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INTEGRATION BY PARTS AND VECTOR DIFFERENTIAL FORMS IN HIGHER ORDER VARIATIONAL CALCULUS ON FIBRED MANIFOLDS

Roman Ya. Matsyuk

R.Ya. Matsyuk. *Integration by parts and vector differential forms in higher order variational calculus on fibred manifolds*, Matematychni Studii, **11** (1999) 85–107.

Infinitesimal variation of Action functional in classical (non-quantum) field theory with higher derivatives is presented in terms of well-defined intrinsic geometric objects independent of the particular field which varies. "Integration by parts" procedure for this variation consists in application of nonlinear Green formula to the vertical differential of the Lagrangian. Euler-Lagrange expressions and the Green operator are calculated by simple pull-backs of certain vector bundle valued differential forms associated with the given variational problem.

Р.Я. Мацюк. *Интегрирование по частям и векторные дифференциальные формы в вариационном исчислении с высшими производными на расслоенных многообразиях* // Математичні Студії. – 1999. – Т.11, №1. – 85–107.

Инфинитезимальная вариация функционала действия в классической (неквантовой) теории поля с высшими производными представлена в понятиях бескоординатного дифференциального исчисления с помощью внутренним образом хорошо определённых геометрических объектов, не зависящих от варьируемого поля. Процедура «интегрирования по частям» состоит в применении нелинейной формулы Грина к вертикальной части дифференциала функции Лагранжа. Выражения Эйлера-Лагранжа и оператор Грина получаются простым ограничением вдоль критического сечения определённых дифференциальных форм с коэффициентами в вертикальных касательных расслоениях.

Introduction

Generally posed variational problem demands covariance with respect to the pseudogroup of local transformations which mix the dependent and independent variables. Appropriate calculus have been developed by many authors, to mention DEDECKER, TULCZYJEW, VINOGRADOV, ZHARINOV as some. In physical field theory, however, only those transformations preserving the given fibred structure count. This suggests that geometric objects adapted to this structure may turn out to be useful. It is our opinion that the first to mention are semi-basic differential forms with values in vertical fiber bundles. The same approach gained support in [3] and [14].

The Fréchet derivative of the Action functional at v, where v belongs to the set of cross-sections of some fibred manifold Y, is an r^{th}-order differential operator in the space of variations. Integration-by-parts formula (in other terms,— the first variation formula) for this operator involves two other objects, namely the transpose operator and the Green operator. Their definition depends on v implicitly. Two questions arise therefore: 1)

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By means of what configuration does v enter into those operators and how to make this dependence upon v explicit? 2) Do there exist such intrinsic geometric objects that the above mentioned operators for each individual v might be calculated by means of a simple restriction procedure? We rewrite the decomposition formula of KOLÁŘ [6] in terms of vector bundle valued differential forms to answer these questions.

Our goal is threefold. First, to present a rigorous (from the point of view of Global Analysis) computation of the Fréchet derivative of the Action functional; second, to compute in conceptually the same spirit the transpose and Green operators; third, to make evident that the notion of a vector bundle valued differential form is best suited to the peculiarities of variational calculus in the framework of classical field theory.

The paper is organized in the following way. In Section 1, which has a preliminary character, we fix notations related to the definitions of some actions of base-substituting morphisms upon vector bundle valued differential forms. These definitions apply naturally to the calculus of variations because the prolongations of different orders and various pull-backs to base manifolds happen in that calculus. Intending to operate with the pull-backs of differential forms which take values in some vector bundles, one needs to make intensive use of the philosophy of induced bundles (reciprocal images). We explain some basic properties of the reciprocal image functor and the interplay between it and the notion of vector bundle valued differential forms. Our development is based on [10]. The accompanying notations allow us to give the adequate appearance to the integration-by-parts formula later and also they facilitate the formalization of most proves.

In field theory the notion of the Lagrangian quite naturally falls into the ramification of the concept of semi-basic differential form with respect to the independent variables. We turn to the discussion of the Lie derivative of such a form in Section 2. This supplies us with the adequate tool for the description of the infinitesimal variation of the Action functional.

In Section 3 the Action functional is introduced and the meaning of its variation is established. Further in this sections we give a purely geometric and strictly intrinsic computation of the Action variation in terms of the fibre differential and the Lie derivative operators. In Section 4 the first variational formula from the point of view of the concept of vector bundle valued differential forms is discussed. We also deduce this formula from the Fréchet derivative expression by means of a suitable reformulation of Kolář decomposition formula.

1. Preliminaries on vector bundle valued semi-basic differential forms

First we develop some general features of the behavior of vector bundle valued differential forms under inverse image functor.

1. Action of base-substituting morphisms upon vector bundles and their cross-sections. Consider a vector bundle $\chi\colon W \to X$ and a morphism of manifolds $\mathfrak{g}\colon B \to X$. The reciprocal image $\mathfrak{g}^{-1}\chi$ of the fibre bundle χ is the set $B \times_X W$ of pairs $(b, \boldsymbol{w})$ with $\mathfrak{g}(b) = \chi(\boldsymbol{w})$. We denote $\chi^{-1}(\mathfrak{g})$ the projection from $B \times_X W$ onto W. Given another vector bundle $\chi'\colon W' \to X'$ and a vector bundle homomorphism $\mathsf{k}\colon W \to W'$ over the morphism of bases $\mathfrak{k}\colon X \to X'$ the homomorphism k may be reduced to the homomorphism $\mathsf{k}_X\colon W \to \mathfrak{k}^{-1}W'$ over the base X and we denote by $\mathfrak{g}^{-1}\mathsf{k} \doteqdot \mathfrak{g}^{-1}\mathsf{k}_X$ the reciprocal image of k_X with respect to $\mathfrak{g}$; it maps $\mathfrak{g}^{-1}W$ into $\mathfrak{g}'^{-1}W'$ where $\mathfrak{g}' = \mathfrak{k} \circ \mathfrak{g}$ and it commutes with k_X via the pair of projections $\chi^{-1}\mathfrak{g}$ and $\chi'^{-1}\mathfrak{g}'$ (see fig.12).

Denote by $\tilde{}$ the one-to-one correspondence between the cross-sections $\mathfrak{s} \in \Gamma\{\mathscr{g}^{-1}W\}$ and vector fields $\mathfrak{g}: B \to W$ along $\mathscr{g}$, so that $\tilde{\mathfrak{g}} \in \Gamma\{\mathscr{g}^{-1}W\}$, and, reciprocally, $\underset{\sim}{\mathfrak{s}}: B \to W$ becomes a morphism along $\mathscr{g}$. Let, in addition, $\mathfrak{w} \in \Gamma\{W\}$ and let $\mathscr{g}^{-1}\mathfrak{w}$ be the reciprocal image of the cross-section $\mathfrak{w}$ with respect to $\mathscr{g}$. The following relations hold—due to relevant definitions—$\underset{\sim}{\mathfrak{s}} = (\chi^{-1}\mathscr{g}) \circ \mathfrak{s}$; $(\chi^{-1}\mathscr{g}) \circ \tilde{\mathfrak{g}} = \mathfrak{g}$; $\mathscr{g}^{-1}\mathfrak{w} = (\mathfrak{w} \circ \mathscr{g})\tilde{}$.

Homomorphism k acts upon $\Gamma\{\mathscr{g}^{-1}W\}$ by $\mathsf{k}_{\#}: \Gamma\{\mathscr{g}^{-1}W\} \to \Gamma\{\mathscr{g}'^{-1}W'\}$, $\mathsf{k}_{\#}\mathfrak{s} = (\mathscr{g}^{-1}\mathsf{k}) \circ \mathfrak{s}$ (this of course applies to $\mathscr{g} = id$ too). This action commutes with the suspension operation $\tilde{}$ in the sense that $\mathsf{k}_{\#}\tilde{\mathfrak{g}} = (\mathsf{k} \circ \mathfrak{g})\tilde{}$ and also $\mathsf{k} \circ \underset{\sim}{\mathfrak{s}} = (\mathsf{k}_{\#}\mathfrak{s})_{\sim}$. One could also write $\mathsf{k}_{\#} = (\mathscr{g}^{-1}\mathsf{k}_X)_{\#}$ since $\mathscr{g}^{-1}\mathsf{k}_X$ maps over the identity in B.[1] Given one more vector bundle $E \to B$ and a vector bundle homomorphism $\mathsf{g}: E \to W$ over $\mathscr{g}$, homomorphism $\mathsf{g}' = \mathsf{k} \circ \mathsf{g}$ acts from $\Gamma\{E\}$ to $\Gamma\{\mathscr{g}'^{-1}W'\}$.

To prove the legitimacy of the usual covariance property, $(\mathsf{k} \circ \mathsf{g})_{\#} = \mathsf{k}_{\#}\mathsf{g}_{\#}$, one applies to both sides of it the projection $\chi'^{-1}\mathscr{g}'$ (which is one-to-one on the fibers). Homomorphism $\mathsf{k} \circ \mathsf{g}$ acts through the composition with $(\mathsf{k} \circ \mathsf{g})_B$ and we have $\chi'^{-1}\mathscr{g}' \circ (\mathsf{k} \circ \mathsf{g})_B = \mathsf{k} \circ \mathsf{g}$ (see again fig.12). On the other hand, by $\mathscr{g}'^{-1}W' \approx \mathscr{g}^{-1}\mathscr{k}^{-1}W'$ we have $\chi'^{-1}\mathscr{g}' \circ \mathscr{g}^{-1}\mathsf{k}_X \circ \mathsf{g}_B \approx \chi'^{-1}\mathscr{k} \circ (\mathscr{k}^{-1}\chi')^{-1}\mathscr{g} \circ \mathscr{g}^{-1}\mathsf{k}_X \circ \mathsf{g}_B = \chi'^{-1}\mathscr{k} \circ \mathsf{k}_X \circ \chi^{-1}\mathscr{g} \circ \mathsf{g}_B = \mathsf{k} \circ \mathsf{g}$, q.e.d.

2. Action of base-substituting morphisms upon vector bundle valued differential forms. A differential form on B with values in a vector bundle E is a cross-section of the vector bundle $E \otimes \wedge T^*B$. Of course, one can take $\mathscr{g}^{-1}W$ in place of E and speak of differential forms which take values in W. We shall use the notation $\Omega^d(B; E)$ for $\Gamma\{E \otimes \wedge^d T^*B\}$ and also write sometimes $\Omega(B; W)$ instead of $\Omega(B; \mathscr{g}^{-1}W)$. If we take $\wedge E^*$ in place of E, the module $\Omega(B; \wedge E^*)$ acquires the structure of a bigraded algebra. If (locally) $\omega_l = \varphi_l \otimes \boldsymbol{\alpha}_l$, $l = 1, 2$, then $\omega_1 \wedge \omega_2 = \varphi_1 \wedge \varphi_2 \otimes \boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2$. The interior product $\mathbf{i}: \Gamma\{E\} \times \Gamma\{\wedge E^*\} \to \Gamma\{\wedge E^*\}$ defines a coupling $\wedge_{\mathbf{i}}$ from $\Omega(B; E) \times \Omega(B; \wedge^d E^*)$ into $\Omega(B; \wedge^{d-1}E^*)$; if $\boldsymbol{\rho} = \mathfrak{e} \otimes \boldsymbol{\alpha}$, $\mathfrak{e} \in \Gamma\{E\}$ and $\boldsymbol{\alpha} \in \Gamma\{\wedge T^*B\}$, then $\boldsymbol{\rho} \wedge_{\mathbf{i}} \omega_1 = \mathbf{i}(\mathfrak{e})\varphi_1 \otimes \boldsymbol{\alpha} \wedge \boldsymbol{\alpha}_1$. We denote $\langle \boldsymbol{\rho}, \omega \rangle \doteq \boldsymbol{\rho} \wedge_{\mathbf{i}} \omega$ when $\omega \in \Omega(B; E^*)$.

Dual $\mathscr{k}$-comorphism $(\mathsf{k}_X)^*$ acts upon $\Omega(B; W'^*)$ by composition with vector bundle homomorphism $\mathscr{g}^{-1}\mathsf{k}_X{}^* \otimes \mathrm{id}$. We write $\mathsf{k}^{\#}\boldsymbol{\theta}'$ for the cross-section $(\mathscr{g}^{-1}\mathsf{k}_X{}^* \otimes \mathrm{id}) \circ \boldsymbol{\theta}'$ and it is of course true that

$$\langle \mathsf{k}_{\#}\mathfrak{s}, \boldsymbol{\theta}' \rangle = \langle \mathfrak{s}, \mathsf{k}^{\#}\boldsymbol{\theta}' \rangle \tag{1}$$

for $\mathfrak{s} \in \Omega^0(B; W)$ and $\boldsymbol{\theta}' \in \Omega(B; W'^*)$.

Inspired by the considerations, heretofore delivered, we introduce some definitions.

Definition 1. Let $\mathscr{k}$ be a morphism of manifolds from X into X', and let W and W' be fibred manifolds over X and X' respectively.

If there exists a natural lift of $\mathscr{k}$ to a fibred morphism $\mathsf{F}_{\mathscr{k}}: W \to W'$, then $\mathscr{k}$ acts upon $\Gamma\{W\}$ as follows, $\mathscr{k}_{\#}: \Gamma\{W\} \to \Gamma\{\mathscr{k}^{-1}W'\}$, $\mathscr{k}_{\#}\mathfrak{w} = (\mathsf{F}_{\mathscr{k}} \circ \mathfrak{w})\tilde{}$.

Let W and W' be vector bundles and assume a couple of morphisms $\mathscr{g}: B \to X$ and $\mathscr{g}': B \to X'$ be given such that $\mathscr{k} \circ \mathscr{g} = \mathscr{g}'$. Then the following modules and operators between them are defined:

$$\mathscr{k}_{\#}: \Omega(B; W) \to \Omega(B; W'), \qquad \mathscr{k}_{\#}\boldsymbol{\eta} = (\mathscr{g}^{-1}\mathsf{F}_{\mathscr{k}} \otimes \mathrm{id})_{\#}\boldsymbol{\eta},$$

$$\mathscr{k}^{\#}: \Omega(B; W'^*) \to \Omega(B; W^*), \qquad \mathscr{k}^{\#}\boldsymbol{\theta}' = (\mathscr{g}^{-1}(\mathsf{F}_{\mathscr{k}})_X{}^* \otimes \mathrm{id})_{\#}\boldsymbol{\theta}'.$$

[1] This definition and the notation used generalize those of STERNBERG [19]

If there does not exist but merely a $\mathscr{k}$-comorphism $\mathsf{F}^*_{\mathscr{k}}$ from $\mathscr{k}^{-1}W'$ into W over the identity in X then the action $\mathscr{k}^\#$ still can be defined as

$$\mathscr{k}^\#\colon \Omega(B;W') \to \Omega(B;W), \qquad \mathscr{k}^\#\boldsymbol{\eta}' = (\mathscr{g}^{-1}\mathsf{F}^*_{\mathscr{k}} \otimes \mathsf{id})_\#\boldsymbol{\eta}' \tag{2}$$

Of course, one may set $\mathscr{g} = \mathit{id}$ everywhere in the above.

Assume be given a morphism δ from a manifold Z into B. We recall that the definition of the reciprocal image $\delta^{-1}\boldsymbol{\rho}$ applies to a cross-section $\boldsymbol{\rho} \in \Gamma\{E \otimes \wedge T^*B\}$ by means of $\delta^{-1}\colon \Omega(B;E) \to \Gamma\{\delta^{-1}E \otimes \delta^{-1} \wedge T^*B\}$, $\delta^{-1}\boldsymbol{\rho} = (\boldsymbol{\rho}\circ\delta)\tilde{}$.

Let us accept the following brief notations for the mappings, induced over Z by the tangent functor T, $\delta^T \doteqdot (T\delta)_Z$, $\delta^* \doteqdot (\delta^T)^*$.

Definition 2. Let F' be a vector bundle over Z. The action $\delta^{(\cdot,\#)}$ of the morphism δ upon the module $\Gamma\{F' \otimes \delta^{-1} \wedge T^*B\}$ is defined by $\delta^{(\cdot,\#)}\colon \Gamma\{F' \otimes \delta^{-1} \wedge T^*B\} \to \Omega(Z;F')$, $\delta^{(\cdot,\#)} = (\mathsf{id} \otimes \wedge\delta^*)_\#$.

The pull-back of a differential form $\boldsymbol{\rho} \in \Omega(B;E)$ is hereupon defined by (see also [1]) $\delta^*\colon \Omega(B;E) \to \Omega(Z;E)$, $\delta^*\boldsymbol{\rho} = \delta^{(\cdot,\#)}\delta^{-1}\boldsymbol{\rho}$.

Definition 3. If there exists a natural δ-comorphism F^*_δ from $\delta^{-1}E$ to a vector bundle F over Z then $\delta^\#$ will mean the total of the "twofold" backward action

$$\delta^\#\colon \Gamma\{\delta^{-1}E \otimes \delta^{-1} \wedge T^*B\} \to \Omega(Z;F), \qquad \delta^\# = (\mathsf{F}^*_\delta \otimes \wedge\delta^*)_\#.$$

The definitions introduced heretofore correlate. For instance, a posteriori given some δ-comorphism F^*_δ, we can apply the operation $\delta^\#$ in the spirit of the Definition 1 to the differential form $\delta^*\boldsymbol{\rho}$ by renaming in (2) B as Z, X' as B, $\mathscr{k}$ as δ, and putting $\mathscr{g} = \mathit{id}$, $W = F$, $W' = E$ and $\mathsf{F}^*_{\mathscr{k}} = \mathsf{F}^*_\delta$, which will then produce $\delta^\#\delta^*\boldsymbol{\rho} = (\mathsf{F}^*_\delta \otimes \mathsf{id})_\#\delta^*\boldsymbol{\rho} = (\mathsf{F}^*_\delta \otimes \mathsf{id})_\#(\mathsf{id} \otimes \wedge\delta^*)_\#\delta^{-1}\boldsymbol{\rho} = \delta^\#\delta^{-1}\boldsymbol{\rho}$, wherein the operation $\delta^\#$ to the extreme right is defined in the spirit of the Definition 3 this time.

In what comes later, we shall *not* bother to indicate explicitly the "partial" character of the $\delta^{(\cdot,\#)}$ operation at the occasions like that of the Definition 2 any more, and shall exploit the same brief notation $\delta^\#$ in place of the more informative one, $\delta^{(\cdot,\#)}$, because hardly any confusion will ever arise.

Definition 4. Given a differential form $\boldsymbol{\theta} \in \Omega(B;W^*)$ and a cross-section $\boldsymbol{\mathfrak{w}} \in \Gamma\{W\}$, we define the contraction $\langle \boldsymbol{\mathfrak{w}}, \boldsymbol{\theta}\rangle$ by $\langle \boldsymbol{\mathfrak{w}}, \boldsymbol{\theta}\rangle = \langle \mathscr{g}^{-1}\boldsymbol{\mathfrak{w}}, \boldsymbol{\theta}\rangle$.

One can easily verify that the following formula holds for $\boldsymbol{\omega} \in \Omega(B;E^*)$ and $\boldsymbol{\mathfrak{e}} \in \Omega^0(B;E)$

$$\langle \delta^{-1}\boldsymbol{\mathfrak{e}}, \delta^{-1}\boldsymbol{\omega}\rangle = \delta^{-1}\langle \boldsymbol{\mathfrak{e}}, \boldsymbol{\omega}\rangle. \tag{3}$$

3. Semi-basic differential forms. Let $\pi\colon B \to Z$ be a surmersion of manifolds. The reciprocal image $\pi^{-1}TZ$ of the tangent bundle TZ is incorporated in the commutative diagram of vector bundle homomorphisms of fig.1.

The existence of the short exact sequence of vector bundle homomorphisms

$$0 \to T(B/Z) \xrightarrow{\iota} TB \xrightarrow{\pi^T} \pi^{-1}(TZ) \to 0 \tag{4}$$

$$\begin{tikzcd}
TB \arrow[dr, "\pi_T"] \arrow[drr, "T\pi"] \arrow[dd, "\tau_B"'] & & \\
& \pi^{-1}(TZ) \arrow[r, "\tau^{-1}\pi"'] \arrow[dl, "\pi^{-1}\tau"'] & TZ \arrow[d, "\tau"] \\
B \arrow[rr, "\pi"] & & Z
\end{tikzcd}$$

FIGURE 1

allows us to define the module $\mathfrak{V}_B = \Gamma\{T(B/Z)\}$ of vertical (with respect to π) vector fields on the manifold B. Let $\mathfrak{F}_B$ denote the ring of C^∞ functions over the manifold B. The $\mathfrak{F}_B$-module of all-direction vector fields over the manifold B will be denoted by $\mathfrak{X}_B$.

Utilizing the partition of unity over the manifold B, one can split the sequence (4),

$$0 \leftarrow T(B/Z) \xleftarrow{\overleftarrow{\iota}} TB \xleftarrow{\overleftarrow{\pi}^T} \pi^{-1}TZ \leftarrow 0.$$

The restriction of the vector bundle homomorphism $\overleftarrow{\iota}$ to the subbundle $\operatorname{Im}\iota$ is the inverse to the mapping ι. In fact, if $\mathbf{t} = \iota(\mathbf{v}) \in \operatorname{Im}\iota$, then $\iota \circ \overleftarrow{\iota}(\mathbf{t}) = \iota \circ \overleftarrow{\iota} \circ \iota(\mathbf{v}) = \iota(\mathbf{v}) = \mathbf{t}$ and so $\iota \circ \overleftarrow{\iota} = \mathrm{id}$, q.e.d.

The reciprocal image $\pi^{-1}(T^*Z) \approx (\pi^{-1}(TZ))^*$ of the cotangent bundle T^*Z is incorporated in the commutative diagram of fig.2

$$\begin{tikzcd}
\wedge T^*B \arrow[dr, "\overset{*}{\tau}_B"'] & \pi^{-1}(\wedge T^*Z) \arrow[l, "\wedge\pi^*"'] \arrow[r, "\overset{*}{\tau}^{-1}\pi"] \arrow[d, "\pi^{-1}\overset{*}{\tau}"'] & \wedge T^*Z \arrow[d, "\overset{*}{\tau}"] \\
& B \arrow[r, "\pi"] & Z
\end{tikzcd}$$

FIGURE 2

Comorphism $\wedge\pi^*$ is dual to the morphism $T\pi$ in the sense that $\wedge\pi^* = \wedge(\pi^T)^*$. We denote the effect of $\wedge\pi^*$ on the sections of the induced bundle $\pi^{-1}\wedge T^*Z$ by $\pi^\#$, so $\pi^\#\beta = (\wedge^d\pi^*)\circ\beta$ if $\beta \in \Gamma\{\pi^{-1}\wedge^d T^*Z\}$.

Let us show that the sequence of the homomorphisms of the modules of cross-sections, which corresponds to the exact sequence (4),

$$0 \to \mathfrak{V}_B \xrightarrow{\iota_\#} \mathfrak{X}_B \xrightarrow{\pi_\#} \Gamma\{\pi^{-1}TZ\} \to 0, \tag{5}$$

is exact as well.

Let $\mathfrak{x} \in \operatorname{Ker}\pi_\#$. The exactness of sequence (4) implies that $\mathfrak{x} \in \operatorname{Im}\iota$. Then the cross-section $\overleftarrow{\iota} \circ \mathfrak{x}$ is being mapped into $\mathfrak{x}$ under the homomorphism $\iota_\#$, because we have $\iota_\#(\overleftarrow{\iota} \circ \mathfrak{x}) \equiv \iota \circ \overleftarrow{\iota} \circ \mathfrak{x} = \mathfrak{x}$. Thus $\mathfrak{x} \in \operatorname{Im}\iota_\#$ and so $\operatorname{Im}\iota_\# \supset \operatorname{Ker}\pi_\#$. Examining the surjectivity of $\pi_\#$ one sees easily that for every $\mathfrak{h} \in \Gamma\{\pi^{-1}TZ\}$ the cross-section $\overleftarrow{\pi}^T \circ \mathfrak{h}$ is mapped by the homomorphism $\pi_\#$ into the cross-section $\mathfrak{h}$ because of $\pi_\#(\overleftarrow{\pi}^T \circ \mathfrak{h}) \equiv \pi \circ \overleftarrow{\pi}^T \circ \mathfrak{h} = \mathfrak{h}$. The injectivity of the homomorphism $\iota_\#$ and the inclusion $\operatorname{Im}\iota_\# \subset \operatorname{Ker}\pi_\#$ are still more obvious, q.e.d.

Due to the exactness of sequence (5) one can identify the module $\Gamma\{\pi^{-1}TZ\}$ with the quotient module $\mathfrak{X}_B/\mathfrak{V}_B$. We introduce the notation $\Omega(B)$ for the graded algebra of differential forms, $\Omega(B) = \sum_{d=0}^{\dim B} \Omega^d(B)$, so that $\mathfrak{F}_B = \Omega^0(B)$. Let also $\mathbf{A}^d$ mean the functor of skew-symmetric multilinear forms of degree d on some module. For a vector bundle E we shall exploit the moduli isomorphism $\mathbf{A}^d(\Gamma\{E\}) \approx \Gamma\{\wedge^d E^*\}$. As we have just seen, the $\mathfrak{F}_B$-algebras $\mathbf{A}(\mathfrak{X}_B/\mathfrak{V}_B)$ and $\Gamma\{\pi^{-1} \wedge T^*Z\}$ can be identified with each other. Cross-sections of the bundle $\pi^{-1} \wedge T^*Z$ are known as semi-basic (with respect to π) differential forms on the manifold B. The graded algebra of these forms will be denoted as $\Omega_B(Z)$. Remind that the exactness of the sequence (4) means that the vector bundles $\pi^{-1}TZ$ and $TB/T(B/Z)$ are isomorphic. Passing to the dual bundles we obtain the moduli isomorphism $\Gamma\{\pi^{-1} \wedge T^*Z\} \approx \Gamma\{\wedge(TB/T(B/Z))^*\}$ and finish up with the double isomorphism of $\mathfrak{F}_B$-algebras $\Omega_B^d(Z) \approx \mathbf{A}^d(\mathfrak{X}_B/\mathfrak{V}_B) \approx \Gamma\{\wedge^d(TB/T(B/Z))^*\}$.

The elements of the middle-term algebra will hereinafter be called the horizontal (with respect to π) differential forms and they will be identified by means of the second isomorphism with the forms $\boldsymbol{\alpha} \in \Omega^d(B)$ such that $\boldsymbol{\alpha}(\mathbf{t}_1, \ldots, \mathbf{t}_d) = 0$ every time when at least one of the tangent vectors $\mathbf{t}_1, \ldots, \mathbf{t}_d$ is vertical.

Let $\boldsymbol{\alpha} \in \mathbf{A}^d(\mathfrak{X}_B/\mathfrak{V}_B)$. The differential form $\boldsymbol{\beta} \in \Omega_B^d(Z)$ such that $\boldsymbol{\beta}(\mathfrak{h}_1, \ldots, \mathfrak{h}_d) = \boldsymbol{\alpha}(\mathfrak{x}_1, \ldots, \mathfrak{x}_d)$ if $\mathfrak{h}_i = \pi_\# \mathfrak{x}_i$, is the image of $\boldsymbol{\alpha}$ under the isomorphism $\mathbf{A}^d(\mathfrak{X}_B/\mathfrak{V}_B) \approx \Omega_B^d(Z)$. Hereinafter we shall use one and the same notation $\Omega_B(Z)$ for both algebras and we shall write $\Omega_r(Z)$ when the manifold Y_r will be considered in place of B. We also introduce a separate notation $\mathfrak{H}_B \equiv \mathfrak{H}_B(Z)$ for the module of cross-sections of the bundle $\pi^{-1}TZ$; thus $\Omega_B^1(Z) = \mathfrak{H}_B{}^*$. The elements $\mathfrak{h}$ of the module $\mathfrak{H}_B$ by means of the vector bundle homomorphism $\tau^{-1}\pi$ are identified with the corresponding lifts $\mathfrak{h}$ of the morphism π, conventionally known as vector fields along π.

2. Infinitesimal variations and the Lie derivative

1. The Fréchet derivative of the base substitution. Let $\delta: Z \to B$ be a morphism of manifolds and let $\boldsymbol{\alpha} \in \Omega(B)$ be a differential form on the manifold B. Assume Z compact. The map ${}^\star\boldsymbol{\alpha}: C^\infty(Z, B) \to \Omega(Z)$ takes any morphism δ over to the differential form $\delta^\star\boldsymbol{\alpha} \in \Omega(Z)$. The space $C^\infty(Z, B)$ of C^∞ mappings from Z into B has as its tangent vector at the point $\delta \in C^\infty(Z, B)$ some lift $\mathfrak{b}: Z \to TB$ of the morphism δ. Consider a one-parametric family δ_t of deformations of the morphism δ, $\delta_t \in C^\infty(Z, B)$, $\delta_0 = \delta$. The mapping $\gamma_\delta: t \to \delta_t$ defines a smooth curve in the manifold $C^\infty(Z, B)$. Let the lift $\mathfrak{b}$ be the tangent vector to this curve at the point δ. Accordingly to the definition of the tangent map, the Fréchet derivative of the application ${}^\star\boldsymbol{\alpha}$ at the point δ is being evaluated on the tangent vector $\mathfrak{b}$ in the following way: $(\mathbf{D}\,{}^\star\boldsymbol{\alpha})(\delta)\,.\,\mathfrak{b} = (d/dt)({}^\star\boldsymbol{\alpha} \circ \gamma_\delta)(0)$.

2. The Fréchet derivative and the Lie derivative. The operation of the Lie derivative in the direction of the lift $\mathfrak{b} = \tau^{-1}\delta \circ \tilde{\mathfrak{b}}$ will be introduced via the formula (compare with [1]) $\mathbf{L}(\mathfrak{b}) = \mathbf{d}\delta^\#\mathbf{i}(\tilde{\mathfrak{b}})\delta^{-1} + \delta^\#\mathbf{i}(\tilde{\mathfrak{b}})\delta^{-1}\mathbf{d}$ which obviously generalizes the conventional one. The derivation $\mathbf{i}(\tilde{\mathfrak{b}})$ is defined in terms of the interior product of the cross-section $\tilde{\mathfrak{b}} \in \Gamma\{\delta^{-1}TB\}$. The differential form $\delta^{-1}\boldsymbol{\alpha}$ is in $\Gamma\{\delta^{-1} \wedge T^*B\}$ whenever $\boldsymbol{\alpha} \in \Omega(B)$.

The proof of the formula $\mathbf{L}(\mathfrak{b})\boldsymbol{\alpha} = (d/dt)\,{}^\star\boldsymbol{\alpha} \circ \gamma_\delta\,(0)$ closely follows the lines of the proof of the corresponding analogue for the conventional Lie derivative. It suffices to verify the effect of it upon functions and Pfaff forms alone since $\mathbf{L}(\mathfrak{b})$ acts as a derivation.

Indeed, first we convince ourselves that the construction $\delta^\#\mathbf{i}(\tilde{\mathfrak{b}})\delta^{-1}$ acts as a derivation of degree -1, $\delta^\#\mathbf{i}(\tilde{\mathfrak{b}})\delta^{-1}(\boldsymbol{\alpha} \wedge \boldsymbol{\alpha}') = \delta^\#\mathbf{i}(\tilde{\mathfrak{b}})(\delta^{-1}\boldsymbol{\alpha} \wedge \delta^{-1}\boldsymbol{\alpha}') = \delta^\#\mathbf{i}(\tilde{\mathfrak{b}})\delta^{-1}\boldsymbol{\alpha} \wedge \delta^\star\boldsymbol{\alpha}' +$

$(-1)^{\deg(\alpha)}\delta^\star\boldsymbol{\alpha}\wedge\delta^{\#}\mathbf{i}(\tilde{\mathfrak{b}})\delta^{-1}\boldsymbol{\alpha}'$. Then we remind that the exterior differential $\mathbf{d}$ acts as a derivation of degree $+1$ and finally we notice that the Lie derivative $\mathbf{L}(\mathfrak{b})$ appears to be their commutator and by this fact is forced to act as a derivation of degree 0 from the algebra $\Omega(B)$ into the algebra $\Omega(Z)$ along the homomorphism $\delta^\star$, $\mathbf{L}(\mathfrak{b})(\boldsymbol{\alpha}\wedge\boldsymbol{\alpha}') = \mathbf{L}(\mathfrak{b})\boldsymbol{\alpha}\wedge\delta^\star\boldsymbol{\alpha}'+\delta^\star\boldsymbol{\alpha}\wedge\mathbf{L}(\mathfrak{b})\boldsymbol{\alpha}'$, q.e.d.

The operator $(d/dt)\delta_t{}^\star$ also acts as a derivation, $(d/dt)\delta_t{}^\star(\boldsymbol{\alpha}\wedge\boldsymbol{\alpha}') = (d/dt)\delta_t{}^\star\boldsymbol{\alpha}\wedge\delta_t{}^\star\boldsymbol{\alpha}'+\delta_t{}^\star\boldsymbol{\alpha}\wedge(d/dt)\delta_t{}^\star\boldsymbol{\alpha}'$. So one concludes that these operators coincide and comes up to the following computational formula

$$(\mathbf{D}^\star\boldsymbol{\alpha})(\delta)\,.\,\mathfrak{b} = \mathbf{L}(\mathfrak{b})\,\boldsymbol{\alpha}. \tag{6}$$

3. Fibre differential. A semi-basic differential form $\beta\in\Omega_B(Z)$ may be considered as a fibred manifolds morphism $\beta\colon B\to\wedge T^*Z$ over Z. In a more general way, let $\zeta\colon F\to Z$ be a vector bundle and let $\underset{\sim}{\beta}\colon B\to F$ be a fibred morphism over the base Z, $\zeta\circ\underset{\sim}{\beta}=\pi$ (see fig.3)

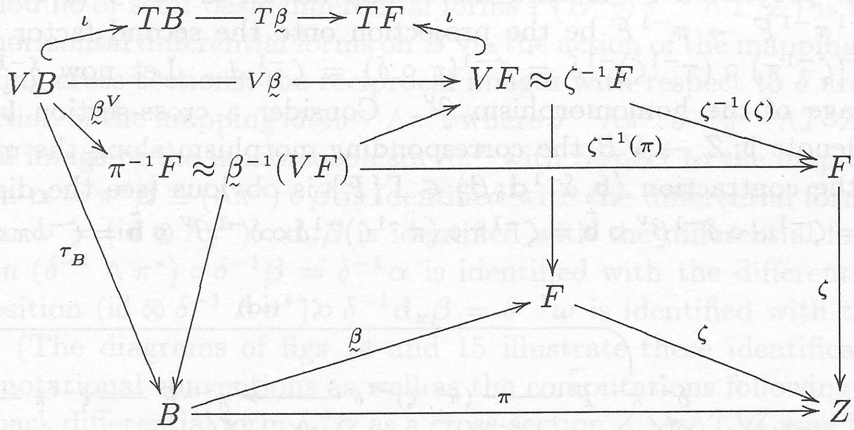

FIGURE 3

The restriction of the tangent mapping $T\underset{\sim}{\beta}$ to the bundle of vertical tangent vectors gives rise to the vector bundle homomorphism $V\underset{\sim}{\beta}\colon VB\to VF$ over the morphism $\underset{\sim}{\beta}$ (we use more economical notations VB and VF for the bundles of vertical tangent vectors $T(B/Z)$ and $T(F/Z)$ over Z). Let $\sigma_{\mathfrak{f}}{}'$ denote the tangent vector to the curve $\sigma_{\mathfrak{f}}(t)$ which belongs completely to the fibre F_z of F over $z\in Z$ and which starts from $\mathfrak{f}\in F$, $\sigma_{\mathfrak{f}}(0)=\mathfrak{f}$; then the derivative $(d\sigma_{\mathfrak{f}}/dt)(0)$ also belongs to the vector space F_z. The vertical tangent vector $\sigma_{\mathfrak{f}}{}'$ is identified with the pair $\big(\mathfrak{f};(d\sigma_{\mathfrak{f}}/dt)(0)\big)$ of the induced bundle $\zeta^{-1}F$ and so the well-known isomorphism $VF\approx\zeta^{-1}F$ over F holds. The homomorphism $V\underset{\sim}{\beta}$ may be reduced to the base B and the morphism so defined, $(\underset{\sim}{\beta})^V\colon VB\to\underset{\sim}{\beta}^{-1}VF$, after the identification of $\underset{\sim}{\beta}^{-1}VF\approx\underset{\sim}{\beta}^{-1}\zeta^{-1}F$ with $\pi^{-1}F$ acts upon a vertical tangent vector $\boldsymbol{v}\in VB$ as follows. Suppose vector $\boldsymbol{v}$ be tangent to a curve σ_b in the manifold B and $b=\sigma_b(0)$. Then $\underset{\sim}{\beta}^V(\boldsymbol{v}) = \big(b;(d/dt)(\underset{\sim}{\beta}\circ\sigma_b)(0)\big)$. This mapping $\underset{\sim}{\beta}^V$ is linear at the fibers of F and may be thus thought of as a cross-section $\mathbf{d}_\pi\beta$ of the bundle $(VB)^*\otimes\pi^{-1}F$,

$$\langle\mathfrak{v},\mathbf{d}_\pi\beta\rangle\approx(V\beta\circ\mathfrak{v})^{\sim} \tag{7}$$

The Lie derivative and fibrewise differentiation of semi-basic forms. In what follows and to the end of current Section we shall be busy with establishing the relationship between the Lie derivative and the fibre differential of a semi-basic differential form

Consider a vector bundle $E \to B$ and its dual bundle $E^* \to B$. If some $\omega \in \Omega^d(B; E^*)$, and if $\tilde{\mathfrak{d}}$ is a cross-section of the vector bundle $\delta^{-1}E$, then, by definition, $\langle \tilde{\mathfrak{d}}, \delta^*\omega \rangle (\mathfrak{u}_1, \dots, \mathfrak{u}_d) = \langle \tilde{\mathfrak{d}}(z), \delta^*\omega(\mathfrak{u}_1, \dots, \mathfrak{u}_d) \rangle$, where $\mathfrak{u}_1, \dots, \mathfrak{u}_d \in T_z Z$.

Set $F = \wedge T^*Z$. By means of the imbedding $\mathsf{id} \otimes \wedge\pi^*$ the cross-sections of the bundle $E^* \otimes \pi^{-1} \wedge T^*Z$ are identified with horizontal (with respect to π) E^*-valued differential forms on the manifold B. The $\mathfrak{F}_B$-module of these forms will be denoted by $\Omega_B(Z; E^*)$. The identification of it with a submodule in $\Omega(B; E^*)$ is carried out by the monomorphism $\pi^{\#}$.

Let both δ and δ_t be cross-sections of the fibred manifold B over Z, i.e. $\pi \circ \delta_t = \pi \circ \delta = \mathit{id}$. In this case the vector $\mathfrak{b}(z)$, which is tangent to the curve $\sigma_{\delta(z)}(t) = \delta_t(t)$, will be vertical.

Set $E = VB$. As long as δ is a cross-section of the projection π, the reciprocal image $\delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta}$ of the cross-section $\mathbf{d}_\pi\boldsymbol{\beta}$ with respect to the mapping δ will be a cross-section of the bundle $\delta^{-1}(VB)^* \otimes F$, since the bundle $\delta^{-1}\pi^{-1}F \approx (\pi \circ \delta)^{-1}F$ has to be identified with F by the projection $\zeta^{-1}\mathit{id}$ onto the second factor, $\mathit{id}^{-1}F \ni (z; \mathsf{f}) \mapsto \mathsf{f} \in F$. Let $(\pi^{-1}\zeta)^{-1}\delta \colon \delta^{-1}\pi^{-1}F \to \pi^{-1}F$ be the projection onto the second factor. It is straightforward that $(\zeta^{-1}\pi) \circ (\pi^{-1}\zeta)^{-1}\delta = \zeta^{-1}(\pi \circ \delta) = \zeta^{-1}\mathit{id}$. Let now $\delta^{-1}\underset{\sim}{\beta}^V$ denote the reciprocal image of the homomorphism $\underset{\sim}{\beta}^V$. Consider a cross-section $\tilde{\mathfrak{b}}$ of the bundle $\delta^{-1}VB$ and denote $\mathfrak{b} \colon Z \to VB$ the corresponding morphism along the mapping δ. The definition of the contraction $\langle \tilde{\mathfrak{b}}, \delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta} \rangle \in \Gamma\{F\}$ is obvious (see the diagram of fig.4): $\langle \tilde{\mathfrak{b}}, \delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta} \rangle = \zeta^{-1}\mathit{id} \circ \delta^{-1}\underset{\sim}{\beta}^V \circ \tilde{\mathfrak{b}} = \zeta^{-1}\pi \circ (\pi^{-1}\zeta)^{-1}\delta \circ \delta^{-1}\underset{\sim}{\beta}^V \circ \tilde{\mathfrak{b}} = \zeta^{-1}\pi \circ \underset{\sim}{\beta}^V \circ \mathfrak{b}$.

$$
\begin{tikzcd}
 & \delta^{-1}\pi^{-1}F \arrow[rr, "(\pi^{-1}\zeta)^{-1}\delta"] \arrow[rrr, bend left, "\zeta^{-1}(\mathit{id})"] \arrow[ddl] & & \pi^{-1}F \arrow[r, "\zeta^{-1}\pi"] \arrow[ddl, "\pi^{-1}\zeta"] & F \arrow[dd, "\zeta"] \\
\delta^{-1}(VB) \arrow[ur, "\delta^{-1}(\underset{\sim}{\beta}^V)"] \arrow[rr] & & VB \arrow[ur, "(\underset{\sim}{\beta}^V)"] \arrow[d] & & \\
Z \arrow[u, "\tilde{\mathfrak{b}}"] \arrow[urr, dotted, "\mathfrak{b}"] \arrow[rr, "\delta"] \arrow[rrrr, bend right, "\mathit{id}"'] & & B \arrow[rr, "\pi"] & & Z
\end{tikzcd}
$$

FIGURE 4

If we take $\wedge T^*Z$ in place of the fibre bundle F and if $\mathfrak{u}_1, \dots, \mathfrak{u}_d \in T_z Z$, then one can easily calculate: $\langle \tilde{\mathfrak{b}}, \delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta} \rangle (\mathfrak{u}_1, \dots, \mathfrak{u}_d) = (\zeta^{-1}\pi \circ \underset{\sim}{\beta}^V \circ \mathfrak{b}(z))(\mathfrak{u}_1, \dots, \mathfrak{u}_d) = (d/dt)(\boldsymbol{\beta} \circ \sigma_{\delta(z)})(\mathfrak{u}_1, \dots, \mathfrak{u}_d)(0)$.

Let $(\pi^{-1}\tau)^{-1}\delta \colon \delta^{-1}\pi^{-1}TZ \to \pi^{-1}TZ$ denote the standard projection onto the second factor and let $\tau^{-1}\mathit{id}$ denote the obvious identification $\mathit{id}^{-1}TZ \approx TZ$ so that $(\tau^{-1}\pi) \circ (\pi^{-1}\tau)^{-1}\delta = \tau^{-1}\mathit{id}$ (see again the diagram of fig.4 with τ in place of ζ this time). Let τ_B denote the projection $TB \to B$. One computes (see fig.5): $T\pi \circ \tau_B{}^{-1}\delta = (\tau^{-1}\pi) \circ \pi^T \circ \tau_B{}^{-1}\delta = (\tau^{-1}\pi) \circ (\pi^{-1}\tau)^{-1}\delta \circ (\delta^{-1}\pi^T) = \tau^{-1}\mathit{id} \circ \delta^{-1}\pi^T$. Composing with the mapping

$$\begin{tikzcd}
TZ \arrow[rrr, "\mathrm{id}"] \arrow[d, "\delta^T"'] \arrow[ddr, "T\delta"] & & & TZ & \\
\delta^{-1}TB \arrow[rrr, dashed, "\delta^{-1}\pi^T"] \arrow[dr, "\tau_B{}^{-1}\delta"'] & & & \delta^{-1}\pi^{-1}TZ \arrow[u, "\tau^{-1}\iota d"'] \arrow[dr, "(\pi^{-1}\tau)^{-1}\delta"'] & \\
& TB \arrow[uurr, "T\pi"] \arrow[rrr, "\pi^T"'] & & & \pi^{-1}TZ \arrow[uul, "\tau^{-1}\pi"']
\end{tikzcd}$$

FIGURE 5. This is the upper part of the complete picture of fig.13.

δ^T it gives $\tau^{-1}\iota d \circ (\delta^{-1}\pi^T) \circ \delta^T = T\pi \circ T\delta = \mathrm{id}$; performing the transition to the dual mappings, one obtains $\wedge\delta^* \circ (\delta^{-1} \wedge \pi^*) = \overset{*}{\tau}{}^{-1}\iota d$.

While the module of semi-basic differential forms $\Gamma\{E^* \otimes \pi^{-1} \wedge T^*Z\}$ is identified with the module of horizontal differential forms on B via the action of the mapping $\mathrm{id}\otimes\wedge\pi^*$ upon the corresponding cross-sections, the reciprocal images with respect to δ are identified by means of the action of the mapping $\mathrm{id}\otimes\delta^{-1}\wedge\pi^*$, where $\delta^{-1}\wedge\pi^*\colon \delta^{-1}\pi^{-1}\wedge T^*Z \to \delta^{-1}\wedge T^*B$ is the reciprocal image of the monomorphism $\wedge\pi^*$ with respect to the map δ. Thus, if the differential form $\boldsymbol{\alpha} = \pi^\#\boldsymbol{\beta} \equiv (\wedge\pi^*) \circ \boldsymbol{\beta}$ is identified with the differential form $\boldsymbol{\beta}$ and if the differential form $\boldsymbol{\omega} = (\mathrm{id} \otimes \wedge\pi^*) \circ \mathbf{d}_\pi\boldsymbol{\beta}$ is identified with the differential form $\mathbf{d}_\pi\boldsymbol{\beta}$, then the composition $(\delta^{-1} \wedge \pi^*) \circ \delta^{-1}\boldsymbol{\beta} = \delta^{-1}\boldsymbol{\alpha}$ is identified with the differential form $\delta^{-1}\boldsymbol{\beta}$ and the composition $(\mathrm{id} \otimes \delta^{-1} \wedge \pi^*) \circ \delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta} = \delta^{-1}\boldsymbol{\omega}$ is identified with the differential form $\delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta}$. (The diagrams of figs 14 and 15 illustrate these identifications and the accompanying notational conventions as well as the computations following herein.)

The pulled-back differential form $\delta_t{}^\star\boldsymbol{\alpha}$ as a cross-section $Z \to \wedge T^*Z$ may be represented as follows:

$$\delta_t{}^\star\boldsymbol{\alpha} = \delta_t{}^\#\delta_t{}^{-1}\boldsymbol{\alpha} \equiv \wedge\delta_t^* \circ \delta_t{}^{-1}\boldsymbol{\alpha} \overset{\sim}{\leftrightarrow} \wedge\delta_t^* \circ (\delta_t{}^{-1} \wedge \pi^*) \circ \delta_t{}^{-1}\boldsymbol{\beta} = \overset{*}{\tau}{}^{-1}\iota d \circ \delta_t{}^{-1}\boldsymbol{\beta} \approx \delta_t{}^{-1}\boldsymbol{\beta}. \quad (8)$$

The pulled-back form $\delta^\star\boldsymbol{\omega}$ as a cross-section $Z \to \delta^{-1}(VB)^* \otimes \wedge T^*Z$ may similarly be represented as $\delta^\star\boldsymbol{\omega} = \delta^\#\delta^{-1}\boldsymbol{\omega} \equiv (\mathrm{id}\otimes\wedge\delta^*)\circ\delta^{-1}\boldsymbol{\omega} \overset{\sim}{\leftrightarrow} (\mathrm{id}\otimes\wedge\delta^*)\circ(\mathrm{id}\otimes\delta^{-1}\wedge\pi^*)\circ\delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta} = (\mathrm{id}\otimes\overset{*}{\tau}{}^{-1}\iota d)\circ\delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta} \approx \delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta}$. In (8) we may also carry out explicitly the composition of the map $\overset{*}{\tau}{}^{-1}\iota d$ with $\delta_t{}^{-1}\boldsymbol{\beta}$. We insert the identity $\overset{*}{\tau}{}^{-1}\iota d = (\overset{*}{\tau}{}^{-1}\pi) \circ (\pi^{-1}\overset{*}{\tau})^{-1}\delta_t$ into (8) and employ the definition of the reciprocal image $\delta_t{}^{-1}\boldsymbol{\beta}$ together with the definition of the π-morphism $\underset{\sim}{\beta}$, which read $(\overset{*}{\tau}{}^{-1}\pi) \circ (\pi^{-1}\overset{*}{\tau})^{-1}\delta_t \circ \delta_t{}^{-1}\boldsymbol{\beta} = \underset{\sim}{\beta} \circ \delta_t$, to obtain simply $\delta_t{}^\star\boldsymbol{\alpha} = \underset{\sim}{\beta} \circ \delta_t$.

Remark. We have in fact proved the following assertion. If $\boldsymbol{\beta} \in \Omega_B(Z)$ is a semi-basic differential form with respect to a fibration $\pi\colon B \to Z$ and if $\delta\colon Z \to B$ is a cross-section of that fibration, then

$$\delta^\star\boldsymbol{\beta} = \underset{\sim}{\beta} \circ \delta. \quad (9)$$

Now it follows easily that $\mathbf{L}(\mathfrak{b})\alpha = \langle\tilde{\mathfrak{b}}, \delta^\star\omega\rangle$. Indeed, on the right-hand side here we have

$$\langle\tilde{\mathfrak{b}}, \delta^\star\omega\rangle(\mathfrak{u}_1, \ldots, \mathfrak{u}_d) = \langle\tilde{\mathfrak{b}}, \delta^{-1}\mathbf{d}_\pi\boldsymbol{\beta}\rangle(\mathfrak{u}_1, \ldots, \mathfrak{u}_d) = (d/dt)(\underset{\sim}{\beta} \circ \sigma_{\delta(z)})(\mathfrak{u}_1, \ldots, \mathfrak{u}_d)(0),$$

whereas on the left-hand side we proceed as follows,

$$\mathbf{L}(\mathfrak{b})\boldsymbol{\alpha} = (d/dt)(\delta_t{}^\star\boldsymbol{\alpha})(0) = (d/dt)(\overset{*}{\tau}{}^{-1}\mathit{id} \circ \delta_t{}^{-1}\boldsymbol{\beta})(0)$$
$$= (d/dt)(\overset{*}{\tau}{}^{-1}\pi \circ (\pi^{-1}\overset{*}{\tau})^{-1}\delta_t \circ \delta_t{}^{-1}\boldsymbol{\beta})(0) = (d/dt)(\boldsymbol{\beta} \circ \delta_t)(0),$$

so that by evaluating on the vectors $\mathfrak{u}_1, \ldots, \mathfrak{u}_d$ one regains that same result,

$$(\mathbf{L}(\mathfrak{b})\boldsymbol{\alpha})(\mathfrak{u}_1, \ldots, \mathfrak{u}_d) = (d/dt)(\boldsymbol{\beta} \circ \delta_t(z))(\mathfrak{u}_1, \ldots, \mathfrak{u}_d)(0) = (d/dt)(\underset{\sim}{\boldsymbol{\beta}} \circ \sigma_{\delta(z)})(\mathfrak{u}_1, \ldots, \mathfrak{u}_d)(0),$$

q.e.d.

Not indicating explicitly the above mentioned identification of the differential forms $\boldsymbol{\alpha}$ and ω with the cross-sections $\boldsymbol{\beta}$ and $\mathbf{d}_\pi\boldsymbol{\beta}$, one can write

$$\mathbf{L}(\mathfrak{b})\boldsymbol{\beta} = \langle \tilde{\mathfrak{b}}, \delta^\star \mathbf{d}_\pi \boldsymbol{\beta} \rangle \tag{10}$$

3. Lagrange structure and the first variation

1. Jet bundle structure. By a classical field we mean a cross-section $v\colon Z \to Y$ of a fibred manifold $\pi\colon Y \to Z$ over the base Z in the category C^∞. The jets of order r of such sections, each denoted $j_r v$, constitute the manifold Y_r which is called the r^{th}-order jet prolongation of the manifold Y and we put $Y = Y_0$. Projections ${}^r\pi_s\colon Y_s \to Y_r$ for $r < s$ and $\pi_r\colon Y_r \to Z$ all are surjective submersions and commute, $\pi_r \circ {}^r\pi_s = \pi_s$. Let $\mathfrak{F}_r$ stand for the ring $\mathfrak{F}_{Y_r}$ of C^∞ functions over the manifold Y_r. Monomorphisms ${}^r\pi_s{}^\star\colon \mathfrak{F}_r \to \mathfrak{F}_s$ and $\pi_s{}^\star\colon \mathfrak{F}_Z \to \mathfrak{F}_s$ allow us to identify the rings $\mathfrak{F}_r$ and $\mathfrak{F}_Z$ with the subrings ${}^r\pi_s{}^\star\mathfrak{F}_r$ and $\pi_s{}^\star\mathfrak{F}_Z$ of the ring $\mathfrak{F}_s$.

Given another fibred manifold Y' over the same base Z and a base-preserving morphism $\phi\colon Y \to Y'$, the morphism

$$J_r\phi\colon j_r v(z) \to j_r(\phi \circ v)(z) \tag{11}$$

from the manifold Y_r to the manifold Y'_r is called the r^{th}-order prolongation of the morphism ϕ [18].

2. The variation of the Action functional. A Lagrangian is a semi-basic (with respect to π) differential form $\boldsymbol{\lambda}$ of maximal degree, $\boldsymbol{\lambda} \in \Omega^p_r(Z)$, $p = \dim Z$. Suppose again that the manifold Z is compact. Let $\mathcal{Y}_r$ denote the space of smooth (C^∞) cross-sections of Y_r. The differential form $\boldsymbol{\lambda}$, thought of as a morphism $\underset{\sim}{\lambda}\colon Y_r \to \wedge^p T^*Z$ along the projection $\pi_r\colon Y_r \to Z$, defines a nonlinear differential operator $\check{\underset{\sim}{\lambda}}$ in the space $\mathcal{Y} = \Gamma\{Y\}$ as follows:

$$\check{\lambda}(v) = (j_r v)^\star \boldsymbol{\lambda} = \underset{\sim}{\lambda} \circ j_r v. \tag{12}$$

The Action functional $S = \int_Z (j_r v)^\star \boldsymbol{\lambda}$ splits into the composition of three mappings, i.e. $S = \int_Z \circ {}^\star\boldsymbol{\lambda} \circ j_r$. Here j_r means the r^{th}-order prolongation operator $j_r\colon \mathcal{Y} \to \mathcal{Y}_r$, $v \mapsto j_r v$; ${}^\star\boldsymbol{\lambda}$ maps the space $\mathcal{Y}_r$ into the space of cross-sections of the determinant bundle $\wedge^p T^*Z$, ${}^\star\boldsymbol{\lambda}\colon \mathcal{Y}_r \to \Omega^p(Z)$, $v_r \mapsto v_r{}^\star\boldsymbol{\lambda}$, $v_r \in \mathcal{Y}_r$; $\int_Z$ is a linear functional on the Banach space $\Omega^p(Z)$, $\int_Z\colon \Omega^p(Z) \to \mathbb{R}$, $\boldsymbol{\beta} \mapsto \int_Z \boldsymbol{\beta}$.

The Euler-Lagrange equations for an extremal cross-section v arise as the condition upon the Fréchet derivative $\mathbf{D}S(v)$ at the point v to be equal to zero. According to the chain rule,

$$\mathbf{D}(S)(v) = (\mathbf{D}\textstyle\int_Z)\big((j_r v)^\star\boldsymbol{\lambda})\big) \cdot (\mathbf{D}\,{}^\star\boldsymbol{\lambda})(j_r v) \cdot (\mathbf{D}j_r)(v)\,. \tag{13}$$

Since the functional $\int_Z$ is linear, its derivative $\mathbf{D}\int_Z$ equals $\int_Z$ regardless of the point $(j_r v)^\star\boldsymbol{\lambda}$.

We pass now to the computation of $\mathbf{D}(S)(v)$ in strictly consistent and *formal* manner.

3. Differential of the map $^\star\boldsymbol{\lambda}$. In the classical field theory the variations of the Action functional are computed with respect to those variations of functions, which are fields themselves, that means, which are cross-sections of the corresponding fibred manifolds. Thus, in the notations of Section 2, the mappings δ and δ_t due to be cross-sections of the fibred manifold $B \to Z$. In this case, and assuming also that the differential form $\boldsymbol{\alpha}$ is semi-basic, one can write, according to (10), $\mathbf{L}(\mathfrak{b})\boldsymbol{\alpha} = \langle \tilde{\mathfrak{b}}, \delta^\star \mathbf{d}_\pi \boldsymbol{\alpha} \rangle$.

Applying this formula along with the formula (6) to the operator $^\star\boldsymbol{\lambda}$ by putting $B = Y_r$, $\boldsymbol{\alpha} = \boldsymbol{\lambda}$, $\delta = j_r v$, and substituting $\tilde{\mathfrak{b}}$ by some $\tilde{\mathfrak{y}}_r \in \Gamma\{v_r{}^{-1}VY_r\}$, we come up finally to the desired calculative formula

$$\mathbf{D}(^\star\boldsymbol{\lambda})(j_r v)\,.\,\mathfrak{y}_r = \langle \tilde{\mathfrak{y}}_r, (j_r v)^\star \mathbf{d}_\pi \boldsymbol{\lambda} \rangle \tag{14}$$

4. The permutation of the partial differentiations (Schwarz lemma). In the following two Paragraphs we reproduce for the sake of the subsequent quotation the well-known technical trick of the exchange in the order of applying the operation of the infinitesimal variation and that of partial differentiation. The tangent space to the manifold $\mathcal{Y}_r$ at the point v_r is the space of cross-sections of the fibre bundle $v_r{}^{-1}V_r$ (from here on we introduce the more economical notation V_r in place of $V(Y_r)$). The manifold V_r along with being fibred over the base Y_r by means of the surmersion $\tau_r\colon T(Y_r) \supset V_r \to Y_r$ is also fibred over the base Z by means of the surmersion $\pi_r \circ \tau_r\colon V_r \to Z$; every time the latter is implied we shall use the notation $(V_r)_Z$. Cross-sections of the bundle $v_r{}^{-1}V_r$ are identified with those cross-sections of the fibred manifold $(V_r)_Z$ which project onto the mapping v_r, the totality of them denoted as $\Gamma_{v_r}\{(V_r)_Z\}$. By means of the application $J_r(\tau_Y)\colon J_r(V_Z) \to J_r(Y) \equiv Y_r$ the manifold $J_r(V_Z)$ while fibred over the base Z appears to carry another fibred structure over the base Y_r. Say $\mathfrak{y}$ be a lift of the cross-section $v\colon Z \to Y$ to the vertical bundle V, then the cross-section $j_r\mathfrak{y}$ of the fibred manifold $J_r(V_Z)$ is projected onto the cross-section $j_r v$ under the mapping $J_r(\tau_Y)$ (see fig.7 of Appendix 1).

The isomorphism is between the manifolds V_s and $J_s(V_Z)$ over the base Y_s is obtained by means of the following procedure. To a vector $\sigma_{y_s}{}' \in V_{y_s}(Y_s)$, tangent at the point $y_s = j_s v(z_0) \in Y_s$ to the curve $\sigma_{y_s}\colon t \mapsto j_s v_t(z_0)$, the jet $j_s\mathfrak{y}(z_0) \in J_s(V_Z)$ is put into correspondence the lift $\mathfrak{y}$ being defined by the family of v_t, i.e. $\mathfrak{y}(z) = \sigma_{v(z)}{}'$ where for each z the curve $\sigma_{v(z)}\colon t \mapsto v_t(z)$ is contained in the fibre Y_z of the fibred manifold Y.

Conversely, given a jet $j_s\mathfrak{y}(z_0)$ of some lift $\mathfrak{y}\colon Z \to V$ along the cross-section $v = \tau_Y \circ \mathfrak{y}\colon Z \to Y$, we construct for each vertical tangent vector $\mathfrak{y}(z)$ an integral curve $\sigma_{v(z)}(t)$ and hence the family of cross-sections $v_t\colon z \mapsto \sigma_{v(z)}(t)$. Then under the mapping is the vertical tangent vector $(j_s v_t(z_0))'$ is sent to the s^{th}-order jet at z_0 of the lift $z \mapsto \sigma_{v(z)}{}' = \mathfrak{y}(z)$.

This very isomorphism is acts upon the cross-sections of the corresponding fibred manifolds over the base Z: if $\mathfrak{y}_r \in \Gamma_{v_r}\{(V_r)_Z\}$, then $\mathsf{is}_\#(\mathfrak{y}_r) \equiv \mathsf{is} \circ \mathfrak{y}_r \in \Gamma_{v_r}\{J_r(V_Z)\}$, where Γ_{v_r} in the second membership relation means that only those cross-sections of $J_r(V_Z)$ count, which project onto v_r under the application $J_r(\tau_Y)$.

5. The differential of j_r. Now we are going to prove the legitimacy of the diagram of fig.6 below.

The differential of the mapping j_r takes a vector $\gamma_v{}' = \mathfrak{y}$, tangent to the curve $\gamma_v\colon t \mapsto v_t$ at the point $v = \gamma_v(0) \in \mathcal{Y}$, over to the vector $\mathfrak{y}_r = (j_r \circ \gamma_v)'$, tangent to the curve $t \mapsto j_r v_t$ at the point $j_r v \in \mathcal{Y}_r$, hence $(\mathbf{D}j_r)(v)\colon \Gamma_v\{V_Z\} \to \Gamma_{j_r v}\{(V_r)_Z\}$. The cross-section $\mathfrak{y}_r\colon z \mapsto \sigma_{j_r v(z)}{}'$ is mapped under $\mathsf{is}_\#$ into the cross-section $\mathsf{is}_\#(\mathfrak{y}_r)\colon z \mapsto j_r\mathfrak{y}(z)$ of the fibred

$$\begin{tikzcd}[column sep=small]
 & T_{j_r v}(\mathcal{Y}_r) = \Gamma_{j_r v}\{(V_r)_Z\} \arrow[dr, "\mathsf{is}_\#"] & \\
T_v(\mathcal{Y}) = \Gamma_v\{V_Z\} \arrow[ur, "\mathbf{D}(j_r)(v)"] \arrow[rr, "j_r"'] & & \Gamma_{j_r v}\{J_r(V_Z)\}
\end{tikzcd}$$

FIGURE 6

manifold $J_r(V_Z)$, that is, $\mathsf{is}_\#(\mathfrak{y}_r) = j_r\mathfrak{y}$. Thus in order to compute the Fréchet differential of the jet prolongation operator j_r one may utilize the following permutation formula

$$\mathbf{D}(j_r)(v)\,.\,\mathfrak{y} = \mathsf{is}_\#^{-1}\, j_r\mathfrak{y}\,. \tag{15}$$

6. The first variation. From (14) and (15) we obtain the differential of the composed mapping, ${}^\star\boldsymbol{\lambda}\circ j_r$,

$$\mathbf{D}\,({}^\star\boldsymbol{\lambda}\circ j_r)\,(v)\,.\,\mathfrak{y} = (\,\mathbf{D}{}^\star\boldsymbol{\lambda})(j_r v)\cdot(\mathbf{D}j_r)(v)\,.\,\mathfrak{y} = \left\langle (\mathsf{is}_\#^{-1}\, j_r\mathfrak{y})^\sim,\ (j_r v)^\star\mathbf{d}_\pi\boldsymbol{\lambda}\right\rangle\,, \tag{16}$$

and, finally, from (13),— the desired expression for the differential of the Action functional

$$\mathbf{D}S(v)\,.\,\mathfrak{y} = \int_Z \left\langle (\mathsf{is}_\#^{-1}\, j_r\mathfrak{y})^\sim,\ (j_r v)^\star\mathbf{d}_\pi\boldsymbol{\lambda}\right\rangle.$$

4. INTEGRATING BY PARTS

To proceed further we need to extend the definition of the fibre differential $\mathbf{d}_\pi$ to the module of semi-basic $\wedge^d V_r^*$-valued differential forms of arbitrary degree d, $\Omega_r(Z;\wedge^d V_r^*)$, and to introduce the notion of total (global) differential $\mathbf{d}_t$. This is being done in Appendix 2. Our considerations there as well as within this Section essentially follow those of [10].

1. As far as we shall work with differential forms of different orders, we shall frequently need to bring them together to the same base manifold (if a differential form belongs to $\Omega_s(Z;V_r^*)$ we call the pair (s,r) be the order of that form). We recall that the homomorphism $({}^r\pi_s)^V$ maps V_s into ${}^r\pi_s{}^{-1}V_r$. The dual homomorphism $({}^r\pi_s{}^V)^*$ acts upon the cross-sections of the dual vector bundle ${}^r\pi_s{}^{-1}V_r^*$ through the composition and hence it acts upon the module $\Omega_s(Z;{}^r\pi_s{}^{-1}\wedge V_r^*)$ by means of composing its elements (viewed as cross-sections) with $\wedge({}^r\pi_s{}^V)^*\otimes\mathrm{id}$. Given a morphism $\mathfrak{g}$ from whatsoever the source be to the manifold Y_s, by $({}^r\pi_s)^\#$ the reciprocal image of this action with respect to $\mathfrak{g}$ will be denoted, namely, if $\omega\in\Gamma\{({}^r\pi_s\circ\mathfrak{g})^{-1}\wedge V_r^*\otimes(\pi_s\circ\mathfrak{g})^{-1}\wedge T^*Z\}$, then $({}^r\pi_s)^\#\doteqdot(V\,{}^r\pi_s)^\#\doteqdot(\mathfrak{g}^{-1}({}^r\pi_s{}^V))^\#\doteqdot(\mathfrak{g}^{-1}(\wedge{}^r\pi_s{}^{V*}))_\#\doteqdot(\mathfrak{g}^{-1}(\wedge{}^r\pi_s{}^{V*})\otimes\mathrm{id})_\#$, and $({}^r\pi_s)^\#\omega = (\mathfrak{g}^{-1}(\wedge{}^r\pi_s{}^{V*})\otimes\mathrm{id})\circ\omega\in\Gamma\{\mathfrak{g}^{-1}\wedge V_s^*\otimes(\pi_s\circ\mathfrak{g})^{-1}\wedge T^*Z\}$. Let δ be another morphism which composes with $\mathfrak{g}$ on the left. Then

$$\delta^\star({}^r\pi_s)^\#\omega = ({}^r\pi_s)^\#\delta^\star\omega\,. \tag{17}$$

Indeed, by definitions, $\delta^\star({}^r\pi_s)^\#\omega = (\mathrm{id}\otimes\wedge\delta^*)\circ(\delta^{-1}({}^r\pi_s{}^\#\omega))$. But $\delta^{-1}({}^r\pi_s{}^\#\omega) = \delta^{-1}((\mathfrak{g}^{-1}\wedge{}^r\pi_s{}^{V*}\otimes\mathrm{id})\circ\omega) = ((\mathfrak{g}\circ\delta)^{-1}\wedge{}^r\pi_s{}^{V*}\otimes\mathrm{id})\circ(\delta^{-1}\omega)$ and also $(\mathrm{id}\otimes\wedge\delta^*)\circ((\mathfrak{g}\circ\delta)^{-1}\wedge{}^r\pi_s{}^{V*}\otimes\mathrm{id}) = (\mathfrak{g}\circ\delta)^{-1}\wedge{}^r\pi_s{}^{V*}\otimes\wedge\delta^*$. On the other hand, $({}^r\pi_s)^\#\delta^\star\omega = ((\mathfrak{g}\circ\delta)^{-1}\wedge{}^r\pi_s{}^{V*}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\wedge\delta^*)\circ(\delta^{-1}\omega)$ and also $((\mathfrak{g}\circ\delta)^{-1}\wedge{}^r\pi_s{}^{V*}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\wedge\delta^*) = (\mathfrak{g}\circ\delta)^{-1}\wedge{}^r\pi_s{}^{V*}\otimes\wedge\delta^*$. So one concludes that (17) holds, q.e.d.

2. We are now ready to write down the decomposition formula[2] of KOLÁŘ in terms of semi-basic differential forms which take values in vector bundles V^* and V^*_{r-1}.

Proposition 1. *Given a Lagrangian $\boldsymbol{\lambda} \in \Omega^p_r(Z)$ there exist semi-basic differential forms $\epsilon \in \Omega^p_{2r}(Z;V^*)$ and $\boldsymbol{\kappa} \in \Omega^{p-1}_{2r-1}(Z;V^*_{r-1})$ such that*

$$ {}^r\pi_{2r}{}^* \mathbf{d}_\pi \boldsymbol{\lambda} = ({}^0\pi_r)^\# \epsilon + \mathbf{d}_t \boldsymbol{\kappa}. \tag{18}$$

The form ϵ is unique inasmuch as its order is fixed and equals $(2r,0)$ [6]. *The form $\boldsymbol{\kappa}$ is defined by this decomposition up to a $\mathbf{d}_t$-exact term* [5].

The proof may be carried out in the explicit local coordinates form by the undetermined coefficients method. Non-existence of a formally intrinsic proof is closely related to non-uniqueness of the differential form $\boldsymbol{\kappa}$.

3. By the Nonlinear Green Formula we mean herein the expression of the Fréchet derivative at the point $v \in \Gamma\{Y\}$ of the operator $\check{\lambda} = {}^*\boldsymbol{\lambda} \circ j_r$ (see (12)) in terms of its transpose ${}^t(\mathbf{D}\check{\lambda}(v))$ and of the Green operator $\mathbf{G}$ [1]

$$\langle (\mathbf{D}\check{\lambda})(v)(\mathfrak{y}), 1\rangle = \langle \mathfrak{y}, {}^t(\mathbf{D}\check{\lambda}(v))(1)\rangle + \mathbf{d}(\mathbf{G}(\mathfrak{y},1)). \tag{19}$$

We recall that the transpose operator ${}^t(\mathbf{D}\check{\lambda}(v))$ is of the type $(\wedge^p T^*Z)^* \otimes \wedge^p T^*Z \to v^{-1}V^* \otimes \wedge^p T^*Z$ whereas $\mathbf{D}\check{\lambda}(v)$ is of the type $v^{-1}V \to \wedge^p T^*Z$ and therefore the Green operator has to be of the type $(v^{-1}V,\ (\wedge^p T^*Z)^* \otimes \wedge^p T^*Z) \to \wedge^{p-1}T^*Z$. Also the isomorphism $(\wedge^p T^*Z)^* \otimes \wedge^p T^*Z \approx \mathbb{R}_z$ holds and under it the contraction on the left-hand side of (19) locally looks like

$$\langle \mu, 1\rangle = \langle \mu_0 \mathbf{d}\xi^1 \wedge \cdots \wedge \mathbf{d}\xi^p, \partial/\partial\xi^1 \wedge \cdots \wedge \partial/\partial\xi^p \otimes \mathbf{d}\xi^1 \wedge \cdots \wedge \mathbf{d}\xi^p\rangle = \mu_0 \mathbf{d}\xi^1 \wedge \cdots \wedge \mathbf{d}\xi^p.$$

That this Green formula (otherwise called the "integration-by-parts" formula) is obtained by so to say "evaluating" the KOLÁŘ decomposition formula (18) along the submanifold $j_{2r}v$, becomes clear to the end of present Section. The demonstration will be carried out in three steps.

(i). Applying $j_{2r}v^*$ to (18) gives

$$(j_r v)^* \mathbf{d}_\pi \boldsymbol{\lambda} = (j_{2r}v)^* ({}^0\pi_r)^\# \epsilon + (j_{2r}v)^* \mathbf{d}_t \boldsymbol{\kappa}. \tag{20}$$

On the other hand, by (16)

$$(\mathbf{D}\check{\lambda})(v)(\mathfrak{y}) = \langle (\mathsf{is}_\#^{-1} j_r \mathfrak{y})^\sim, (j_r v)^* \mathbf{d}_\pi \boldsymbol{\lambda}\rangle\,. \tag{21}$$

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[2] This is the generalization to an arbitrary order of the decomposition formula adduced by TRAUTMAN in [15]. It has its counterpart in the algebra $\Omega(Y_r)$, where it is known under the name of the first variation formula [9]. As long as the field theory is concerned and thereby the splitting of the set of variables into independent and dependent ones by π is recognized, it is our opinion that the bigraded algebra $\Omega_s(Z;V^*_r)$ is a more appropriate object to work with than the complete skew-symmetric algebra $\Omega(Y_r)$.

(ii). In what concerns the first addend of the right-hand side of (20), one first applies (17) to get $(j_{2r}v)^\star({}^0\pi_r)^\#\epsilon = ({}^0\pi_r)^\#(j_{2r}v)^\star\epsilon$ and then consecutively (1) and (A3) together with (A1) to arrive at

$$\left\langle(\mathsf{is}_\#^{-1}j_r\mathfrak{y})^{\tilde{}},({}^0\pi_r)^\#(j_{2r}v)^\star\epsilon\right\rangle = \left\langle({}^0\pi_r{}^V)_\#(\mathsf{is}_\#^{-1}j_r\mathfrak{y})^{\tilde{}},(j_{2r}v)^\star\epsilon\right\rangle = \langle\tilde{\mathfrak{y}},(j_{2r}v)^\star\epsilon\rangle\,. \tag{22}$$

(iii). It remains to carry out some work upon the expression $\left\langle(\mathsf{is}_\#^{-1}j_r\mathfrak{y})^{\tilde{}},(j_{2r})^\star\mathbf{d}_t\boldsymbol{\kappa}\right\rangle$.

Suppose the vertical vector field $\mathfrak{y}$ along v be extended to a vertical field $\mathfrak{v}$ on Y, so that $\mathfrak{y} = \mathfrak{v}\circ v$. As an intermediate step we first prove the following relationship:

$$\left\langle(\mathsf{is}_\#^{-1}j_r\mathfrak{y})^{\tilde{}},(j_{2r}v)^\star\mathbf{d}_t\boldsymbol{\kappa}\right\rangle = (j_{2r}v)^\star\left\langle J_r(\mathfrak{v}),\mathbf{d}_t\boldsymbol{\kappa}\right\rangle. \tag{23}$$

By (A2), $(\mathsf{is}_\#^{-1}j_r\mathfrak{y})^{\tilde{}} = (J_r(\mathfrak{v})\circ j_rv)^{\tilde{}} = j_rv^{-1}(J_r(\mathfrak{v}))$. After the definition of the pull-back, $(j_{2r}v)^\star\mathbf{d}_t\boldsymbol{\kappa} = (j_{2r}v)^\# j_{2r}v^{-1}\mathbf{d}_t\boldsymbol{\kappa}$. So, on the left-hand side of (23) we come up to the expression $\left\langle j_rv^{-1}(J_r(\mathfrak{v})),(j_{2r}v)^\# j_{2r}v^{-1}\mathbf{d}_t\boldsymbol{\kappa}\right\rangle$. On the other hand, by (3), $j_{2r}v^{-1}\langle J_r(\mathfrak{v}),\mathbf{d}_t\boldsymbol{\kappa}\rangle = \left\langle j_{2r}v^{-1}\,{}^r\pi_{2r}{}^{-1}J_r(\mathfrak{v}),\, j_{2r}v^{-1}\mathbf{d}_t\boldsymbol{\kappa}\right\rangle$.

Next we apply to this the $(j_{2r}v)^\#$ operation (in the only sensible way, i.e. with respect to Z-variables in $\mathbf{d}_t\boldsymbol{\kappa}$; so it doesn't effect the term $j_{2r}v^{-1}\,{}^r\pi_{2r}{}^{-1}J_r(\mathfrak{v})$ of the contraction) and obtain on the right-hand side of (23) $(j_{2r}v)^\star\left\langle J_r(\mathfrak{v}),\mathbf{d}_t\boldsymbol{\kappa}\right\rangle = \left\langle j_{2r}v^{-1}\,{}^r\pi_{2r}{}^{-1}J_r(\mathfrak{v}),\right.$ $\left.(j_{2r}v)^\# j_{2r}v^{-1}\mathbf{d}_t\boldsymbol{\kappa}\right\rangle = \left\langle j_rv^{-1}J_r(\mathfrak{v}),(j_{2r}v)^\# j_{2r}v^{-1}\mathbf{d}_t\boldsymbol{\kappa}\right\rangle$, q.e.d.

In (23) we resort now consecutively to (A6) and (A4) to obtain the expected formula,

$$\left\langle(\mathsf{is}_\#^{-1}j_r\mathfrak{y})^{\tilde{}},(j_{2r}v)^\star\mathbf{d}_t\boldsymbol{\kappa}\right\rangle = \mathbf{d}(j_{2r-1}v^\star\left\langle J_{r-1}(\mathfrak{v}),\boldsymbol{\kappa}\right\rangle). \tag{24}$$

Comparing (19) with (18) by means of (21), (22), and (24), and applying an analogue of (23) with $\boldsymbol{\kappa}$ in place of $\mathbf{d}_t\boldsymbol{\kappa}$, $\left\langle(\mathsf{is}_\#^{-1}j_{r-1}\mathfrak{y})^{\tilde{}},(j_{2r-1}v)^\star\boldsymbol{\kappa}\right\rangle = (j_{2r-1}v)^\star\left\langle J_{r-1}(\mathfrak{v}),\boldsymbol{\kappa}\right\rangle$, the Reader easily convinces himself that *under the identification* $j_s\mathfrak{y} \overset{\approx}{\leftrightarrow} (\mathsf{is}_\#^{-1}j_s\mathfrak{y})^{\tilde{}} \in j_sv^{-1}V_s$ *in accordance with the isomorphism* $\mathsf{is}^{-1}: J_s(V_ZY) \overset{\approx}{\to} V(Y_s)$ the following assertion is true:

Proposition 2. *Let* $\boldsymbol{\lambda}$ *be an* r^{th}*-order Lagrangian for a nonlinear field* $v \in \Gamma\{Y \to Z\}$*. The variational derivative of the Action density* $\check{\lambda}(v) = (j_rv)^\star\boldsymbol{\lambda}$ *at* v *is an* r^{th}*-order differential operator* $\mathbf{D}\check{\lambda}(v)$ *in the space* $\Gamma_v\{V_ZY\}$ *of the variations of the field* v*. Let* ${}^t(\mathbf{D}\check{\lambda}(v))$ *denote the transpose operator and let* $\mathbf{G}$ *denote the Green operator for* $\mathbf{D}\check{\lambda}(v)$*. Then there exist semi-basic differential forms* ϵ *on* $J_{2r}(Y)$ *and* $\boldsymbol{\kappa}$ *on* $J_{2r-1}(Y)$ *which take values in vector bundles* $(VY)^*$ *and* $\left(J_{r-1}(V_ZY)\right)^*$ *respectively (as in Proposition 1) such that*

$$(\mathbf{D}\check{\lambda})(v)(\mathfrak{y}) = \langle j_r\mathfrak{y},(j_rv)^\star\mathbf{d}_\pi\boldsymbol{\lambda}\rangle\,;$$

$${}^t(\mathbf{D}\check{\lambda}(v))(1) = (j_{2r}v)^\star\epsilon\,;$$

$$\mathbf{G}(\mathfrak{y})(1) = \langle j_{r-1}\mathfrak{y},(j_{2r-1}v)^\star\boldsymbol{\kappa}\rangle\,.$$

Whereas the Green operator is defined up to a $\mathbf{d}$*-closed term, the differential form* $\boldsymbol{\kappa}$ *is defined up to a* $\mathbf{d}_t$*-exact term.*

The differential form ϵ *is defined uniquely and the Euler-Lagrange equations arise as a local expression of the exterior differential equation*

$$(j_{2r}v)^\star\epsilon = 0 \tag{25}$$

Discussion. One would wish to introduce some intrinsically defined operator $\mathbf{E}$ to give an explicit expression to the Euler-Lagrange form ϵ by means of $\epsilon = \mathbf{E}(\boldsymbol{\lambda})$. Considerable efforts were made, mainly by TULCZYJEW [16] and KOLÁŘ [7] in this direction which amount to defining the operator $\mathbf{E}$ in terms of some order-reducing derivations $\imath^i$ of degree 0, acting in the exterior algebra of fibre differential forms over the r^{th}-order prolongation manifold Y_r. These derivations act as trivial ones in the ring of functions $\mathfrak{F}_r$ over the manifold Y_r and are defined by prescribing their action upon one-forms in a local chart $(\xi^i;\psi^a_{\mathrm{N}})$ as follows

$$\imath^i \mathbf{d}\psi^a_{\mathrm{N}} = \begin{cases} \mathbf{d}\psi^a_{\mathrm{N}-1_i}, & \text{if } \mathrm{N} = (\nu_1,\dots,\nu_p) \geq 1_i \\ 0, & \text{otherwise}. \end{cases}$$

We recall also the local expressions of partial total derivatives $\mathbf{D}_i = \mathbf{D}_t(\partial/\partial\xi^i)$ (see Appendix 2), $\mathbf{D}_i = \partial/\partial\xi^i + \psi^a_{\mathrm{N}+1_i}\partial/\partial\psi^a_{\mathrm{N}}$. Let $\deg(\varphi)$ = degree of the fibre differential form φ. Neither $\mathbf{D}_i$ nor $\imath^i$ have any intrinsic meaning, but has the operator

$$\mathbf{E} = \deg \circ\, \mathbf{d}_\pi + \sum_{\|\mathrm{N}\|>0} \frac{1}{\mathrm{N}!}\mathbf{D}_{\mathrm{N}}\imath^{\mathrm{N}}\mathbf{d}_\pi\,. \tag{26}$$

As far as this operator $\mathbf{E}$, defined initially in $\Omega^0_r(Z;\wedge V^*_r)$, acts trivially upon the subring $\mathfrak{F}_Z$ of the ring $\mathfrak{F}_r$, its action can be extended to the whole of $\Omega_r(Z;\wedge V^*_r)$ remaining trivial over the subalgebra $\Omega(Z)$ of the algebra $\Omega_r(Z;\wedge V^*_r)$. Indeed, for a *parallelizable* Z (which *locally* is always true) in course of considerations, similar to those of Paragraph 2 of Appendix 2, we can profit by *local* isomorphism $\Omega_r(Z;\wedge V^*_r) \approx \Omega^0_r(Z;\wedge V^*_r)\otimes_{\mathfrak{F}_Z}\Omega(Z)$ to define $\mathbf{E}(\varphi\otimes\boldsymbol{\mu}) = \mathbf{E}(\varphi)\otimes\boldsymbol{\mu}$, whenever $\varphi\in\Omega^0(Z;\wedge V^*_r)$ and $\boldsymbol{\mu}\in\Omega(Z)$.

Comparing $\mathbf{E}(\boldsymbol{\lambda})$ in (26) with (18), it becomes evident that the definition of the operator $\mathbf{E}$ amounts to the choice of the form $\boldsymbol{\kappa}$. Since there is no natural way to make such choice intrinsic, this seems to be the reason why the efforts to explicitly present a consistently intrinsic definition of the Euler-Lagrange operator $\mathbf{E}$ failed as far. Of course, the problem melts down when passing to some quotient spaces of differential forms. Immense development took place in that direction at the level of cohomologies of bigraded complexes with the theory becoming still more abstract and still more deviating from the original Euler-Lagrange expression.

We wish to emphasize, that it was due to the existence of the projection $Y \to Z$ that the global splitting of variables into dependent and independent ones became possible, which, in turn, led to the natural interpretation of the term ϵ in (18) as a *semi-basic* differential form *with values in a vector bundle*. In a more general framework,— that of a contact manifold in place of the jet prolongation Y_r of a global surmersion $Y \to Z$,— such interpretation would never be possible. Even the Lagrangian itself could not be globally presented as a semi-basic form with respect to independent variables, and thus could not be specified in a canonical way [9]. In practical computations, however, one makes use of the local isomorphism between an r^{th}-order contact manifold $C^p_r(M)$ and the jet bundle $J_r(\mathbb{R}^p;\mathbb{R}^q)$, $p+q=\dim M$, so even in this case it is possible to profit by the advantages of the representation (25). We now pass to the discussion of these advantages.

First, we see that the representation (25) is natural. Indeed, the form ϵ originated from a Lagrangian, which ought to have been integrated over the base manifold, hence ought to have manifested itself as a *differential p-form*. But the variation was undertaken with respect to a vertical field $\mathfrak{y}$ and that vertical field enters as an argument to the linear transformation, associated with ϵ, which has nothing to do with the properties of ϵ as a semi-basic p-form. So, ϵ must manifest itself as a *vector valued* p-form (more precisely—with values in the dual to the vector bundle of infinitesimal variations).

Now, the expression (25) is deprived of any other inessential parameters which may have appeared during the process of variation. In particular, no trace of any auxiliary vertical vector field (as in [9]) remains. This allows representation of the solutions of the Euler-Lagrange equations in the form of the integral manifolds of an exterior vector valued differential system. Once recognized, such approach suggests the framework of linear algebra: first, in investigating the symmetries of the Euler-Lagrange equations, second, in solving the inverse problem of variational calculus. In both cases the method consists in transforming the problem of equivalence of two systems of differential equations, one of them generated by the left-hand side of (25), into an algebraic problem of the equivalence of the corresponding modules of vector differential forms using the Lagrange multipliers. To apply this approach consistently, in the case of studying symmetries, one needs to generalize the notion of the Lie derivative of a differential form to a derivative of a vector bundle valued differential form (see [7], [14], [10], [11], [13]).

Appendix 1. Graded structure of vertical tangent bundles and the prolongation of fiber transformations

1. Within the notations of Section 3 (Paragraph 4), let ${}^r\mathsf{pr}_s\colon J_s(V_Z)\to J_r(V_Z)$ denote the projection $j_s\mathfrak{y}(z)\mapsto j_r\mathfrak{y}(z)$. Applying $J_r(\tau_Y)$ to the target of the application ${}^r\mathsf{pr}_s$, we get: $J_r(\tau_Y)\circ{}^r\mathsf{pr}_s\ (j_s\mathfrak{y}(z)) = (J_r\tau_Y)(j_r\mathfrak{y}(z)) = j_r v(z)$. On the other hand, ${}^r\pi_s\circ J_s(\tau_Y)\ (j_s\mathfrak{y}(z)) = {}^r\pi_s(j_s(\tau_Y\circ\mathfrak{y})(z)) = {}^r\pi_s(j_s v(z)) = j_r v(z)$; hence ${}^r\mathsf{pr}_s$ is fibred over ${}^r\pi_s$. As soon as ${}^r\mathsf{pr}_s$ is so fibred, it acts upon the cross-sections of the reciprocal image $j_s v^{-1}J_s(V_Z)$ of the fibred manifold (in fact a vector bundle) $J_s(\tau_Y)\colon J_s(V_Z)\to Y_s$. After the general definitions, if we denote by $(j_s\mathfrak{y})\tilde{}$ the cross-section corresponding to the morphism $j_s\mathfrak{y}$ along $j_s v$, and by $({}^r\mathsf{pr}_s)_{Y_s}$ the reduction of ${}^r\mathsf{pr}_s$ to the base Y_s, then $({}^r\mathsf{pr}_s)_\#(j_s\mathfrak{y})\tilde{}\in\Gamma\{j_r v^{-1}J_r(V_Z)\}$, and $({}^r\mathsf{pr}_s)_\#(j_s\mathfrak{y})\tilde{}\doteqdot(j_s v^{-1}({}^r\mathsf{pr}_s)_{Y_s})\circ(j_s\mathfrak{y})\tilde{} = ({}^r\mathsf{pr}_s\circ j_s\mathfrak{y})\tilde{} = (j_r\mathfrak{y})\tilde{}$. If abandon the tilde in the superscripts, the projection $({}^r\mathsf{pr}_s)_\#$ will obtain a slightly different meaning as one acting from $\Gamma\{J_s(V_Z)\}$ into $\Gamma\{J_r(V_Z)\}$ "over Z",

$$({}^r\mathsf{pr}_s)_\#(j_s\mathfrak{y}) = j_r\mathfrak{y} \tag{A1}$$

2. The isomorphism between the manifolds V_s and $J_s(V_Z)$ allows us to write down a useful relationship between the jet of a restricted vertical vector field $\mathfrak{v}$, viewed as a cross-section of V_Z, and the prolongation $J_r(\mathfrak{v})$ of this field, obtained by prolonging its one-parametric local group as follows.

Consider a field $\mathfrak{v}\in\mathfrak{V}$ of vertical tangent vectors, generated by its local group $e^{t\mathfrak{v}}$. The r^{th}-order prolongation $J_r(\mathfrak{v})$ of $\mathfrak{v}$ is the vector field generated by the local group $J_r(e^{t\mathfrak{v}})\colon y_r\mapsto j_r(e^{t\mathfrak{v}}\circ v)(z)$, if $y_r = j_r v(z)$. Consider thereto a restriction $\mathfrak{v}\circ v$ of the vector field $\mathfrak{v}$ to some submanifold $v(Z)$ of Y ; it is an element $\mathfrak{y}$ from $\Gamma_v\{V_Z\}$ (see fig.7).

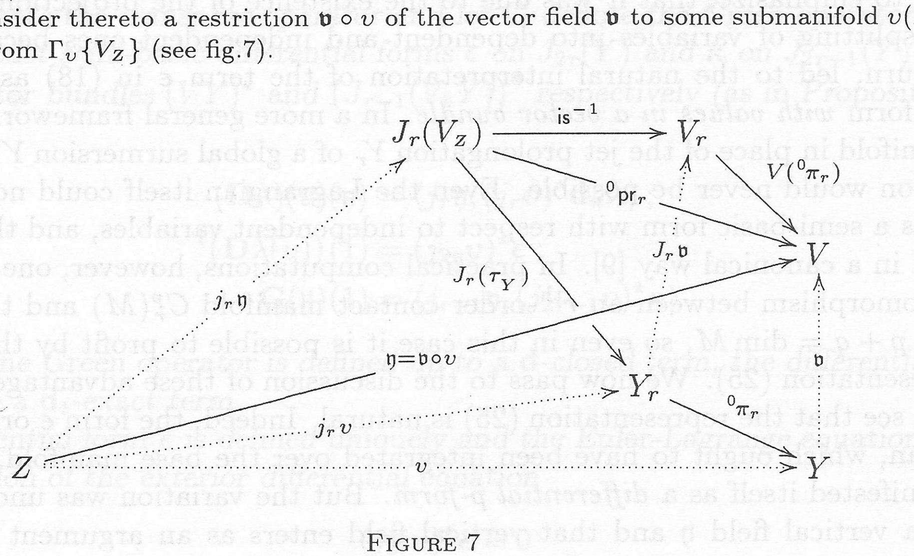


Figure 7

Under the application is^{-1} the point $j_r(\mathfrak{v} \circ \upsilon)(z) \in J_r(V_Z)$ transforms into the vertical tangent vector $(j_r(e^{t\mathfrak{v}} \circ \upsilon)(z))'$, which is nothing but exactly the value of the vertical field $J_r(\mathfrak{v})$ at the point y_r. We conclude thereof that the following formula (employed while demonstrating the veracity of (23)) holds:

$$\mathsf{is}^{-1} \circ j_r(\mathfrak{v} \circ \upsilon) = J_r(\mathfrak{v}) \circ j_r \upsilon. \tag{A2}$$

This relationship may be viewed as an alternative definition either of $J_r(\mathfrak{v})$ or of is, as it is evidently clear from fig.7 again

3. Let $V({}^r\pi_s)$ denote the restriction of the tangent mapping $T({}^r\pi_s)$ to the bundle V_s of vertical vectors tangent to the fibred manifold $Y_s \to Z$. The mapping is is graded with respect to the pair of mappings, $V({}^r\pi_s)$ and ${}^r\mathrm{pr}_s$, over ${}^r\pi_s$ (see fig.8).

$$\begin{tikzcd}
V(Y_s) \arrow[rr, "V({}^r\pi_s)"] \arrow[dr, "\mathsf{is}"] \arrow[dd, "\tau_s"'] & & V(Y_r) \arrow[dr, "\mathsf{is}"] \arrow[dd] & \\
& J_s(V_Z) \arrow[rr, "{}^r\mathrm{pr}_s"] \arrow[dl, "J_s(\tau_Y)"] & & J_r(V_Z) \arrow[dl, "J_r(\tau_Y)"] \\
Y_s \arrow[rr, "{}^r\pi_s"] & & Y_r &
\end{tikzcd}$$

FIGURE 8

Indeed, under the tangent mapping $T({}^r\pi_s)$ the vertical vector $\sigma_{y_s}{}'$ projects onto the vector $\sigma_{y_r}{}'$, tangent to the curve $\sigma_{y_r}\colon t \mapsto j_r\upsilon_t(z_0)$ at the point $y_r = {}^r\pi_s(y_s) = j_r\upsilon(z_0)$, and that vector is identified with the jet $j_r\mathfrak{y}(z_0)$ which is of course the image of $\mathsf{is}(\sigma_{y_s}{}')$ under the projection ${}^r\mathrm{pr}_s$, q.e.d.

Again, if we accept for a moment the slight difference between $\Gamma_{\upsilon_r}\{J_r(V_Z)\}$ and $\Gamma\{\upsilon_r{}^{-1}J_r(V_Z)\}$, the mapping $\mathsf{is}_{\#}^{-1}$ will appear to act upon every $(j_r\mathfrak{y})^{\tilde{}} \in \Gamma\{j_r\upsilon^{-1}J_r(V_Z)\}$. Let $({}^r\pi_s)^V$ stand for the reduction of $V({}^r\pi_s)$ to the base Y_s by means of the reciprocal image functor ${}^r\pi_s{}^{-1}$. According to the general philosophy, we denote by $({}^r\pi_s{}^V)_{\#}$ its action upon the cross-sections of the bundle $\upsilon_s{}^{-1}V_s$ consisting in composing them with $\upsilon_s{}^{-1}({}^r\pi_s{}^V)$, $({}^r\pi_s{}^V)_{\#}\colon \Gamma\{\upsilon_s{}^{-1}(V_s)\} \to \Gamma\{({}^r\pi_s \circ \upsilon_s)^{-1}(V_r)\}$. If $\tilde{\mathfrak{y}}_s \in \Gamma\{\upsilon_s{}^{-1}(V_s)\}$ then $({}^r\pi_s{}^V)_{\#}\tilde{\mathfrak{y}}_s = ({}^r\pi_s{}^V \circ \mathfrak{y}_s)^{\tilde{}} = (V({}^r\pi_s) \circ \mathfrak{y}_s)^{\tilde{}}$. Because of $V({}^r\pi_s) \circ \mathsf{is}^{-1} = \mathsf{is}^{-1} \circ {}^r\mathrm{pr}_s$ we have

$$({}^r\pi_s{}^V)_{\#}\mathsf{is}_{\#}^{-1} = \mathsf{is}_{\#}^{-1}({}^r\mathrm{pr}_s)_{\#}. \tag{A3}$$

APPENDIX 2. DERIVATIONS OVER JET BUNDLES

In this Appendix we recall some very few preliminary properties of differentiation technique in the graded modules over fibred manifolds for the sake of comprehension and also to support several references encountered here and there in the text. An interested Reader may appreciate at least three equivalent but conceptually differing definitions of the operator of total differential each revealing a separate property quoted elsewhere and still all three intrinsic.

1. The fibre differential. Let as usual some vector bundle F be fibred over the base Z by means of the projection ζ, and consider a manifold B, fibred over Z by means of the surmersion π. In Section 2 (formula (7)) we have already defined the fiber differential $\mathbf{d}_\pi\beta \in \Gamma\{(VB)^* \otimes \pi^{-1}F\}$ of a cross-section $\beta \in \Gamma\{\pi^{-1}F\}$. The Lie algebra $\mathfrak{V}_B$ of vertical vector fields on B is a subalgebra in $\mathfrak{X}_B$ and hence acts as an algebra of derivations of the ring $\mathfrak{F}_B$. For a vertical vector field $\mathfrak{v} \in \mathfrak{V}_B$ and a function $f \in \mathfrak{F}_B$ it is obvious that (cf. the exact sequence (4)) $\langle \mathfrak{v}, \mathbf{d}_\pi f\rangle = \langle \iota_{\#}\mathfrak{v}, \mathbf{d}f\rangle$. For some fixed $\mathfrak{v}$ define an $\mathbb{R}$-endomorphism $\mathbf{D}_\pi(\mathfrak{v})$ of the module $\Gamma\{\pi^{-1}F\}$ by the rule $\mathbf{D}_\pi(\mathfrak{v})\beta \doteq \langle \mathfrak{v}, \mathbf{d}_\pi\beta\rangle$. The map $\mathbf{D}_\pi\colon \mathfrak{v} \mapsto \mathbf{D}_\pi(\mathfrak{v})$ is in fact a homomorphism of modules over $\mathfrak{F}_B$. It has the crucial property of $\mathbf{D}_\pi(\mathfrak{v})(f \cdot \beta) = \langle \iota_{\#}\mathfrak{v}, \mathbf{d}f\rangle + f \cdot \mathbf{D}_\pi(\mathfrak{v})\beta$ and thus may be called the law of derivation of the elements of the $\mathfrak{F}_B$-module $\Gamma\{\pi^{-1}F\}$ in the direction of the elements of the algebra $\mathfrak{V}_B$. Exploiting this derivation law the differential $\mathbf{d}_\pi$ is being extended to

$$\begin{tikzcd}
\mathbf{A}(\mathfrak{F}_r \otimes_{\mathfrak{F}_Z} \mathfrak{X}_Z;\, \Phi_r) \arrow[rr, "(\pi_r{}^{-1})^*"] & & \mathbf{A}(\mathfrak{X}_Z;\, \Phi_r) \\
\Phi_r \otimes_{\mathfrak{F}_r} \mathbf{A}(\mathfrak{F}_r \otimes_{\mathfrak{F}_Z} \mathfrak{X}_Z) \arrow[u] & & \Phi_r \otimes_{\mathfrak{F}_Z} \mathbf{A}(\mathfrak{X}_Z) \arrow[ll] \arrow[u]
\end{tikzcd}$$

FIGURE 11

whenever $\beta \in \Omega_r^d(Z)$.

One more way to make operator $\mathbf{d}_t$ act over the whole of the graded module $\mathbf{A}(\mathfrak{H}_r;\Phi_r)$ is to first extend the derivation, defined in $\mathfrak{F}_r$ by $\mathbf{D}_t(\mathfrak{h}) \doteqdot (\mathbf{d}_t f)\,.\,\mathfrak{h}$, $\mathfrak{h} \in \mathfrak{H}_{r+1}(Z)$, to the whole of $\Phi_r \approx \Omega_r^0(Z;\wedge V_r^*)$ engaging the similar procedure as above, and then to extend the total differential $\mathbf{d}_t\colon \Phi_r \to \mathbf{A}^1(\mathfrak{H}_{r+1};\Phi_{r+1})$, defined by $(\mathbf{d}_t\varphi)\,.\,\mathfrak{h} \doteqdot \mathbf{D}_t(\mathfrak{h})\varphi$, $\mathfrak{h} \in \mathfrak{H}_{r+1}$, to the module $\mathbf{A}(\mathfrak{H}_r;\Phi_r)$ through the property $\mathbf{d}_t{}^2 = 0$ together with the property (A5).[3]

Let us compute the total differential of a contraction (see also [5]).

Lemma. *Let $\mathfrak{v} \in \mathfrak{V}$, $\omega \in \Omega_r(Z;V_r^*)$, and let $J_r(\mathfrak{v})$ denote the r^{th}-order prolongation of the vector field $\mathfrak{v}$ by means of prolonging its local one-parametric group. The following formula holds*

$$\mathbf{d}_t\,\langle J_r(\mathfrak{v}),\omega\rangle = \langle J_{r+1}(\mathfrak{v}),\mathbf{d}_t\omega\rangle\,. \tag{A6}$$

We give a brief proof. Locally the algebra $\Omega_s(Z;V_s^*)$ is generated over $\Omega_s(Z)$ by $\Omega_s^0(Z;V_s^*)$. Also $\langle \mathfrak{v}_s, \beta\wedge\varphi\rangle = \beta\wedge\langle\mathfrak{v}_s,\varphi\rangle$ for $\beta\in\Omega_s(Z)$, $\varphi\in\Omega_s^0(Z;V_s^*)$ and $\mathfrak{v}_s\in\mathfrak{V}_s$, so one may restrict oneself to the case $\omega = \mathbf{d}_\pi f$. Let $\beta = \mathbf{d}_t f$. Since $\mathbf{d}_t$ and $\mathbf{d}_\pi$ commute, $\mathbf{d}_t\omega$ equals $\mathbf{d}_\pi\beta$. We compute:

$$\langle J_r(\mathfrak{v}),\mathbf{d}_\pi f\rangle\,(y_r) = (Tf)\big(J_r(\mathfrak{v})\big)(y_r) = (d/dt) f\big(J_r(e^{t\mathfrak{v}})(y_r)\big)(0);$$
$$\langle J_{r+1}(\mathfrak{v}),\mathbf{d}_\pi\beta\rangle_{\sim}\,(y_{r+1}) = (T\underset{\sim}{\beta})\big(J_{r+1}(\mathfrak{v})(y_{r+1})\big) = (d/dt)\underset{\sim}{\beta}\big(J_{r+1}(e^{t\mathfrak{v}})(y_{r+1})\big)(0).$$

We recall that if some $\mathbf{u}\in T_zZ$ with $z=\pi_{r+1}(y_{r+1})$ is tangent to the curve $\sigma_z^{\mathbf{u}}(s)$ and if $y_{r+1} = j_{r+1}v(z)$, then $(\mathbf{d}_t f)_{\sim}\,(y_{r+1})\,.\,\mathbf{u} = (d/ds)(f\circ j_r v\circ\sigma_z^{\mathbf{u}})(0)$. Now take $\langle J_r(\mathfrak{v}),\mathbf{d}_\pi f\rangle$ in place of f to obtain

$$\begin{aligned}(\mathbf{d}_t\,\langle J_r(\mathfrak{v}),\mathbf{d}_\pi f\rangle)\,.\,(y_{r+1},\mathbf{u}) &= (\mathbf{d}_t\,\langle J_r(\mathfrak{v}),\mathbf{d}_\pi f\rangle)_{\sim}\,(y_{r+1})\,.\,\mathbf{u} = (d/ds)\,(\langle J_r(\mathfrak{v}),\mathbf{d}_\pi f\rangle\circ j_r v\circ\sigma_z^{\mathbf{u}})\,(0)\\ &= (d/ds)\,(d/dt)\,f\circ J_r(e^{t\mathfrak{v}})\circ j_r v\circ\sigma_z^{\mathbf{u}}(s)\big|_{s=t=0}\,.\end{aligned}$$

On the other hand,

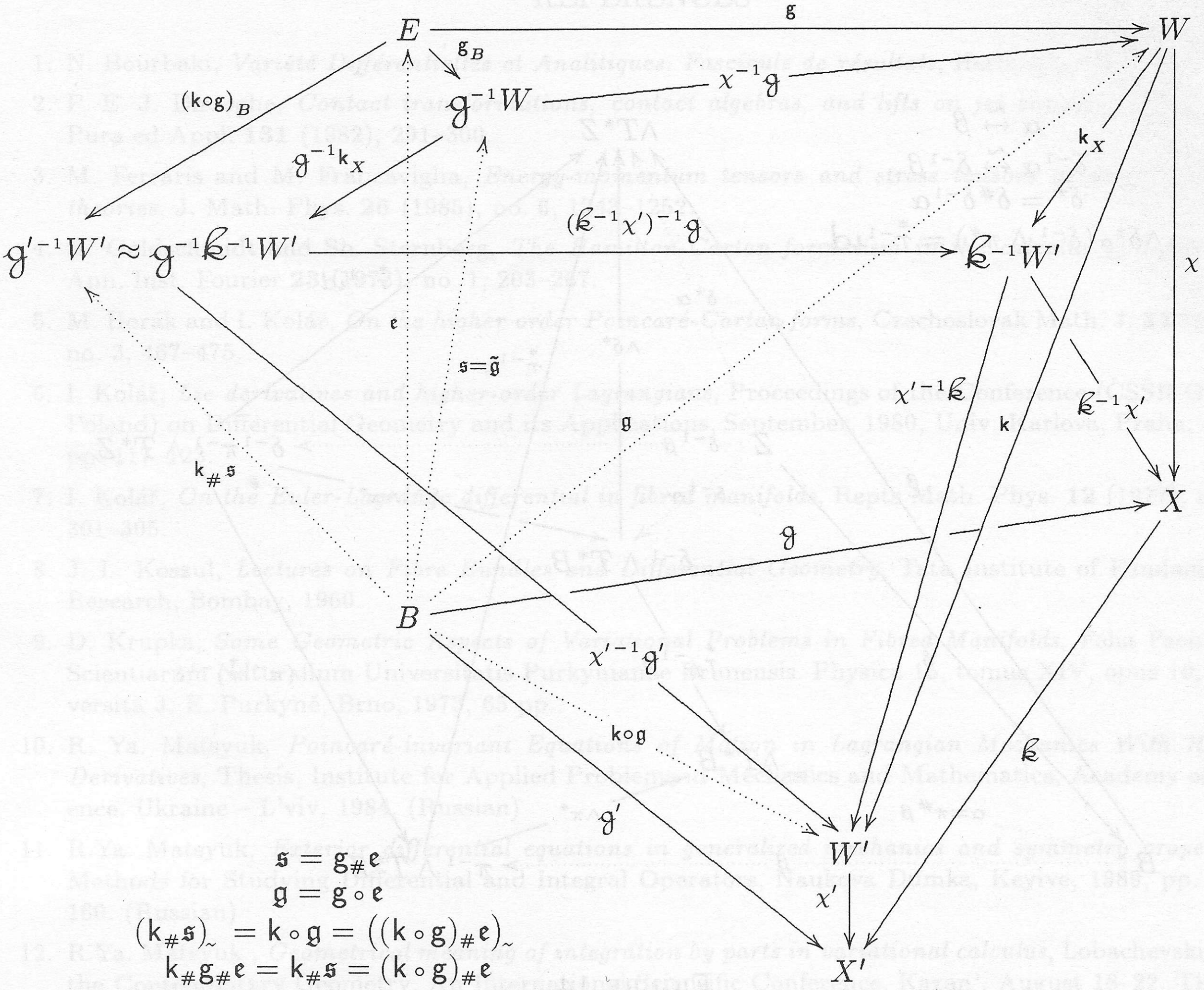
$$\langle J_{r+1}(\mathfrak{v}),\mathbf{d}_\pi\mathbf{d}_t f\rangle\,.\,(y_{r+1},\mathbf{u}) = \langle J_{r+1}(\mathfrak{v}),\mathbf{d}_\pi\beta\rangle_{\sim}\,(y_{r+1})\,.\,\mathbf{u} = (d/dt)\big((\mathbf{d}_t f)_{\sim}\,(J_{r+1}(e^{t\mathfrak{v}})(y_{r+1}))\,.\,\mathbf{u}\big)(0).$$

At this stage it is necessary to put in the property of the prolongation procedure, namely, $J_{r+1}(e^{t\mathfrak{v}})\circ j_{r+1}v = j_{r+1}(e^{t\mathfrak{v}}\circ v)$, in order to arrive at

$$\begin{aligned}\langle J_{r+1}(\mathfrak{v}),\mathbf{d}_\pi\mathbf{d}_t f\rangle\,.\,(y_{r+1},\mathbf{u}) &= (d/dt)\,(d/ds)\,f\circ j_r(e^{t\mathfrak{v}}\circ v)\circ\sigma_z^{\mathbf{u}}(s)\big|_{t=s=0}\\ &= (d/dt)\,(d/ds)\,f\circ J_r(e^{t\mathfrak{v}})\circ j_r v\circ\sigma_z^{\mathbf{u}}(s)\big|_{t=s=0}\,.\end{aligned}$$

Thus both sides of (A6) when evaluated at arbitrary $(y_{r+1},\mathbf{u})\in\pi_{r+1}{}^{-1}TZ$ provide one and the same expression, q.e.d.

[3] The bigraded algebra $\mathbf{A}(\mathfrak{H}_{r-1};\Phi_{r-1})\approx\Omega_{r-1}(Z;\wedge V_{r-1}^*)$ may be converted into exterior one by applying the dual of the Cartan contact form $T(Y_r)\to{}^{r-1}\pi_r{}^{-1}(V_{r-1})$ with subsequent alternation. To the operators $\mathbf{d}_\pi$ and $\mathbf{d}_t$ defined in $\Omega_{r-1}(Z;\wedge V_{r-1}^*)$ correspond under this conversion the operators $\mathbf{d}_V$ and $\mathbf{d}_H$ of TULCZYJEW [16] defined in $\Omega(Y_r)$.



$$\mathfrak{s} = \mathfrak{g}_{\#}\mathfrak{e}$$
$$\mathfrak{g} = \mathfrak{g}\circ\mathfrak{e}$$
$$(\mathsf{k}_{\#}\mathfrak{s})_{\sim} = \mathsf{k}\circ\mathfrak{g} = ((\mathsf{k}\circ\mathfrak{g})_{\#}\mathfrak{e})_{\sim}$$
$$\mathsf{k}_{\#}\mathfrak{g}_{\#}\mathfrak{e} = \mathsf{k}_{\#}\mathfrak{s} = (\mathsf{k}\circ\mathfrak{g})_{\#}\mathfrak{e}$$

FIGURE 12

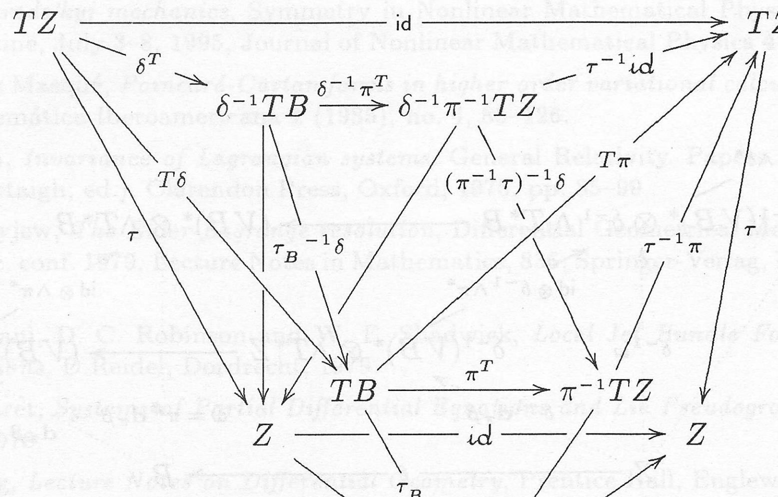


FIGURE 13

This is the complete picture underlying that of fig.5

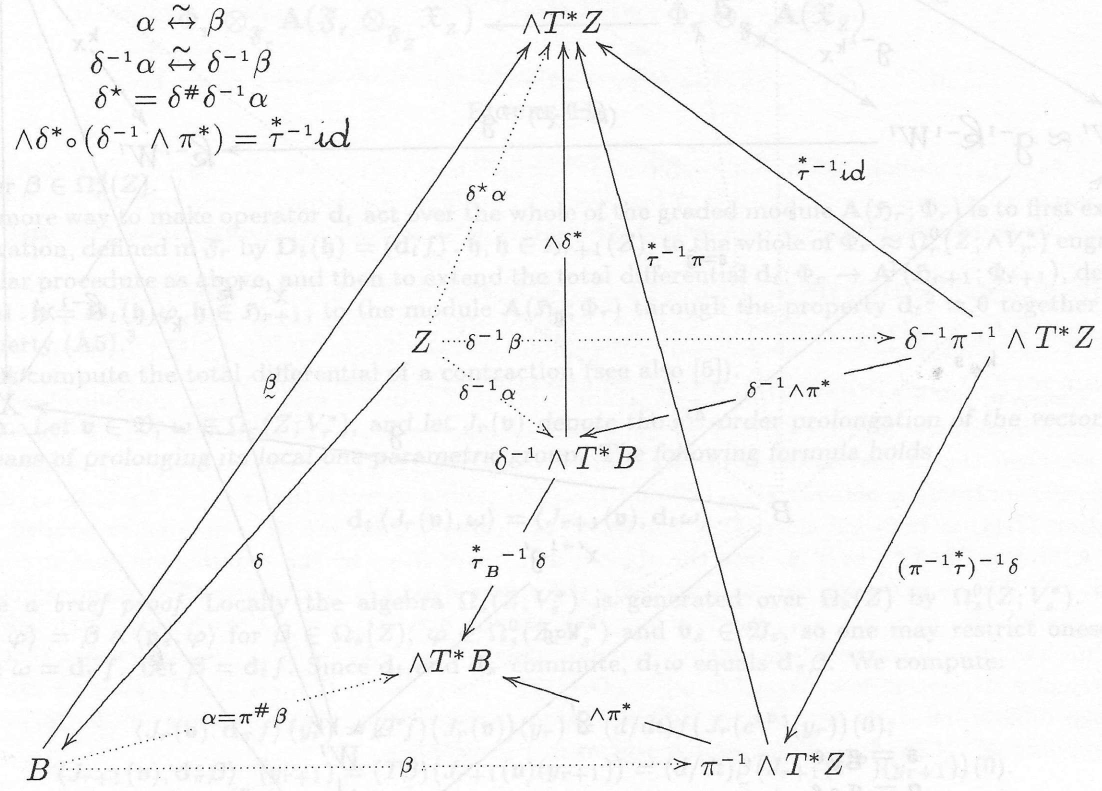


FIGURE 14

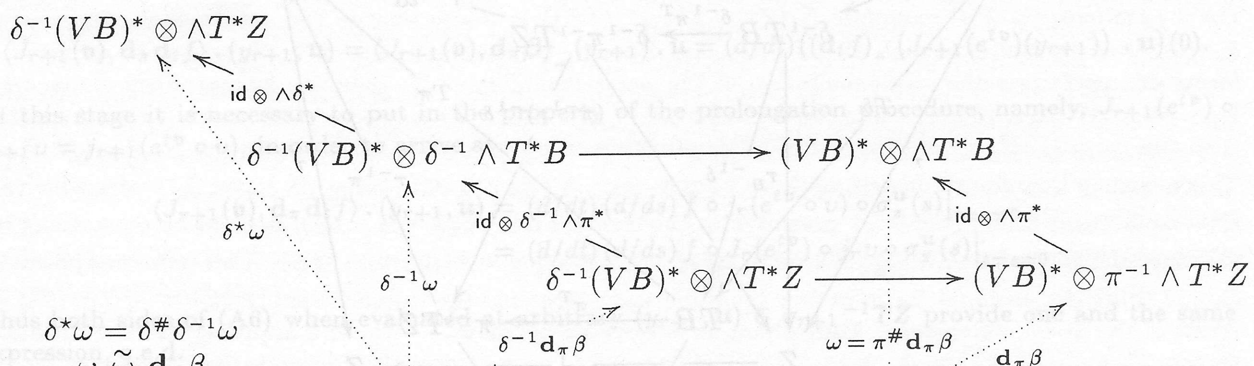


FIGURE 15

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Institute for Applied Problems in Mechanics and Mathematics, L'viv

matsyuk@lms.lviv.ua

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