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LOCALLY FINITE LIE ALGEBRAS WITH COMPLEMENTED ONE-DIMENSIONAL SUBALGEBRAS

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It is proved that all one-dimensional subalgebras of a locally finite-dimensional Lie algebra L with countable basis over a perfect field of characteristic $\neq 2$ are complemented iff L can be isomorphically embedded into a Cartesian sum of 3-dimensional simple Lie algebras of type A_1. It is also shown that all one-dimensional subalgebras of a locally finite-dimensional Lie algebra L over an arbitrary field of characteristic $p = 2$ are complemented in L iff L is solvable and can be isomorphically embedded into a Cartesian sum of 2-dimensional non-abelian Lie algebras.

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Доказано, что все одномерные подалгебры локально конечномерной алгебры Ли L со счетным базисом над совершенным полем характеристики $\neq 2$ дополняемы если и только если L может быть изоморфно вложено в некоторою декартову сумму трехмерных простых алгебр Ли типа A_1. Показано также, что все одномерные подалгебры локально конечномерной алгебры Ли L над произвольным полем характеристики $p = 2$ дополняемо в L если и только если L разрешима и может быть изоморфна вложена в декартову сумму двумерных неабелевых алгебр Ли.

Analogously to group theory, a subalgebra A of a Lie algebra L will be called *complemented* in L if there exists a subalgebra $B \subseteq L$ such that $L = A + B$, $A \cap B = 0$. Finite groups with complemented subgroups were studied by Ph. Hall [1] and N.V. Chernikova [2] and the complete description of such groups both finite and infinite was given in [2]. In the paper [3] finite-dimensional Lie algebras with complemented subalgebras were described (under some restrictions on the ground field), and it was shown that the following three conditions for a finite-dimensional Lie algebra L are equivalent: (1) all subalgebras of L are complemented in L; (2) all one-dimensional subalgebras from L are complemented in L; (3) the intersection $\Phi_1(L)$ of all subalgebras of codimension 1 in L is zero. Note that finite-dimensional Lie algebras L of characteristic 0 with $\Phi_1(L) = 0$ were studied by K. Hofmann [4] and D. Poguntke [5] in connection with classification of Lie semi-algebras. Therefore, it seems to be interesting to consider the question about

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structure of infinite-dimensional Lie algebras L with complemented one-dimensional subalgebras (the latter condition is also equivalent to the equality $\Phi_1(L) = 0$).

In this paper, we give a description of locally finite-dimensional Lie algebras with complemented one-dimensional subalgebras which have a countable basis over a perfect field of characteristic $\neq 2$ and over an arbitrary field of characteristic $p = 2$. It is also shown that in contrast to the finite-dimensional case, the conditions of being complemented for all subalgebras and one-dimensional subalgebras are not equivalent even for solvable infinite-dimensional Lie algebras (note that for periodic groups Yu. Gorchakov [6] proved the existence of groups with complemented subgroups of prime orders which contain non-complemented subgroups).

We mostly use standard notations, the semi-direct product of two Lie algebras A and B will be denoted by $A \rtimes B$. Recall that a finite-dimensional Lie algebra L over a field K is called split if L has a Cartan subalgebra H such that all eigenvalues of transformations $\operatorname{ad} h$, $h \in H$ belong to the field K (see [7, p. 121]). For any elements $g_1, \dots, g_n$ of the Lie algebra L, the subalgebra generated by these elements will be denoted by $\langle g_1, \dots, g_n \rangle$.

For an arbitrary set of indices I and a family of Lie algebras $\{L_i\}_{i \in I}$ we will denote by $\bigoplus_{i \in I} L_i$ the direct sum of Lie algebras L_i, $i \in I$ and by $\prod_{i \in I} L_i$ their Cartesian sum. Following [3], we will call a Lie algebra L a *Lie c-algebra* if all subalgebras from L are complemented in L. Further, a Lie algebra L with complemented one-dimensional subalgebras will be called a Lie c_1-algebra. For convenience, the zero Lie algebra will be also considered as a Lie c_1-algebra. By [3, Lemma 1], these two classes of Lie algebras coincide in the finite-dimensional case. It is easily seen that every quotient algebra and subalgebra of a Lie c-algebra (c_1-algebra) is a Lie c-algebra (resp. c_1-algebra).

The following Lemma is a counterpart of Lemma 1 from [3] and can be proved by analogy.

Lemma 1. *Let L be a Lie algebra over an arbitrary field. Then the following assertions are equivalent:*

(1) all one-dimensional subalgebras of L are complemented in L;

(2) all finite-dimensional subalgebras of the algebra L are complemented in L;

(3) the intersection $\Phi_1(L)$ of all subalgebras of codimension 1 of L is zero.

Lemma 2. *Let $\{L_i\}_{i \in I}$ be a family of Lie c_1-algebras. Then the Cartesian sum $L = \prod_{i \in I} L_i$ is also a Lie c_1-algebra.*

Proof. Obviously, we can assume that $L \neq 0$. Let $\langle g \rangle$ be an arbitrary one-dimensional subalgebra from L. Then $g = \{g_i\}_{i \in I}$, where $g_i \in L_i$ is the i-th coordinate of the element g. Since $g \neq 0$, we have $g_{i_0} \neq 0$ for some $i_0 \in I$. Let us take an arbitrary complementation D_{i_0} for the subalgebra $\langle g_{i_0} \rangle$ in L_{i_0} and consider the subalgebra D from L, consisting of all elements of the form $h = \{h_i\}_{i \in I}$, where $h_{i_0} \in D_{i_0}$. It is easy to see that $D \cap \langle g \rangle = 0$. Since $\dim L/D = 1$, we get $D + \langle g \rangle = L$ and D is a complementation to $\langle g \rangle$ in L. The proof is complete.

Lemma 3. *Every nonzero abelian ideal N of a Lie c-algebra L can be decomposed in a direct sum of one-dimensional ideals of the algebra L.*

Proof. Let us show that the ideal N contains a one-dimensional ideal of the algebra L. Let $\langle d \rangle$ be a one-dimensional subalgebra of N and D an arbitrary complementation to $\langle d \rangle$ in L. Obviously, $D \cap N$ is a subalgebra of codimension 1 in N. Denote by D_1 an arbitrary complementation to $D \cap N$ in L and by N_1 the intersection $N_1 = D_1 \cap N$. It is easily seen that N_1 is an ideal in L and $\dim N_1 = 1$. Repeating the considerations above we can construct an ideal M_1 of the algebra L such that $N = M_1 \oplus N_1$. The decomposition of N in the required direct sum of one-dimensional ideals can be implemented by transfinite induction. The proof is complete.

Lemma 4. *All one-dimensional subalgebras of a locally solvable Lie algebra L over an arbitrary field are complemented in L iff L can be isomorphically embedded into the Cartesian sum of 2-dimensional non-abelian Lie algebras.*

Proof. (a) First, let L be a solvable Lie algebra with complemented one-dimensional subalgebras. We shall show that $L^{''} = 0$. Indeed, otherwise there exists a solvable Lie c_1-algebra of derived length ≥ 3. Since every quotient algebra of a Lie c_1-algebra is also a c_1-algebra, there exist solvable Lie c_1-algebras of derived length 3. Denote by L such an arbitrary algebra and let $\langle g \rangle$ be an arbitrary one-dimensional subalgebra from the abelian ideal $L^{''}$ of L. The subalgebra $\langle g \rangle$ has a complementation D in the algebra L and therefore $N = D \cap L^{''}$ is an abelian ideal of the algebra L such that $\dim L^{''}/N = 1$. Obviously, the derived length of the quotient algebra $\bar{L} = L/N$ is equal to 3 and $\dim(\bar{L})^{''} = 1$. Therefore without lost of generality, one can at once assume that L is a Lie c_1-algebra, $L^{(3)} = 0$, and $\dim L^{''} = 1$. Since L' is not abelian and $L^{''}$ is complemented in L (and in L'), we see that L' is non-nilpotent. It follows from this that there exists an element $h \in L'$ such that $L^{''} \leftthreetimes \langle h \rangle = S$ is a non-abelian 2-dimensional ideal of L'. Then it is known that S is a direct summand of L', i. e. $L' = S \oplus C$, where $C = Z(L')$ is the center of L'. In the quotient algebra L/C the ideal $S + C/C$ is also the direct summand $L/C = (S + C)/C \oplus T/C$, where, obviously, T/C is an abelian ideal from L/C and therefore, $[L, L] \subseteq L^{''} + C$. But then $L^{''} + C$ is an abelian ideal of the algebra L with abelian quotient algebra $L/(L^{''} + C)$. This contradicts to the choice of the algebra L. Therefore $L^{''} = 0$.

Now let L be a locally solvable Lie c_1-algebra. Choose arbitrary elements x_1, x_2, x_3, x_4 from L and consider the subalgebra H from L, generated by all these elements. Since H is solvable, as has been shown above $H^{''} = 0$ and therefore $[[x_1, x_2], [x_3, x_4]] = 0$. The latter means that L is solvable of the length ≤ 2.

Let $\langle g \rangle$ be an arbitrary one-dimensional subalgebra of the abelian ideal L' of the algebra L. Then there exists a complementation C_g to $\langle g \rangle$ in L and $D_g = C_g \cap L'$ is an ideal of the algebra L with $\dim L'/D_g = 1$. It is easy to see that the quotient algebra L/D_g is either abelian or a direct sum of an abelian algebra and 2-dimensional non-abelian algebra. Since $\bigcap_{g \in L'} D_g = 0$, it is easy to make sure that the algebra L can be isomorphically embedded into Cartesian sum of 2-dimensional non-abelian Lie algebras.

(b) By Lemma 2 every Cartesian sum of non-abelian 2-dimensional Lie algebras is a Lie c_1-algebra. As was mentioned above every subalgebra of this sum is also a Lie c_1-algebra. The proof is complete.

Theorem 1. *All subalgebras of a locally finite solvable Lie algebra L over an arbitrary field K are complemented in L iff L can be decomposed in a semi-direct product $L = A \leftthreetimes B$, where A is an abelian ideal from L, A is either zero or decomposable in a direct sum of one-dimensional ideals of the algebra L, and B is an abelian subalgebra of the algebra L.*

Proof. (a) By Lemma 4, $L'' = 0$ and therefore L' is an abelian ideal of the algebra L. Denote by A the ideal L' and consider an arbitrary complementation B to A in L. Then B is an abelian subalgebra from L, and the ideal A can be decomposed in a direct sum of one-dimensional ideals of the algebra L, by Lemma 3 in the case $A \neq 0$.

(b) Let H be an arbitrary subalgebra of $L = A \rightthreetimes B$, where $A = \bigoplus_{i \in I} \langle a_i \rangle$ is the direct sum of one-dimensional ideals $\langle a_i \rangle$ (obviously, one can assume that $A \neq 0$). Set $A_1 = A \cap H$, $B_1 = B \cap (A + H)$. Further, denote by A_2 an arbitrary complementation to the subalgebra A_1 in A which can be represented as a sum of one-dimensional ideals from the family $\{\langle a_i \rangle\}_{i \in I}$ (it is easy to prove that there exists such a complementation). Take an arbitrary complementation B_2 to the subalgebra B_1 in B and show that $H \cap (A_2 + B_2) = 0$. Indeed, let otherwise $g = a_2 + b_2$, $a_2 \in A_2$, $b_2 \in B_2$ be a nonzero element from this intersection. Since $g \in H$, we have $b_2 \in B_1$, and therefore $b_2 \in B_1 \cap B_2$, i. e. $b_2 = 0$. It gives $g = a_2 \in A$ and thus $g \in H \cap A = A_1$. It follows from this that $g = a_2 \in A_1 \cap A_2$, i.e. $g = 0$. By the choice of the element g, the latter is impossible. The obtained contradiction shows that $H \cap (A_2 + B_2) = 0$. Now, let us show that $H + (A_2 + B_2) = L$. Indeed, since $A_1 \subseteq H$ and $A_2 \subseteq A_2 + B_2$, we have $A \subseteq H + (A_2 + B_2)$. The subalgebras A and A_2 are ideals of L and the following equalities hold true:

$$H + (A_2 + B_2) = (A + H) + B_2 = (A + H) + (A + B_2).$$

The equality $A + H = A + B_1$ implies $H + (A_2 + B_2) = A + B_1 + B_2 = A + B = L$. Thus $A_2 + B_2$ is a complementation to the subalgebra H in L. The proof is complete.

Lemma 5. *The class of Lie c-algebras is strictly contained in the class of Lie c_1-algebras, i. e. there exists a Lie algebra with complemented one-dimensional subalgebras which has an uncomplemented infinite-dimensional subalgebra.*

Proof. Let $L = \prod_{i=1}^{\infty} L_i$ be the Cartesian sum of two-dimensional nonabelian Lie algebras of the form $L_i = \langle a_i \rangle \rightthreetimes \langle b_i \rangle$, $[a_i, b_i] = a_i$, $i = 1, \ldots$. By Lemma 2, L is a Lie c_1-algebra. Show that L is not a Lie c-algebra. By Lemma 3, it suffices to prove that the abelian ideal $I = \prod_{i=1}^{\infty} \langle a_i \rangle$ of the algebra L cannot be decomposed in any direct sum of one-dimensional ideals of the algebra L. Indeed, assume the contrary and let I be such a direct sum. Denote by I_0 the direct sum $I_0 = \oplus_{i=1}^{\infty} \langle a_i \rangle$. Obviously, I_0 is an ideal of L and therefore the ideal I/I_0 contains one-dimensional ideals of the algebra L/I_0. Let $\langle g \rangle + I_0/I_0$ be such a one-dimensional ideal. Then $g = \{g_i\}_{i=1}^{\infty}$, $g_i \in \langle a_i \rangle \subseteq L_i$ and infinitely many of components $g_{i_1}, g_{i_2}, \ldots, g_{i_n}, \ldots$ are nonzero. Take an element $h = \{h_i\}_{i=1}^{\infty}$, with $h_i = \lambda_i b_i$, where only $\lambda_{i_2}, \lambda_{i_4}, \ldots, \lambda_{i_{2n}}, \ldots$ (arbitrary elements from the ground field) are nonzero. It is easy to see that $[g, h] \neq 0$ and therefore $[g, h] + I_0 = \langle \mu g \rangle + I_0$, where μ is a nonzero element from the ground field. The latter is impossible, because the element $[g, h]$ has nonzero components exactly in algebras $L_{i_2}, L_{i_4}, \ldots, L_{i_{2n}}, \ldots$, but every element $\mu g + i$, $\mu \neq 0$, $i \in I_0$ has nonzero components in every L_{i_k} for sufficiently large k. The proof is complete.

Theorem 2. *All one-dimensional subalgebras of a locally finite-dimensional Lie algebra L with countable basis over a perfect field K of characteristic $\neq 2$ are complemented iff the algebra L can be isomorphically embedded into Cartesian sum of 3-dimensional simple Lie algebras of type A_1.*

Proof. (a) Let $\{e_i\}_{i=1}^{\infty}$ be a basis of L and L_n a subalgebra from L generated by the elements $e_1, \ldots, e_n$. Then $L = \bigcup_{i=1}^{\infty} L_i$ is a union of increasing tower of finite-dimensional Lie algebras. It follows from the main theorem in [3] that every finite-dimensional subalgebra L_i is decomposable into the direct sum $L_i = L_i^{''} \oplus S_i$, where $L_i^{''}$ is the direct sum of 3-dimensional simple Lie algebras of type A_1 (or $L_i^{''} = 0$), and S_i is the solvable radical of the subalgebra L_i. Further, $L^{''}$ is an ideal of L and, as it is easily seen, $L^{''} = \bigcup_{i=1}^{\infty} L_i^{''}$. Since $L_i \subseteq L_{i+1}$ for $i = 1, 2, \ldots$, we have $L_i^{''} \subseteq L_{i+1}^{''}$ for $i = 1, 2, \ldots$. Show that $L_1^{''}$ is complemented in $L^{''}$ by some ideal I of L such that $I \subseteq L^{''}$. If $L_1^{''} = 0$, then it is obvious. Let $L_1^{''} \neq 0$. As $L_1^{''}$ is a direct sum of 3-dimensional simple Lie algebras of type A_1 (see [3]) and $L_2^{''}$ has the same structure, then there exists a complementation B_2 to $L_1^{''}$ in $L_2^{''}$, consisting of direct summands of the algebra $L_2^{''}$. By the choice, B_2 is an ideal of the algebra L_2. Analogously, one can construct a complementation B_3 to the subalgebra $L_2^{''}$ in $L_3^{''}$ such that B_3 is an ideal of the algebra L_3 etc. Denote by C_i the sum $C_i = B_2 + \cdots B_i$, $i = 2, 3, \ldots$ It is easily verified that C_i is a subalgebra of $L^{''}$ and $C = \bigcup_{i=2}^{\infty} C_i$ is a complementation to $L_1^{''}$ in $L^{''}$. Show that C is an ideal of the algebra L. Using induction on i, show that $[B_i, L] \subseteq C$ for $i = 1, 2, \ldots$ Indeed, let $i = 2$. Since B_2 is an ideal of the subalgebra L_2, we have $[B_2, S_1] \subseteq B_2$. It follows from the inclusion $L_2^{''} \subseteq L_k^{''}$ for $k \geq 2$ that $[B_2, S_k] = 0$ for $k \geq 2$. Thus, $[B_2, L_k] \subseteq C$ для $k = 2, 3, \ldots$ and therefore $[B_2, L] \subseteq C$. Suppose we have proved that $[B_i, L] \subseteq C$, show that the inclusion $[B_{i+1}, L] \subseteq C$ holds. To do this, it is sufficient to prove that $[B_{i+1}, S_k] \subseteq B_{i+1}$ for $k = 1, 2, \ldots$ Since B_{i+1} is an ideal in L_{i+1}, we have $[B_{i+1}, S_k] \subseteq B_{i+1}$ for $k \leq i+1$. Further, $B_{i+1} \subseteq L_k^{''}$ for $k \geq i+1$ and therefore $[B_{i+1}, S_k] = 0$ by $k \geq i+1$. As $L_i = L_i^{''} + S_i$, we have $[B_{i+1}, L_k] \subseteq C$ for all $k = 1, 2, \ldots$ Thus, $[B_{i+1}, L] \subseteq C$. It follows from this that C is an ideal of the algebra L. Now let H be an arbitrary 3-dimensional simple subalgebra from $L^{''}$. Setting

$$M_1 = H,\ M_2 = \langle L_2, H \rangle, \ldots, M_i = \langle L_i, H \rangle, \ldots$$

we obtain $L = \bigcup_{i=1}^{\infty} M_i$, and here $M_1^{''} = M_1$. As has been shown above, the subalgebra $H = M_1$ is complemented in $L^{''}$ by means of an ideal I_H of the algebra L, i. e. $L^{''} = H \oplus I_H$, $I_H \subseteq L^{''}$. Since $L^{''}$ is an union $L^{''} = \bigcup_{i=1}^{\infty} L_i^{''}$ of subalgebras $L_i^{''}$, which are decomposable into direct sum of 3-dimensional simple Lie algebras (of type A_1), and all finite-dimensional subalgebras from $L^{''}$ are complemented in L'' (by Lemma 1), it is easily shown that $\bigcap_{H \subseteq L''} I_H = 0$, where H goes through all 3-dimensional simple subalgebras from L''. But then the Lie algebra L can be isomorphic embedded into Cartesian sum $\prod_{H \subseteq L''} L/I_H$ of the algebras L/I_H which are Lie c_1-algebras and contain 3-dimensional simple split ideal $L^{''}/I_H$ with solvable quotient algebra (the latter is isomorphic to $L/L^{''}$). It is easy to see that L/I_H is a direct sum of 3-dimensional simple Lie algebra (of type A_1) and a solvable Lie c_1-algebra. Then the quotient algebra L/I_H can be isomorphic embedded into Cartesian sum of 3-dimensional simple Lie algebras of type A_1 for $H \subseteq L^{''}$. From this it follows that there exists an analogous embedding for the algebra L.

(b) It is sufficient to apply Lemma 2 and to note that every subalgebra of a Lie c_1-algebra is also a c_1-algebra.

Theorem 3. *All one-dimensional subalgebras of a locally finite-dimensional Lie algebra L over an arbitrary field of characteristic $p = 2$ are complemented iff L can be isomorphically embedded into a a Cartesian sum of two-dimensional non-abelian Lie algebras (in particular, L is solvable of length ≤ 2).*

Proof. (a) Since the Lie c_1-algebra L is locally finite, it is locally solvable by the main Theorem from [3] and therefore, by Lemma 4, it can be isomorphically embedded into Cartesian sum of two-dimensional non-abelian Lie algebras.

(b) It follows from Lemma 2.

REFERENCES

[1] Ph. Hall, *Complemented groups*, Journ. London Math. Soc. **12** (1937), 201–204.

[2] N.V. Chernikova, *Groups with complemented subgroups*, Mat. sbornik **3** (1956), 273–292. (in Russian)

[3] A.P. Petravchuk, *Lie algebras with complemented one-dimensional subalgebras*, Visnyk Kyiv univ. Mat. i mech (1999). (in Ukrainian, to appear)

[4] K.H. Hofmann, *Hyperplane subalgebras of real algebras*, Geometriae Dedicata **36** (1990), no. 2-3, 207–224.

[5] D. Poguntke, *The theorem of Lie and hyperplane subalgebras in Lie algebras*, Geometriae Dedicata **43** (1992), no. 1, 83–91.

[6] Yu.M. Gorchakov, *Primitively factorized groups*, Uchenye zapiski Permsk. univ. **17** (1960), 15–31. (in Russian)

[7] N.Jacobson, *Lie algebras*, Nauka, Moscow, 1964. (translated into Russian)

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