f is absolutely continuous.

We denote by Δ the set $x \in (a, b)$ for which there exist finite f', f_k', $1 \le k \le n$. From the absolute continuity of the functions f, f_k it follows that the set $[a, b] \setminus \Delta$ has null measure.

Let $x \in \Delta$ and $f'(x) > 0$. Then there exists a sequence $(x_m) \subset (a, b)$ such that

$$\big(x_m \uparrow x\big) \wedge \big(\forall m \in \mathbb{N}\big)\big(f(x) - f(x_m) \ge 0\big).$$

Therefore, in view of (5), we have

$$\frac{f(x) - f(x_m)}{x - x_m} \le \max\left\{\frac{f_k(x) - f_k(x_m)}{x - x_m}:\ 1 \le k \le n\right\}. \tag{7}$$

Tending in (7) $m \to \infty$, in view of continuity of the function

$$\mathbb{R}^n \ni (1, \dots, x_n) \to \max\{x_k:\ 1 \le k \le n\},$$

we obtain $f'(x) \le \max\{f_k'(x):\ 1 \le k \le n\}$ and thus

$$(f'(x))^+ \le \max\{(f_k'(x))^+:\ 1 \le k \le n\}.$$

Lemma 3. *Let $f_1, \dots, f_n$ be positive absolutely continuous functions on $[a,b]$ and*

$$f(x) \stackrel{\text{def}}{=} \max\{f_k(x):\ 1 \le k \le n\}.$$

If almost everywhere on $[a,b]$

$$f_k'(x) \le \varphi(x) f(x), \quad 1 \le k \le n, \tag{8}$$

where φ is a positive continuous function on $[a,b]$, then for all $x \in [a,b]$

$$\ln f(x) \le \ln f(a) + \int_a^x \varphi(t)\ dt. \tag{9}$$

Proof. From (8), in view of Lemma 2, we obtain

$$(f'(x))^+ \le \max\{(f_k'(x))^+ :\ 1 \le k \le n\} \le \varphi(x) f(x) \tag{10}$$

almost everywhere on $[a,b]$. From the absolute continuity of f (see Lemma 2) and its positivity it follows that the function $\ln f(x)$ is absolutely continuous. Hence, in view of (10), we obtain

$$\ln f(x) - \ln f(a) = \int_a^x (\ln f(t))'\ dt = \int_a^x \frac{f'(t)}{f(t)}\ dt \le \int_a^x \frac{(f'(t))^+}{f(t)}\ dt \le \int_a^x \varphi(t)\ dt.$$

3°. For $a \in \mathbb{R}$ we put $a^- = 0$, if $a \ge 0$, and $a^- = -a$, if $a \le 0$.

Theorem 1. *Suppose that a function l is positive continuous on $(-\infty, A)$, continuously differentiable on $[\sigma_0, A)$, satisfies condition (2) and*

$$\varlimsup_{\sigma\uparrow A} \frac{(l'(\sigma))^-}{l^2(\sigma)} = q < +\infty. \tag{11}$$

If Dirichlet series (1) has an abscissa of absolutely convergence A and represents a function of bounded linear l-M-index N then

$$\varlimsup_{\sigma\uparrow A} \frac{\ln M(\sigma, F)}{L(\sigma)} \le (N+1)(1+q), \quad L(\sigma) = \int_{\sigma_0}^{\sigma} l(t) dt. \tag{12}$$

Proof. Put

$$f_k(\sigma) = \frac{M(\sigma, F^{(k)})}{k! l^k(\sigma)}, \quad 0 \le k \le N$$

and $f(\sigma) = \max\{f_k(\sigma):\ 0 \le k \le N\}$.

In view of Lemma 1, almost everywhere on $[\sigma_0, A)$ we have

$$f_k'(\sigma) = \frac{M'(\sigma, F^{(k)})}{k! l^k(\sigma)} - \frac{M(\sigma, F^{(k)})}{k! l^{k+1}(\sigma)} k l'(\sigma) \le$$
$$\le \frac{M(\sigma, F^{(k+1)})}{(k+1)! l^{k+1}(\sigma)} (k+1) l(\sigma) + \frac{M(\sigma, F^{(k)})}{k! l^k(\sigma)} k \frac{(l'(\sigma))^-}{l(\sigma)} \le$$
$$\le f(\sigma) \left((k+1) l(\sigma) + k \frac{(l'(\sigma))^-}{l(\sigma)} \right) \le (N+1) \left(l(\sigma) + \frac{(l'(\sigma))^-}{l(\sigma)} \right) f(\sigma).$$

Putting $\varphi(\sigma) = (N+1)\Big(l(\sigma) + \frac{(l'(\sigma))^-}{l(\sigma)}\Big)$, in view of Lemma 3, we obtain

$$\ln f(\sigma) \le \ln f(\sigma_0) + \int_{\sigma_0}^{\sigma} \varphi(t)\ dt.$$

Hence, it follows

$$\ln M(\sigma, F) \le \ln f(\sigma) \le (N+1)(1+q)(1+o(1)) \int_{\sigma_0}^{\sigma} l(t)\ dt, \quad \sigma \to A.$$

Theorem 1 is proved.

We remark that $q = 0$, if l is nondecreasing. We remark also that from the proof of (12) it follows that condition (11) can be replaced by the condition

$$\overline{\lim_{t\uparrow A}} \frac{\sum_{\sigma_k^* \le t}\{\ln l(\sigma_k) - \ln l(\sigma_k^*)\}}{L(t)} = q < +\infty,$$

where $\big((\sigma_k, \sigma_k^*)\big)$ is a sequence of intervals on which the function $\ln l$ is decreasing $(\sigma_k^* < \sigma_{k+1})$.

4°. Let $-\infty < A \le +\infty$ and $\Omega(A)$ be a class of positive unbounded on $(-\infty, A)$ functions Φ such that the derivative Φ' is continuous positive and increasing to $+\infty$ on $(-\infty, A)$. From now on, by φ we denote the inverse function to Φ', and let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with Φ in the sence of Newton. Put

$$A_\Phi = \overline{\lim_{\sigma\to A}} \frac{\Phi'(\sigma)}{\Phi'(\Psi(\sigma))}.$$

Clearly, $A_\Phi \ge 1$. If $A_\Phi < +\infty$, then we say that the function Φ has regularly varying derivative relatively of Ψ.

The quantity

$$T_\Phi = \overline{\lim_{\sigma\to A}} \frac{\ln M(\sigma, F)}{\Phi(\sigma)}$$

is called the Φ-type of Dirichlet series (1).

Theorem 2. *Let α be a positive continuous increasing to $+\infty$ on $[0, +\infty)$ function such that $\alpha(t) = o(t)$, $t \to +\infty$. Let $-\infty < A \le +\infty$, $\Phi \in \Omega(A)$, Φ' a regularly varying function with respect to the function Ψ associated with Φ in the sence of Newton,*

$$\overline{\lim_{\sigma\to A}} \frac{\Phi'\Big(\sigma + \frac{2}{\alpha(a\Phi'(\sigma))}\Big)}{\Phi'(\sigma)} = \Gamma_{\Phi'}(a) < +\infty \tag{13}$$

for every $a \in (0, +\infty)$ and there exists $\sigma^* \in (-\infty, A-2)$ such that

$$\frac{2\Phi'(\sigma)}{\alpha(\Phi'(\sigma))} < \Phi(\sigma) + (A - \sigma)\Phi'(\sigma),\ \sigma^* \le \sigma < A, \tag{14}$$

Finally, suppose that $\ln n(t) \le t/\alpha(t)$ for $t \ge t_0$, where $n(t) = \sum_{\lambda_n \le t} 1$ is the counting function of the sequence (λ_n), and Dirichlet series (1) has the abscissa of absolutely convergence A and the Φ-type $T_\Phi < +\infty$.

Then for every $\sigma_0 \in (-\infty, A-2)$ there exists $n_0 \in \mathbb{Z}_+$ such that for all $n \geq n_0$ and all $\sigma \in [\sigma_0, A)$ the following inequality is true

$$\frac{M(\sigma, F^{(n)})}{n! M(\sigma, F)} \leq \big\{\Phi'(\sigma)\big\}^n. \tag{15}$$

Proof. Let $T_\Phi < +\infty$. Then for every $T > T_\Phi \geq 0$

$$\varlimsup_{\sigma\to A} \frac{\ln M(\sigma, F)}{\Phi(\sigma)} \leq T. \tag{16}$$

In [5] it is shown that if $\varlimsup\limits_{\sigma\to A} \dfrac{\ln M(\sigma, F)}{\Phi(\sigma)} \leq 1$, then

$$\varlimsup_{\sigma\to A} \frac{M(\sigma, F')}{M(\sigma, F)\Phi'(\Psi^{-1}(\sigma+\beta(\sigma)))} \leq 1, \quad \beta(\sigma) = \frac{2}{\alpha(\Phi'(\Psi^{-1}(\sigma)))}.$$

Hence, considering $T\Phi(x)$ instead of Φ, in view of (16), we obtain

$$\varlimsup_{\sigma\to A} \frac{M(\sigma, F')}{M(\sigma, F)\Phi'(\Psi^{-1}(\sigma+\beta_T(\sigma)))} \leq T, \tag{17}$$

where $\beta_T(\sigma) = 2/\alpha(T\Phi'(\Psi^{-1}(\sigma)))$.

Thus,

$$\varlimsup_{\sigma\to A} \frac{M(\sigma, F')}{M(\sigma, F)\Phi'(\sigma)} = \varlimsup_{\sigma\to A} \frac{M(\sigma, F')}{M(\sigma, F)\Phi'(\Psi^{-1}(\sigma+\beta_T(\sigma)))} \frac{\Phi'(\Psi^{-1}(\sigma+\beta_T(\sigma)))}{\Phi'(\sigma)} \leq$$
$$\leq T \varlimsup_{\sigma\to A} \frac{\Phi'(\Psi^{-1}(\sigma+\beta_T(\sigma)))}{\Phi'(\sigma)} = T \varlimsup_{\sigma\to A} \frac{\Phi'(\Psi^{-1}(\sigma+\beta_T(\sigma)))}{\Phi'(\Psi^{-1}(\sigma))} \frac{\Phi'(\Psi^{-1}(\sigma))}{\Phi'(\sigma)} \leq$$
$$\leq TA_\Phi \varlimsup_{\sigma\to A} \frac{\Phi'(\Psi^{-1}(\sigma+\beta_T(\sigma)))}{\Phi'(\Psi^{-1}(\sigma))} = TA_\Phi \varlimsup_{\sigma\to A} \frac{\Phi'(\Psi^{-1}(\sigma+\beta_T(\sigma)))}{\Phi'(\sigma+\beta_T(\sigma))} \frac{\Phi'(\sigma+\beta_T(\sigma))}{\Phi'(\Psi^{-1}(\sigma))} \leq$$
$$\leq TA_\Phi^2 \varlimsup_{\sigma\to A} \frac{\Phi'(\sigma+\beta_T(\sigma))}{\Phi'(\Psi^{-1}(\sigma))} \leq TA_\Phi^2 \varlimsup_{\sigma\to A} \frac{\Phi'(\sigma+\beta_T(\sigma))}{\Phi'(\sigma)} \leq TA_\Phi^2 \Gamma_\Phi(T) < +\infty.$$

Therefore,

$$\sup\left\{\frac{M(\sigma, F')}{M(\sigma, F)\Phi'(\sigma)} :\ \sigma \in [\sigma_0, A)\right\} = t^* = t^*(\sigma_0) < +\infty.$$

Hence, it follows

$$\frac{M(\sigma, F')}{M(\sigma, F)} \leq t^*\Phi'(\sigma) \quad (\sigma > \sigma_0). \tag{18}$$

Let $\gamma > 0$. Using the inequality $M'(\sigma, F) \leq M(\sigma, F')$, where $M'(\sigma, F)$ is the right-side derivative of $M(\sigma, F)$ (see Lemma 1), from (18) for $\sigma \geq \sigma_0$ we obtain

$$\ln \frac{M(\sigma+\gamma, F)}{M(\sigma, F)} = \int_\sigma^{\sigma+\gamma} \frac{M'(x, F)}{M(x, F)}\, dx \leq \int_\sigma^{\sigma+\gamma} \frac{M(x, F')}{M(x, F)}\, dx \leq$$
$$\leq t^* \int_\sigma^{\sigma+\gamma} \Phi'(x)\, dx = t^*\big\{\Phi(\sigma+\gamma) - \Phi(\sigma)\big\}. \tag{19}$$

Let $\operatorname{Re} s = \sigma$. From the Cauchy formula

$$F^{(n)}(s) = \frac{n!}{2\pi i} \int\limits_{|\xi - s| = \gamma} \frac{F(\xi)}{(\xi - s)^{n+1}}\, d\xi,$$

we obtain easily

$$M(\sigma, F^{(n)}) \le \frac{n!}{\gamma^n} \max\{|F(\xi)| : \ |\xi - s| = \gamma\} \le \frac{n!}{\gamma^n} M(\sigma + \gamma, F).$$

Therefore, using inequality (19), we have

$$\frac{M(\sigma, F^{(n)})}{n! M(\sigma, F)} \le \frac{1}{\gamma^n} \frac{M(\sigma + \gamma, F)}{M(\sigma, F)} = \frac{1}{\gamma^n} \exp\bigl\{t^* \bigl\{\Phi(\sigma + \gamma) - \Phi(\sigma)\bigr\}\bigr\}. \tag{20}$$

From (13) it follows

$$\varlimsup_{\sigma \to A} \frac{\Phi'\Bigl(\sigma + \frac{2}{\Phi'(\sigma)}\Bigr)}{\Phi'(\sigma)} \le \varlimsup_{\sigma \to A} \frac{\Phi'\Bigl(\sigma + \frac{2}{\alpha(\Phi'(\sigma))}\Bigr)}{\Phi'(\sigma)} = \Gamma_{\Phi'}(1) < +\infty.$$

Therefore,

$$\sup\biggl\{\frac{\Phi'\bigl(x + \frac{2}{\Phi'(x)}\bigr)}{\Phi'(x)} : \ x \ge 0\biggr\} = \tau < +\infty.$$

Hence,

$$\Phi\left(\sigma + \frac{2}{\Phi'(\sigma)}\right) - \Phi(\sigma) = \int_\sigma^{\sigma + 2/\Phi'(\sigma)} \Phi'(x)\, dx \le 2 \frac{\Phi'\bigl(\sigma + \frac{2}{\alpha(\Phi'(\sigma))}\bigr)}{\Phi'(\sigma)} \le 2\tau. \tag{21}$$

Putting $\gamma = 2/\Phi'(\sigma)$, from (20) and (21) we have

$$\frac{M(\sigma, F^{(n)})}{n! M(\sigma, F)} \le \bigl\{\Phi'(\sigma)\bigr\}^n \frac{\exp\{2t^*\tau\}}{2^n} = \bigl\{\Phi'(\sigma)\bigr\}^n \exp\{2t^*\tau - n \ln 2\}.$$

If now we put $n_0 = [2t^*\tau/\ln 2] + 1$ ($t^* = t^*(\sigma_0)$), then we obtain inequality (15) for all $n \ge n_0 = n_0(\sigma_0)$. Theorem 2 is proved.

5°. Using Theorems 1 and 2, we prove now the following theorem which is the main result in this paper.

Theorem 3. *Let $-\infty < A \le +\infty$, the functions α and $\Phi \in \Omega(A)$ are such as in Theorem 2 and Φ'' is continuous on $(-\infty, A)$. Let Dirichlet series (1) has the abscissa of absolute convergence A and its exponents satisfies the condition $\ln n(t) \le t/\alpha(t)$ for $t \ge t_0$. Then F has a finite Φ-type if and only if F is of bounded linear l-M-index with $l(\sigma) \equiv \max\{\Phi'(\sigma), 1\}$.*

Proof. We first prove the sufficiency. If Dirichlet series (1) is of bounded linear l-M-index then, in view of nondecreasing of l, (11) holds with $q = 0$. Since l is continuously differentiable on $[\sigma_0, A)$, from Theorem 1 we have relation (12). Since $L(\sigma) \sim \Phi(\sigma)$ as $\sigma \to A$, the function F has finite Φ-type.

We now prove the necessity. Since $\Phi(\sigma)$ satisfies all conditions of Theorem 2 and $l(\sigma) = \Phi'(\sigma)$ for $\sigma \ge \sigma_0$, there exists $n_0 \in \mathbb{Z}_+$ such that for all $n \ge n_0$ and $\sigma \ge \sigma_0$ (15) holds, i. e.

$$\frac{M(\sigma, F^{(n)})}{n! l^n(\sigma)} \le M(\sigma, F). \tag{22}$$

We can assume that $\sigma < A - 2$. Let $\eta = \text{const} > 0$ and

$$K(\eta, \sigma_0) = \max\left\{\frac{M(\sigma+\eta, F)}{M(\sigma, F)} : \ -\infty < \sigma \le \sigma_0\right\}.$$

If $a_0 \ne 0$ then $F(\sigma) = a_0 + o(1)$ as $\sigma \to -\infty$, $M(\sigma, F) = |a_0| + o(1)$ as $\sigma \to= -\infty$ and, thus, $K(\eta, \sigma_0) < +\infty$. If $a_0 = 0$, but, for example, $a_1 \ne 0$ then $F(s) = a_1 \exp(s\lambda_1)(1 + o(1))$, $\sigma \to -\infty$, that is $M(\sigma, F) = |a_1| \exp(\sigma\lambda_1)(1 + o(1))$ and $M(\sigma + \eta, F)/M(\sigma, F) = (1 + o(1)) \exp(\eta\lambda_1)$, $\sigma \to -\infty$. Therefore, in this case $K(\eta, \sigma_0) < +\infty$. Since $l(\sigma) \ge 1$ for all $\sigma \in (-\infty, \sigma_0]$, for such σ, as in the proof of Theorem 2, we have

$$\frac{M(\sigma, F^{(n)})}{n! l^n(\sigma) M(\sigma, F)} \le \frac{M(\sigma, F^{(n)})}{n! M(\sigma, F)} = \left(\frac{1}{2}\right)^n \frac{M(\sigma+2, F)}{M(\sigma, F)} \le \frac{1}{2^n} K(2, \sigma_0) \le 1, \quad n \ge n_1(\sigma_0),$$

i. e. we obtain again (22), but for $n \ge n_1(\sigma_0)$. Therefore (22) holds for all $\sigma \in (-\infty, A)$ and all $n \ge \max\{n_0, n_1\}$. Thus, l-M-index of F does not exceed $\max\{n_0, n_1\}$. The proof of Theorem 3 is complete.

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