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∞-CONVEX SETS AND THEIR APPLICATIONS TO THE PROOF OF CERTAIN CLASSICAL THEOREMS OF FUNCTIONAL ANALYSIS

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Using the notion of an ∞-convex set we present a new geometric proof of two classical theorems of functional analysis: Banach Open Mapping Principle and Banach-Mazur Theorem on quotients of l_1.

Т. Банах, В.Е. Лянце, Я.В. Микитюк. ∞-*Выпуклые множества и их приложения к доказательству некоторых классических теорем функциональнього анализа* // Математичнi Студiї. – 1999. – Т.11, № 1. – C.83–84.

На основе понятия ∞-выпуклого множества найдено новое геометрическое доказательство двух классических теорем функционального анализа: теоремы Банаха об открытом операторе и теоремы Банаха-Мазура о фактор-пространствах l_1.

Each lecturer teaching the standard course of functional analysis is familiar with difficulties accompanying students trying to conceive the proof of the fundamental Banach Open Mapping Principle. The aim of this note is to recover a simple geometric idea hidden in the standard proof of that Principle and present a transparent geometric proof which seems to be more accessible for students. Our approach is based on the geometric notion of an ∞-convex set. The idea to apply this concept to the proof of Banach Open Mapping Principle belongs to the third author.

A bounded subset C of a Banach space X is called ∞-*convex*, provided C coincides with its ∞-*convex hull*

$$\operatorname{conv}_\infty(C)=\Big\{\sum_{n=1}^\infty \lambda_n x_n:\ x_n\in C,\ \lambda_n\ge 0,\ \sum_{n=1}^\infty\lambda_n=1\Big\}.$$

∞-Convex sets share many properties of usual convex sets. In particular, the following their properties can be established immediately.

(1) *Any open ball in a Banach space is an ∞-convex set;*
(2) *The intersection of bounded ∞-convex sets is an ∞-convex set;*
(3) *For any bounded linear operator $T: X\to Y$ between Banach spaces the image $T(C)$ of any bounded ∞-convex set $C\subset X$ is an ∞-convex set in Y.*

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However there exists a property, specific for ∞-convex sets.

Static Property. *If C is a dense ∞-convex subset of an open ball B of a Banach space X, then $C = B$. In other words, each point $x \in B$ is the center of mass for their suitable distrubution at points of C.*

Proof. Without loss of generality we can suppose that B is the open unit ball $\{x \in X : \ \|x\| < 1\}$. Take an arbitrary $x \in B$ and find a real λ_1 such that $\|x\| < \lambda_1 < 1$. Find also positive reals $\lambda_2, \lambda_3, \dots$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Since C is dense in B and $\frac{1}{\lambda_1} x \in B$, there exists $x_1 \in C$ such that $\|x - \lambda_1 x_1\| < \lambda_2$. Since $\frac{1}{\lambda_2}(x - \lambda_1 x_1) \in B$, there exists $x_2 \in C$ such that $\|x - \lambda_1 x_1 - \lambda_2 x_2\| < \lambda_3$. By induction, for any $n \in \mathbb{N}$ we find points $x_1, \dots, x_n \in \mathbb{N}$ such that $\|x - \lambda_1 x_1 - \dots - \lambda_n x_n\| < \lambda_{n+1}$. Therefore $x = \sum_{n=1}^{\infty} \lambda_n x_n$, i.e. $x \in \mathrm{conv}_{\infty}(C) = C$. $\square$

Now the proof of the Banach Open Mapping Principle becomes quite easy.

Banach Open Mapping Principle. *Any surjective bounded linear operator $T : X \to Y$ between Banach space is open.*

Proof. Let B_X and B_Y denote the open unit balls of the Banach spaces X and Y, respectively. It suffices to show that the image $T(B_X)$ contains a neighborhood of the origin in Y. Because $Y = T(X) = \bigcup_{n=1}^{\infty} T(nB_X)$, we may apply Baire Category Theorem to find $n \in \mathbb{N}$ such that the closure of $T(nB_X)$ in Y contains an ε-ball $y + \varepsilon B_Y$ around some point $y \in Y$, i.e., the intersection $(y + \varepsilon B_Y) \cap T(nB_X)$ is dense in $y + \varepsilon B_Y$. By the properties (1)–(3), the set $(y + \varepsilon B_Y) \cap T(nB_X)$ is ∞-convex and by the Static Property, $(y + \varepsilon B_Y) \cap T(nB_X) = y + \varepsilon B_Y$. Thus the image $T(nB_X)$ contains the ball $y + \varepsilon B_Y$. Observe that $T(2nB_X) \supset T(nB_X - nB_X) \supset \varepsilon B_Y$ and hence $T(B_X)$ contains the neighborhood $\frac{\varepsilon}{2n} B_Y$ of the origin. $\square$

∞-Convex sets can also be applied to simplify the proof of another classical result.

Banach-Mazur Theorem. *Every separable Banach space X is isomorphic to a quotient space of l_1.*

Proof. It suffices to construct a surjective linear bounded operator $T : l_1 \to X$. Fix a countable dense subset $\{b_n\}_{n=1}^{\infty}$ in the open unit ball B of X and define the operator $T : l_1 \to X$ by $T((\lambda_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \lambda_n b_n$ for $(\lambda_n)_{n=1}^{\infty} \in l_1$. One can easily check that the operator T is well-defined, linear and continuous. Denoting by B_1 the open unit ball of the Banach space l_1, we see that the ∞-convex set $T(B_1)$ is dense in the ball B and thus coincides with B according to the Static Property. This implies that the operator T is "onto". $\square$

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