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NONSTANDARD STURM-LIOUVILLE DIFFERENCE OPERATOR, II

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This article extends the investigation with the same title printed in the previous volume of "Mat. studii". We consider the spectrum and spectral expansion of a nonstandard ordinary singular Sturm-Liouville operator L in finite differences. The resolvent of L is considered and its norm is estimated.

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Эта статья продолжает исследование с таким же названием, напечатанное в предыдущем томе "Мат. студий". Рассматриваются спектр и спектральное разложение нестандартного обыкновенного сингулярного оператора Штурма-Лиувилля L в конечных разностях. Рассматривается резольвента L и оценивается ее норма.

This paper is the continuation of the paper "Nonstandard Sturm-Liouville difference operator" (see [16]).[1]

6. Relative standardness. To vavy notions of standardness we use different norms.

6.1. Definition. For $r > 0$ we denote by H_r the Banach space of functions $x \in \mathbb{C}^T$ corresponding to

$$| x |_r := \max_{t \in T} |x(t)| r^{-t}. \tag{6.1}$$

Also by $\mathbf{H}_r$ we denote the Banach space of functions $\xi \in \mathbb{C}^{\mathbb{N}}$ corresponding to

$$| \xi |_r := \sup_{t \in \mathbb{N}} |\xi(t)| r^{-t}.$$

In the sequel we suppose that r is standard, therefore $\mathbf{H}_r$ is standard, too. As before, $\Pi\xi$ denotes the restriction of $\xi \in \mathbb{C}^{\mathbb{N}}$ to T.

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6.2. Definition. A function $x \in \mathbb{C}^T$ is said to be *r–standard* and we write $x \in {}^{nst}H_r$ iff $x = \Pi\xi$ for some $\xi \in {}^{st}\mathbf{H}_r$. Evidently, such ξ is unique. It is called the *standardized image* of x and denoted by ${}^\bullet x$.

6.3. Definition. A function $x \in \mathbb{C}^T$ is said to be *r–nearstandard* and we write $x \in {}^{nst}H_r$ iff $(\exists y \in {}^{st}H_r)$ $(| x - y\|_r \approx 0)$. Such y is denoted by ${}^\circ x$ and called the *shadow* of x. The standardized image of ${}^\circ x$ is denoted by ${}^\bullet x$ and called the *shadow* of x in $\mathbf{H}_r$. Thus ${}^\bullet x \in {}^{st}\mathbf{H}_r$, ${}^\circ x = \Pi{}^\bullet x$, $| x - {}^\circ x |_r \approx 0$.

6.4. Remark. It is easy to see that definitions 6.2, 6.3 are co-ordinated with each other and with previous definitions. For instance, let $x \in {}^{nst}H$ and $\xi = {}^\bullet x$ =the shadow of x in $\mathbf{H}$. Then $\xi \in {}^{st}\mathbf{H} = {}^{st}\ell_2(\mathbb{N})$ and therefore $\|\xi\| \ll \infty$. Hence $\forall r \geq 1$ $\xi \in {}^{st}\mathbf{H}_r$ and $\|x - \Pi\xi\| \approx 0$ implies $\|x - \Pi\xi\|_r \approx 0$. We see that the same ξ is the shadow of x in $\mathbf{H}$ and in $\mathbf{H}_r$, $r \geq 1$.

Denote by $\Xi(\cdot, \rho)$ the solution of the equation $(\mathbf{l} - \lambda)\Xi = 0$ (see (4.20)), which takes the same initial values $\Xi(0, \rho) = 0$, $\Xi(1, \rho) = \rho - \rho^{-1}$ as the solution $Z(\cdot, \rho)$ of the equation $(\ell - \lambda)Z = 0$, $\lambda = -\frac{1}{2}(\rho + \rho^{-1})$.

6.5. Proposition. *Let $r \geq 1$ be standard. For any $\rho \in W$ (see (5.6)), if $|\rho| < r$, then the function $Z(\cdot, \rho)$ is r–nearstandard and*

$$^\bullet Z(\cdot, \rho) = \Xi(\cdot, {}^\circ\rho). \tag{6.3}$$

Proof. From (3.5) and (3.6) we see that

$$Z_r := | Z(\cdot, \rho) |_r \ll \infty. \tag{6.4}$$

To prove $| Z(\cdot, \rho) - \Pi\Xi(\cdot, {}^\circ\rho) |_r \approx 0$, we note that $\Xi(\cdot, \rho)$ is a solution of the "integral" equation (compare with (3.4))

$$\Xi(t, \rho) = \frac{2}{\rho - \rho^{-1}} \sum_{u=1}^{t-1} b(u) Z_0(t - u, \rho) + Z_0(t, \rho), \quad t \geq 2. \tag{6.5}$$

Put $Z(t) := |Z(t, \rho) - \Pi\Xi(t, \rho)|$, $t \in T$. Then (3.4) and (6.5) yield

$$Z(t) \leq \frac{2}{|\rho - \rho^{-1}|} \sum_{u=1}^{t-1} |b(u) Z(u) Z_0(t - u, \rho)| + \\ + \frac{2}{|\rho - \rho^{-1}|} \sum_{u=1}^{t-1} |a(u) - b(u)||Z(u, \rho)|Z_0(t - u, \rho)|.$$

By (6.4), $\forall t \in T$ $|Z(t, \rho)| \leq Z_r r^t$. Therefore,

$$Z(t) \leq \frac{2}{|\rho - \rho^{-1}|} \sum_{u=1}^{t-1} |b(u)| Z(u) Z_r r^{t-u} + \frac{2}{|\rho - \rho^{-1}|} \|a - b\|_1 Z_r^2 r^t.$$

Denote $E(t) := Z(t) r^{-t}$ and note that $| Z(\cdot, \rho) - \Pi\Xi(\cdot, {}^\circ\rho) |_r \leq \|E(\cdot)\|_\infty$. We have $E(t) \leq \frac{2}{|\rho - \rho^{-1}|} \sum_{u=1}^{t-1} |b(u)| E(u) + \gamma$, where $\gamma := \frac{2}{|\rho - \rho^{-1}|} Z_r^2 \|a - b\|_1 \approx 0$. Since

$E(t+1)-E(t) = \frac{2}{|\rho-\rho^{-1}|} Z_r |b(t)| E(t)$, we obtain $E(t+1) \le \gamma \cdot exp \frac{2}{|\rho-\rho^{-1}|} Z_r \|b\|_1 \approx 0$. □

6.6. Remark. Let $\rho \in W$ (see (5.6)) be such that $|e(\rho)| \gg 0$. Then $\forall^{st} r \ge \rho$

$$\hat{e}(\cdot,\rho) \in {}^{nst}H_r. \tag{6.6}$$

This follows from 6.5 and (5.26). Using equation (5.21) we can prove that ${}^{\bullet}\hat{e}(\cdot,\rho)$ is a solution of the equation $(\ell-\lambda)\hat{\varepsilon} = 0$, for which

$$\hat{\varepsilon}(t,\rho) = \rho^t[1+o(1)], \ \ |t| \to \infty \tag{6.7}$$

uniformly relative $|\rho| > 1+\alpha$, for every $\alpha > 0$.

Let us consider also c_0-standardness. By c_0 we denote the Banach space of functions $\xi \in \mathbb{C}^{\mathbb{N}}$ for which $\lim_{t\to\infty} \xi(t) = 0$, with the norm $\|\xi\|_\infty := \max_{t\in\mathbb{N}} |\xi(t)|$.

6.7. Definition. A function $x \in \mathbb{C}^T$ is said to be c_0-*standard* iff $\exists \xi \in {}^{st}c_0 \ x = \Pi\xi$. Such ξ is called the *standardized image* of x and we write $x \in {}^{c_0 st}\mathbb{C}^T$, $\xi = {}^{\bullet}x$. A function $x \in \mathbb{C}^T$ is said to be c_0-*nearstandard* and we write $x \in {}^{c_0 nst}\mathbb{C}^T$ iff $\exists y \in {}^{c_0 st}\mathbb{C}^T$

$\|x-y\|_\infty \approx 0$. Such y is denoted by ${}^\circ x$ and called the *shadow* of x. The standardized image of ${}^\circ x$ is denoted by ${}^{\bullet}x$ and called the *shadow* of x in c_0.

6.8. Proposition. *A function $x \in \mathbb{C}^T$ is c_0-nearstandard iff*

$$\|x\|_\infty \ll \infty \text{ and } \forall t \approx \infty \ \ x(t) \approx 0. \tag{6.8}$$

Proof. ($\Rightarrow$) Let $\xi = {}^{\bullet}x$, i.e. $\xi \in {}^{st}c_0$ and $\|x - \Pi\xi\|_\infty \approx 0$. Then $\|x\|_\infty \le \|\xi\|_\infty + 1 \ll \infty$ and $\forall t \approx \infty \ \ x(t) \approx \xi(t) \approx 0$.

($\Leftarrow$) Let ξ be the standard extension to $\mathbb{N}$ of $({}^\circ[x(t)])_{t\in{}^{st}\mathbb{N}}$. Suppose that the sequence $(\xi(t))_{t\in\mathbb{N}}$ has a point of accumulation $\gamma \ne 0$. Then there exists a strongly increasing sequence $(t_n)_{n\in\mathbb{N}}$ of numbers $t_n \in \mathbb{N}$ such that $\lim_{n\to\infty} \xi(t_n) = \gamma$. By transfer, we may assume that γ and $(t_n)_{n\in\mathbb{N}}$ are standard. Therefore $\forall n \approx \infty$ $t_n \approx \infty$ and $\xi(t_n) \approx \gamma \gg 0$. This is absurd, for $\forall t \approx \infty \ \xi(t) \approx x(t) \approx 0$. □

Propositions 5.5 and 6.8 imply the following assertion.

6.9. Proposition. *Let $\rho \in W_m$ (see (5.12)), then the function $T \ni t \mapsto e(t,\rho) - \rho^{-t}$ is c_0-nearstandard.*

7. Outside eigenvalues

7. Outside eigenvalues. So we call eigenvalues $\lambda = -\frac{1}{2}(\rho+\rho^{-1})$ of the operator L for which $|\rho| \gg 1$ (that is, for which the distance from λ to the segment $[-1,+1] \subset \mathbb{C}$ is an appreciable real number).

7.1. Theorem. *Let λ be an outside eigenvalue of the operator L. Then the eigenfunction $Z(\cdot,\rho)$ corresponding to λ is $\|\cdot\|$-nearstandard. This means that $Z(\cdot,\rho)$ is square integrable:*

$$\|Z(\cdot,\rho)\| \ll \infty \text{ and } \forall t \approx \infty \ \sum_{u=t+1}^{m} |Z(u,\rho)|^2 \approx 0. \tag{7.1}$$

The shadow $^\circ\lambda$ *of* λ *is an eigenvalue of the operator* $\mathbf{L} := {}^\bullet L$ *(see (4.20), (4.21); compare with 6.5) and*

$$^\bullet Z(\cdot,\rho) = \Xi(\cdot,{}^\circ\rho). \tag{7.2}$$

Proof. Up to a constant factor, a unique eigenfunction of L relative to λ is $Z(\cdot,\rho)$. (This means that the geometric multiplicity of any eigenvalue of L is equal to 1). Since $Z(m+1,\rho)=0$, formula (5.26) may be rewritten as follows

$$Z(t,\rho) = \hat{e}(\rho)[f(t,\rho) - e(t,\rho)], \tag{7.3}$$

where

$$f(t,\rho) := \frac{\hat{e}(t,\rho)}{\rho^{m+1}\hat{e}(m+1,\rho)}. \tag{7.3$'$}$$

(We recall that $e(m+1,\rho) = \rho^{-m-1}$.) In view of 5.9, 5.10, $|\hat{e}(\rho)| \ll \infty$ and in virtue of 5.3 $e(\cdot,\rho) \in {}^{nst}H$. By (5.23), there exists a $C \in \mathbb{R}$, $0 < C \ll \infty$, such that $|\hat{e}(t,\rho)| \le C|\rho|^t$. Moreover, $\rho^{-m-1}\hat{e}(m+1,\rho) \approx 1$. Hence $|f(t,\rho)| \le C|\rho|^{-(2m+2-t)}$ and $f(\cdot,\rho)$ is nearstandard trivially: $\|f(\cdot,\rho)\| \approx 0$.

Note that

$$\|L_0\| = 1, \quad \|L\| \le 1 + \|a\|_\infty. \tag{7.4}$$

Since $|\lambda| \le \|L\|$, we have $|\lambda| \ll \infty$, $|\rho| \ll \infty$, hence $^\circ\lambda$, $^\circ\rho$ are determinated. By Remark 4.13, $^\bullet[(L-\lambda)Z(\cdot,\rho)] = ({}^\bullet L - {}^\circ\lambda){}^\bullet Z(\cdot,{}^\circ\rho)$. For $^\bullet L = \mathbf{L}$, $^\bullet Z(\cdot,\rho) = \Xi(\cdot,{}^\circ\rho)$, and
$(L-\lambda)Z(\cdot,\rho) = 0$, we get $(\mathbf{L} - {}^\circ\lambda)\Xi(\cdot,{}^\circ\rho) = 0$. □

Let λ be some eigenvalue of L. If $n \in \mathbb{N}$ and $(L-\lambda)^n x = 0$, where $x \in \mathbb{C}^T \setminus \{0\}$, then x is called a *fundamental* function of L, belonging to λ. Theorem 7.1 can be extended to fundamental functions. We begin with

7.2. Lemma. *Let* $k \in {}^{nst}H$ *and*

$$\forall x \in H \;\; \forall t \in T \;\; Kx(t) := \sum_{u=1}^{t} k(t-u)x(u). \tag{7.5}$$

Then $K \in {}^{nst}\mathcal{B}(H)$ *and*

$$\forall \xi \in \mathbf{H} \;\; \forall t \in \mathbb{N} \;\; ({}^\bullet K)\xi(t) = \sum_{u=1}^{t} ({}^\bullet k)(t-u)\xi(u). \tag{7.6}$$

Proof. Denote $\forall \xi \in \mathbf{H}$ $\forall t \in \mathbb{N}$ $\mathbf{K}\xi(t) := \sum_{u=1}^t ({}^\bullet k)(t-u)\xi(u)$. Then $\mathbf{K} \in {}^{st}\mathcal{B}(\mathbf{H})$. Let $x \in H$, $t \in T$, then $Kx(t) - \Pi\mathbf{K}Qx(t) = \sum_{u=1}^t [k(t-u) - {}^\bullet k(t-u)]x(u)$. Hence $\|(K - \Pi\mathbf{K}Q)x\| \le \|k - \Pi({}^\bullet k)\| \cdot \|x\|$. Since $\|k - \Pi({}^\bullet k)\| \approx 0$, we get $\|K - \Pi\mathbf{K}Q\| \approx 0$.
□

Denote by $s(\cdot,\lambda)$ and $\sigma(\cdot,\lambda)$ respectively solutions of equations $(\ell-\lambda)s = 0$ and $(\mathbf{l}-\lambda)\sigma = 0$ with initial values

$$s(0,\lambda) = \sigma(0,\lambda) = 0, \quad s(1,\lambda) = \sigma(1,\lambda) = 1. \tag{7.7}$$

Obviously, for $\lambda = -\frac{1}{2}(\rho+\rho^{-1})$ we have

$$s(t,\lambda) = \frac{1}{\rho-\rho^{-1}} Z(t,\rho), \quad \sigma(t,\lambda) = \frac{1}{\rho-\rho^{-1}} \Xi(t,\rho). \tag{7.8}$$

Put

$$s_k := \left(\frac{d}{d\zeta}\right)^k s(\cdot,\zeta)_{\zeta=\lambda}, \;\; \sigma_k := \left(\frac{d}{d\zeta}\right)^k \sigma(\cdot,\zeta)_{\zeta={}^\circ\lambda}. \tag{7.9}$$

7.3. Theorem. *Let $\lambda = -\frac{1}{2}(\rho+\rho^{-1})$ be an outside eigenvalue of L. Denote by H_λ the space of fundamental functions of L, belonging to λ. Then*

$$n_\lambda := \dim H_\lambda \ll \infty \tag{7.10}$$

(recall that n_λ is the algebraic multiplicity of λ) and

$$(\forall x \in H_\lambda)\;\; (\|x\| \ll \infty \implies x \in {}^{nst}H). \tag{7.11}$$

Moreover, n_λ coincides with the multiplicity of λ as a root of the equation $s(m+1,\lambda)=0$ (and with the multiplicity of ρ as a root of the equation $Z(m+1,\rho)=0$). Besides, $(s_0,\dots,s_{n_\lambda-1})$ is a total chain of fundamental functions, belonging to λ (that is, a basis of H_λ). At last,

$$^\bullet s_k = \sigma_k, \quad k = 0,\dots,n_\lambda - 1. \tag{7.12}$$

Proof. By Leibnitz's rule $(\ell-\lambda)s_1 = s_0$, $(\ell-\lambda)s_2 = s_1$, ... Functions $s_0, s_1, \dots$ take initial values $s_0(0) = s_1(0) = \cdots = 0$. So $(L-\lambda)s_0 = 0$, $(L-\lambda)^2 s_1 = 0$, $(L-\lambda)^3 s_2 = 0$, ... iff $s_0(m+1)=0$, $s_1(m+1)=0$, $s_2(m+2)=0$, ... Hence in view of (7.9), $s_0,\dots,s_{n-1}$ is a total chain iff the multiplicity of λ as a root of the equation $s(m+1,\lambda)=0$ is equal to n.

Note that the solution of the equation $(\ell-\lambda)x = f$ which takes the initial value $x(0)=0$ can be represented in the following form (compare with (3.2)):

$$x(t) = -2\sum_{u=1}^{t-1} f(u)s(t-u,\lambda) + x(1)s(t,\lambda). \tag{7.13}$$

Hence, by Lemma 7.2, if $f \in {}^{nst}H$, then $x \in {}^{nst}H$. Indeed, for Theorem 7.1, $s(\cdot,\lambda) \in {}^{nst}H$. In view of $s(1,\zeta)\equiv 1$ we have $s_k(1) = 0$ for $k \geq 1$. By $(\ell-\lambda)s_k = s_{k-1}$ and (7.13),

$$s_k(t) = -2\sum_{u=1}^{t-1} s_{k-1}(u)s(t-u,\lambda). \tag{7.13$'$}$$

Let $n \in {}^{st}\mathbb{N}$ and $n \leq n_\lambda$. By the (external) induction, from (6.12′) and Lemma 7.2 we obtain

$$s_0,\dots,s_{n-1} \in {}^{nst}H. \tag{7.14}$$

Note that $\forall k \in {}^{st}\mathbb{N}\; \forall t \in {}^{st}\mathbb{N}\; ({}^\bullet s_k)(t) \approx s_k(t)$. Since $s_k(1)=0$ and ${}^\bullet s_k$ is standard, we have ${}^\bullet s_k(1) = 0$, $k \geq 1$. Since $(L-\lambda)s_1 = s_0$ and ${}^\bullet s_0 = \sigma(\cdot,{}^\circ\lambda)$ (see (7.2)), in virtue of 4.13 we have $(\mathbf{L}-{}^\circ\lambda)^\bullet s_1=\sigma(\cdot,{}^\circ\lambda)$. This implies (7.12) for $k=1$. By the (external) finite induction one can obtain it for $k>1$, $k\ll\infty$. Denote by $\mathbf{H}_\mu$ the space of fundamental functions of L relative to an eigenvalue μ of $\mathbf{L}$. It is known (see [15]) that for any eigenvalue μ of $\mathbf{L}$ one has $\dim H_\mu<\infty$. Hence $\mathbf{L}$ is standard, $\dim\mathbf{H}_\mu\ll\infty$. All this leads us to the conclusion, that is

$$\dim H_\lambda\le\dim\mathbf{H}_{\circ\lambda}\ll\infty. \tag{7.15}$$

8. Resolvent. Let ζ be a regular value of L. Since $\dim H\in\mathbb{N}$, $L-\zeta$ is an isomorphism $H\to H$. The relation

$$R(\zeta)y(t):=(L-\zeta)^{-1}y(t)=\sum_{u=1}^{m}R(t,u,\zeta)y(u),\ t\in T,\ y\in H, \tag{8.1}$$

uniquely determines the matrix $R(\cdot,\cdot,\zeta)$. We call it the *Green function* of L. Let us express this function by solutions of the equation $(\ell-\zeta)x=0$. Besides the solution $s(\cdot,\zeta)$, which satisfies initial conditions $s(0,\zeta)=0$, $s(1,\zeta)=0$, we use also a solution $v(\cdot,\zeta)$ with initial values at the right end of T, $v(m+1,\zeta)=0$, $v(m,\zeta)=c$. It turns out that the constant c may be chosen so that $v(\cdot,\zeta)$ and $e(\cdot,\rho)$ are infinitely close with each other.

8.1. Proposition. *Let $v(\cdot,\zeta)$ be a solution of the equation $\ell v=\zeta v$ such that*

$$v(m+1,\zeta)=0,\quad v(m,\zeta)=\rho^{-m}-\rho^{-m-2}, \tag{8.2}$$

where

$$\zeta=-\frac{1}{2}(\rho+\rho^{-1}),\quad |\rho|\gg1. \tag{8.3}$$

Then

$$\|v(\cdot,\zeta)-e(\cdot,\zeta)\|\approx0. \tag{8.4}$$

Proof. It is easy to verify that

$$\forall t\in T\ v(t,\zeta)=\rho^{-m-1}\hat e(m+1,\rho)e(t,\rho)-\rho^{-2m-2}\hat e(t,\rho). \tag{8.5}$$

By 5.9 and 5.10, we have $\forall t\in T\ |\hat e(t,\rho)|\le C_{|\rho|}|\rho|^t$, where $C_{|\rho|}\ll\infty$. Hence

$$\|\rho^{-2m-2}\hat e(\cdot,\rho)\|\le C_{|\rho|}|\rho|^{-2m-2}[|\rho|^2+\cdots+|\rho|^{2m}]^{1/2}\approx0.$$

Besides, by (5.21), $\rho^{-m-1}\hat e(m+1,\rho)\approx1$. Hence,

$$\forall t\in T\quad v(t,\zeta)\approx e(t,\rho). \tag{8.6}$$

From (8.5) we see also that $v(\cdot,\zeta)\in{}^{nst}H$ and that (8.4) is true. □

8.2. Remark. Let

$$v(\zeta):=v(t,\zeta)s(t+1,\zeta)-v(t+1,\zeta)s(t,\zeta). \tag{8.7}$$

$v(\zeta)$ is the Wronskian of solutions of the equation $\ell x=\zeta x$, therefore it is not depending upon t. If we put in the first place $t=0$ and secondly $t=m$, we obtain

$$v(\zeta)=v(0,\zeta)=(\rho^{-m}-\rho^{-m-2})s(m+1,\zeta)=\rho^{-m-1}Z(m+1,\rho). \tag{8.8}$$

Then for $\zeta\ne\pm1$

$$v(\zeta)=0\Leftrightarrow s(m+1,\zeta)=0\Leftrightarrow Z(m+1,\zeta)=0. \tag{8.9}$$

Therefore $v(\zeta)=0$ iff λ is an eigenvalue of L. If ζ is a regular value of L, functions $v(\cdot,\zeta)$ and $s(\cdot,\zeta)$ form a fundamental system of solutions of the equation $\ell x=\zeta x$.

8.3. Proposition. *Let $\zeta \in \mathbb{C}$ be some regular value of L. Then $\forall t \in T$, $\forall u \in T$*

$$R(t,u,\zeta) = \begin{cases} -\frac{2}{v(\zeta)} s(t,\zeta)v(u,\zeta) & \text{for } t \le u, \\ -\frac{2}{v(\zeta)} s(u,\zeta)v(t,\zeta) & \text{for } t \ge u. \end{cases} \tag{8.10}$$

In particular, $R(\cdot,\cdot,\zeta)$ is symmetric (but not Hermitian):

$$R(t,u,\zeta) = R(u,t,\zeta). \tag{8.11}$$

Proof. From (8.1) we see that $R(\cdot,u,\zeta) = R(\zeta)\delta_u$, where δ_u is the Kronecker delta concentrated at the point u. Therefore,

$$(\ell_t - \zeta)R(t,u,\zeta) = \delta_u(t), \quad R(v,u,\zeta) = R(m+1,\zeta) = 0. \tag{8.12}$$

The first equation shows that

$$R(t,u,\zeta) = \begin{cases} A(u)s(t,\zeta) & \text{for } t < u, \\ B(u)v(t,\zeta) & \text{for } t > u. \end{cases} \tag{8.13}$$

To determine $A(u)$, $B(u)$, $R(u,u,\zeta)$ we substitute successively $t = u-1$, $t = u$, $t = u+1$ in (8.12). We find that

$$-1/2[A(u)s(u-2,\zeta) + R(u,u,\zeta)] + [a(u-1) - \zeta]A(u)s(u-1,\zeta) = 0,$$
$$-1/2[A(u)s(u-1,\zeta) + B(u)v(u+1,\zeta)] + [a(u) - \zeta]R(u,u,\zeta) = 1,$$
$$-1/2[R(u,u,\zeta) + B(u)v(u+2,\zeta)] + [a(u+1) - \zeta]B(u)v(u+1,\zeta) = 0.$$

Since $\ell s = \zeta s$, $\ell v = \zeta v$, the first and third equations yield

$$R(u,u,\zeta) = A(u)s(u,\zeta) = B(u)v(u,\zeta).$$

Thus the second equation can be transformed to

$$-1/2[A(u)s(u+1,\zeta) + B(u)v(u+1,\zeta)] = 1.$$

Hence

$$A(u) = -\frac{2v(u,\zeta)}{v(\zeta)}, \quad B(u) = -\frac{2s(u,\zeta)}{v(\zeta)}.\square \tag{8.14}$$

We want to show that for $\zeta = -\frac{1}{2}(\rho + \rho^{-1})$, $|\rho| \gg 1$, the resolvent $R(\zeta)$ is nearstandard. To this end we use the following assertion.

8.4. Lemma. *Let $U = U_r$, $V = V_r$ be operators defined as follows: $\forall x \in H$ $Ux(1) = 0$ and $\forall t \in T$*

$$Ux(t) = r^{-t}\sum_{u=1}^{t-1} r^{u-1}x(u), \quad Vx(t) = r^{t-1}\sum_{u=t}^{m} r^{-u}x(u). \tag{8.15}$$

Then for $r > 1$

$$\|v\| \le \frac{1}{(r-1)r}, \qquad \|V\| \le \frac{1}{r-1}. \tag{8.16}$$

Proof. It is easy to see that $Ux(t+1) = r^{-1}Ux(t) + r^{-2}x(t)$, $Vx(t+1) = rVx(t) - x(t)$. Therefore $\|Ux\| \le r^{-1}\|Ux\| + r^{-2}\|x\|$, $\|Vx\| \le r\|Vx\| + \|x\|$. $\square$

8.5. Proposition. *Let $\zeta = -\frac{1}{2}(\rho + \rho^{-1})$ be a regular value of L and $1 \ll |\rho| \ll \infty$. Then $R(\zeta) := (L-\zeta)^{-1} \in {}^{nst}\mathcal{B}(H)$, $^\circ\zeta$ is a regular value of $\mathbf{L}$ and $^\bullet[R(\zeta)] = \mathbf{R}(^\circ\zeta) := (\mathbf{L} - {}^\circ\zeta)^{-1}$, where $\mathbf{L} = {}^\bullet L$.*

Proof. By (3.7), (5.8), (8.4), and (8.10) there exists a real $C \ll \infty$ such that $(\forall x \in H)$ $(\forall t \in T)$ $(1 \ll |\rho| < r \implies |Rx(t)| \le C|U_r x(t) + V_r x(t)|$. Therefore by (8.16),

$$\|R(\zeta)\| \le C\frac{r+1}{r-1} = \frac{C_1}{r-1} \ll \infty. \tag{8.17}$$

It is known (see [15]) that $\forall \xi \in \mathbf{H}$ $\forall t \in \mathbb{N}$

$$\mathbf{R}(\zeta) = -\frac{2}{\varepsilon(\zeta)}\left[\sigma(t,\zeta)\sum_{u=1}^{t-1}\varepsilon(u,\zeta)\xi(u) + \varepsilon(t,\zeta)\sum_{u=t}^{\infty}\sigma(u,\zeta)\xi(u)\right],$$

where $\varepsilon(\cdot, {}^\circ\zeta) = {}^\bullet e(\cdot,\zeta)$, $\sigma(\cdot, {}^\circ\zeta) = {}^\bullet s(\cdot,\zeta)$. Hence $\forall x \in H$, $\forall t \in T$ $[R(\zeta) - \Pi\mathbf{R}(\zeta)Q]x(t) = -\frac{2}{v(\xi)}s(t,\zeta)\sum_{u=1}^{t-1}v(u,\zeta)x(u) + \frac{2}{\varepsilon(\zeta)}\sigma(t,\zeta)\sum_{u=1}^{t-1}\varepsilon(u,\zeta)x(u) - \frac{2}{v(\zeta)}v(t,\zeta)\sum_{u=t}^{m}s(u,\zeta)x$ $\frac{2}{\varepsilon(\zeta)}\varepsilon(t,\zeta)\sum_{u=t}^{m}\sigma(u,\zeta)x(u)$. In view of $|\, s(\cdot,\zeta) - v(\cdot,\zeta)\,|_r \approx 0$ and $\|v(\cdot,\zeta) - \varepsilon(\cdot,\zeta)\| \approx 0$ and in virtue of (3.7) and (5.8) (recall that $s = (\rho - \rho^{-1})Z$) there exists a real $C_2 \approx 0$ such that $|[R(\zeta) - \Pi\mathbf{R}(\zeta)Q]x(t)| \le C_2[U_r x(t) + V_r x(t)]$. Thus,

$$\|R(\zeta) - \Pi\mathbf{R}(\zeta)Q\| \le \frac{C_1 C_2}{r-1} \approx 0. \quad \square \tag{8.18}$$

8.6. Remark. As $R(\zeta) \in {}^{nst}\mathcal{B}(H)$, the equality $^\bullet[R(\zeta)] = \mathbf{R}(^\circ\zeta)$ is a simple corollary of 4.16.

Inequality (8.17) gives us an upper estimation when ρ tends in ρ-plain to the circle $|\rho| = 1$. Let us obtain a lower estimation.

8.7. Proposition. *Let $\zeta \in \mathbb{C}$ be a regular value of the operator L. Then*

$$\|R(\zeta)\| \ge 2\frac{\|v(\cdot,\zeta)\|}{|v(\zeta)|}. \tag{8.19}$$

Proof. Let $n \in \mathbb{N}$ and

$$x_n(t) = \begin{cases} s(t,\zeta) & \text{for } t \le n, \\ 0 & \text{for } t > n. \end{cases}$$

By (8.10), $\forall t > n$

$$R(\zeta)x_n(t) = -\frac{2v(t,\zeta)}{v(\zeta)}\sum_{u=1}^{n}|x_n(u)|^2.$$

Hence $|R(\zeta)x_n(t)| = 2\left|\frac{v(t,\zeta)}{v(\zeta)}\right|\|x_n\|^2$ and $\|R(\zeta)\| \ge 2\frac{\|v(\cdot,\zeta)\|}{|v(\zeta)|}\|x_n\|$. This implies (8.19), because $\|x_n\| \ge |s(1,\zeta)| = 1$.

8.8. Corollary. *Suppose that* $\zeta = -\frac{1}{2}(\rho + \rho^{-1})$ *and* $|\rho| \approx 1$. *Then* $\|R(\zeta)\| \approx \infty$. *(On this account we may speak that the halo of the segment* $[-1, +1]$ *contains the continuous spectrum of* L.*)*

Proof. It is easy to verify that for $|\rho| \approx 1$

$$v(t, \zeta) = e(t, \rho) - \rho^{-2m-2}\tilde{e}(t, \rho). \tag{8.20}$$

From the proof of Proposition 5.8 we see that (see (5.12)) $\forall \rho \in W_m \ \|v(\cdot, \zeta)\| \approx \infty$. □

9. Spectral expansion. With any eigenvalue λ of L there is associated the spectral projector:

$$\mathcal{P}_\lambda := -\frac{1}{2\pi i}\oint (L - \zeta)^{-1}\, d\zeta, \tag{9.1}$$

i.e. the residue of the resolvent with respect to its pole λ. The path of integration in (9.1) is, for instance, a small circle (with a positive orientation) with center λ, which separates λ from other eigenvalues of L. We have

$$H_\lambda = \mathcal{P}_\lambda H, \tag{9.2}$$

where H_λ is the space of fundamental functions of L, related to λ. Let $\mathcal{P}(\cdot, \cdot, \lambda)$ be the matrix of $\mathcal{P}_\lambda$ with respect to the natural basis of H, that is

$$\forall x \in H \ \ \forall t \in T \ \ \mathcal{P}_\lambda x(t) = \sum_{u=1}^{m} \mathcal{P}(t, u, \lambda) x(u). \tag{9.3}$$

9.1. Proposition. *Let* m_λ *be the multiplicity of* λ *as a root of the equation* $v(\zeta) = 0$ *(or the equation* $s(m+1, \zeta) = 0$ *which is equivalent). Then*

$$\mathcal{P}(t, u, \lambda) = \frac{2}{(m_\lambda - 1)!}\left(\frac{d}{d\zeta}\right)^{m_\lambda - 1} (\zeta - \lambda)^{m_\lambda}\frac{v(1, \zeta)}{v(\zeta)} s(t, \zeta) s(u, \zeta)\,|_{\zeta = \lambda}\,. \tag{9.4}$$

Proof. Denote by $c(\cdot, \zeta)$ the solution of the equation $\ell c = \zeta c$ with initial values $c(0, \zeta) = 1$, $c(1, \zeta) = 0$. It is evident that

$$\forall t \in T \ \ v(t, \zeta) = v(1, \zeta) s(t, \zeta) + v(\zeta) c(t, \zeta); \tag{9.5}$$

we recall that $v(\zeta) := v(0, \zeta)$ and note that $c(t, \zeta)$ is a polynomial with respect to ζ (of degree $t - 2$). Therefore the last formula can be rewritten as

$$R(t, u, \zeta) = -\frac{2v(1, \zeta)}{v(\zeta)} s(t, \zeta) s(u, \zeta) + \cdots \tag{9.6}$$

where $\cdots$ is some polynomial in ζ. But $\mathcal{P}(t, u, \zeta)$ is the residue of $R(t, u, \zeta)$ at pole λ:

$$\mathcal{P}(t, u, \lambda) = -\frac{1}{2\pi i}\oint R(t, u, \zeta)\, d\zeta. \tag{9.7}$$

Thus, by the elementary formula for the residue of an analytic function, we obtain (9.4). □

9.2. Definition. An operator $A \in \mathcal{B}(H)$ is said to be *S-compact* and we write $A \in \mathcal{B}_\infty(H)$ iff

$$(\forall x \in H)\ \ (\|x\| \ll \infty \implies Ax \in {}^{nst}H). \tag{9.8}$$

9.3. Remark. Let $A \in \mathcal{B}_\infty(H)$, then $\|A\| \ll \infty$.

Indeed, since $\forall y \in {}^{nst}H\ \|y\| \approx \|{}^\circ y\| \ll \infty$, by permanence we obtain $\sup\{\|Ax\| : \|x\| \le 1\} \ll \infty$. □

9.4. Proposition. *Let $\mathcal{P}^2 = \mathcal{P} \in \mathcal{B}_\infty(H)$, then* $\operatorname{rank}\mathcal{P} \ll \infty$. *Besides, $\mathcal{P}^* \in \mathcal{B}_\infty(H)$ iff $\mathcal{P}$ can be represented as follows*

$$\forall x \in H\ \ \mathcal{P}x = \sum_{i \le n} (x|f_i)e_i,\ \ n = \operatorname{rank}\mathcal{P}, \tag{9.9}$$

where $e_i, f_i \in {}^{nst}H$.

Proof. Let $(e_1, \ldots, e_n)$ be a basis of $\operatorname{im}\mathcal{P}$ and $\forall i\ \|e_i\| \ll \infty$. Then $e_i = \mathcal{P}e_i \in {}^{nst}H$. The set $E := \{{}^\bullet e_1, \ldots, {}^\bullet e_n\}$ is standard and all its elements are standard in the usual sense. Therefore $n = \operatorname{card} E \ll \infty$. Evidently, if $\forall i\ e_i, f_i \in {}^{nst}H$, then operator (9.9) and its adjoint are S-compact. Conversely, suppose that $\mathcal{P}^*$ is S-compact too. Choose $(e_1, \ldots, e_n)$ as an orthonormal basis of $\operatorname{im}\mathcal{P}$ and define $\forall i\ f_i = \mathcal{P}^* e_i$. Then we get $f_i \in {}^{nst}H$ and $\forall x \in H\ \sum_{i \le n}(x|f_i)e_i = \sum_{i \le n}(\mathcal{P}x|e_i)e_i = \mathcal{P}x$. □

9.5. Definition. An eigenvalue λ of the operator L is said to be *authentic* iff spectral projectors $\mathcal{P}_\lambda$ and $\mathcal{P}_\lambda^*$ both are S-compact.

9.6. Theorem. *An eigenvalue of the operator L is authentic iff it is outside.*

Proof. Let λ be an outside eigenvalue of L and n_λ its algebraic multiplicity. By (7.11), $\forall k < n_\lambda\ \left(\frac{d}{d\zeta}\right)^k s(\cdot, \zeta)\ |_{\zeta=\lambda} \in {}^{nst}H$. From formula (9.4) and Proposition 9.4 we conclude that $\mathcal{P}_\lambda$ and $\mathcal{P}_\lambda^*$ both are S-compact. Therefore λ is authentic. On the other hand, by 5.8, if λ is not outside, none solution x of the equation $\ell x = \lambda x$ is nearstandard. In particular, $s(\cdot, \lambda) \notin {}^{nst}H$ and $\mathcal{P}_\lambda \notin \mathcal{B}_\infty(H)$. □

9.7. Theorem. *Let λ be an authentic eigenvalue of the operator L. Then there exists a standard neighbourhood $U \subset \mathbb{C}$ of λ such that any eigenvalue λ' of L, belonging to U, is infinitely close to λ: $\lambda' \approx \lambda$. Moreover we have $\sum_{\lambda' \approx \lambda} \mathcal{P}_{\lambda'} \in {}^{nst}\mathcal{B}(H)$ and*

$${}^\bullet\Big(\sum_{\lambda' \approx \lambda} \mathcal{P}_{\lambda'}\Big) = \mathbf{P}_{{}^\circ\lambda}, \tag{9.10}$$

where $\mathbf{P}_{{}^\circ\lambda}$ is the spectral projector of $\mathbf{L} = {}^\circ L$ corresponding to ${}^\circ\lambda$.

Proof. Note that ${}^\circ\lambda$ is an eigenvalue of $\mathbf{L}$ and it is isolated in the spectrum of $\mathbf{L}$. Let Γ be a circle with the center ${}^\circ\lambda$ and with a standard radius enough small to separate ${}^\circ\lambda$ from the remainder of the spectrum of $\mathbf{L}$. Let U be the neighbourhood of ${}^\circ\lambda$ with the boundary Γ. Any eigenvalue $\lambda' \in U$ of $\mathbf{L}$ must be infinitely close to λ. Indeed, otherwise there exists an eigenvalue ${}^\circ(\lambda')$ of $\mathbf{L}$ in U different from ${}^\circ\lambda$. By Proposition 8.5, we have $\forall \zeta \in \Gamma\ (L - \zeta)^{-1} \approx \Pi(\mathbf{L} - \zeta)^{-1}Q$. Thus $\sum_{x \approx \lambda} \mathcal{P}_{\lambda'} = -\frac{1}{2\pi i}\oint (L - \zeta)^{-1}\, d\zeta \approx -\frac{1}{2\pi i}\Pi \oint (\mathbf{L} - \zeta)^{-1}\, d\zeta\, Q = \Pi\mathbf{P}_{{}^\circ\lambda}Q$, hence we obtain (9.10).□

9.8. Corollary. *Outside any standard neighbourhood of the segment* $[-1,+1] \subset \mathbb{C}$ *the set of eigenvalues of* L *(taking into account the algebraic multiplicity) is finite (in the standard sense).*

Denote by $\sigma(L)$ the spectrum of L. Since $\dim H \in \mathbb{N}$, $\sigma(L)$ consists of eigenvalues only and

$$\sum_{\lambda\in\sigma(L)} \mathcal{P}_\lambda = \mathbb{I}_H. \tag{9.11}$$

For simplicity suppose that any eigenvalue of L is simple. Then formula (9.11) may be transformed as follows.

9.9. Proposition. *Let* $\forall\lambda \in \sigma(L)$ $m_\lambda = 1$. *Put*

$$\forall x \in H \;\; \forall\lambda \in \sigma(L) \;\; \tilde{x}(\lambda) := \frac{2v(1,\lambda)}{v'(\lambda)} \sum_{u\in T} x(u)s(u,\lambda). \tag{9.12}$$

(We note that $v'(\lambda) \neq 0$, *whenever* $m_\lambda = 1$.*) Then*

$$\forall t \in T \quad x(t) = \sum_{\lambda\in\sigma(L)} \tilde{x}(\lambda)s(t,\lambda). \tag{9.13}$$

Proof. If $\lambda \in \sigma(L)$ and $m_\lambda = 1$, formula (9.4) takes the form

$$\mathcal{P}(t,u,\lambda) = \frac{2v(1,\lambda)}{v'(\lambda)} s(t,\lambda)s(u,\lambda). \tag{9.14}$$

Therefore, in view of (9.12),

$$\mathcal{P}_\lambda x = \tilde{x}(\lambda)s(\cdot,\lambda), \tag{9.14'}$$

and (9.11) implies (9.13). □

9.10. Remark. To multiplication of operators there corresponds multiplication of their matrices. Since $\mathcal{P}_\lambda^2 = \mathcal{P}_\lambda$, $\mathcal{P}_\lambda\mathcal{P}_\mu = 0$ whenever $\lambda,\mu \in \sigma(L)$ and $\lambda \neq \mu$, we have

$$\sum_{u\in T} \mathcal{P}(t,u,\lambda)\mathcal{P}(u,t',\lambda) = \mathcal{P}(t,t',\lambda), \quad \sum_{u\in T} \mathcal{P}(t,u,\lambda) = \mathcal{P}(u,t',\mu) = 0$$

for $\lambda \neq \mu$. If $\forall\lambda \in \sigma(L)$ $m_\lambda = 1$, this implies

$$\frac{2v(1,\lambda)}{v'(\lambda)} \sum_{u\in T} [s(u,\lambda)]^2 = 1, \tag{9.15}$$

$$\sum_{u\in T} s(u,\lambda)s(u,\mu) = 0 \; for \; \lambda \neq \mu. \tag{9.16}$$

From (9.14) we find

$$\forall\lambda \in \sigma(L) \;\; \|\mathcal{P}_\lambda\| = \frac{2|v(1,\lambda)|}{|v'(\lambda)|} \|s(\cdot,\lambda)\|^2 = \frac{\|s(\cdot,\lambda)\|^2}{\sum_{u\in T}[s(u,\lambda)]^2}. \tag{9.17}$$

If $\operatorname{Im} a(t) \equiv 0$, we have $L = L^*$. Thus $\|\mathcal{P}_\lambda\| = 1$ and

$$\|s(\cdot,\lambda)\| = \sqrt{\frac{|v'(\lambda)|}{2|v(1,\lambda)|}}, \quad \lambda \in \sigma(L). \tag{9.18}$$

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