УДК 517.986

THE EULER FUNCTIONAL IDENTITIES FOR GENERALIZED ČEBYŠEV POLYNOMIALS

W. Kawa

W. Kawa. *The Euler functional identities for generalized Čebyšev polynomials*, Matematychni Studii, **11**(1999) 63–70.

The author studies generalized Čebyšev polynomials and using the well-known Euler identities for homogenous functions of order −1 introduces in this way a family of functions conjugated to the generalized Čebyšev polynomials that appears to be involutions.

В. Кава. *Функциональные тождества Эйлера для обобщенных полиномов Чебышева.* // Математичні Студії. – 1999. – Т.11, № 1. – С.63–70.

Изучаются обобщенные полиномы Чебышева. С использованием хорошо известных тождеств Эйлера для однородных функций порядка −1 вводится семейство функций, сопряженных с обобщенными полиномами Чебышева, которые появляются при инволюциях.

Introduction[1]

In the classical way the first order Čebyšev polynomials are defined by the following formulae

$$T_0(x) = 1, \quad T_1(x) = x, \tag{0.1}$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \dots \tag{0.2}$$

that hold for every $x \in \mathbb{R}$ and $n = 0, \pm 1, \pm 2, \dots$.

Important from a very classical point of view the class of Čebyšev polynomials

1991 *Mathematics Subject Classification.* 41A10,41A50.

[1]This paper is a part of the author's doctoral thesis written under direction of Józef Zając

possesses the following properties:

$$(i)\quad T_n(x)=\begin{cases}\cos(n\arccos x), & x\in\langle -1,1\rangle,\\ (\operatorname{sgn} x)^n\cosh(n\operatorname{areacosh}|x|), & x\notin\langle -1,1\rangle;\end{cases}$$

$$(ii)\quad T_n(x)=T_{-n}(x);$$

$$(iii)\quad (1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=0;$$

$$(iv)\quad T_n(x)=2^{-n}\sum_{k=0}^{n}\binom{2n}{2k}(x+1)^{n-k}(x-1)^k,\quad n=0,\pm1,\pm2,\dots;$$

(*v*) the system $\sqrt{\frac{2}{\pi}}T_n(x)$ is orthonormal as x ranges over $\langle 0,1\rangle$ with weight function g given by

$$g(x)=\frac{1}{\sqrt{1-x^2}}.$$

From properties (i) and (ii) it follows that one may simply generalize $P_n(x)$ for $|x|\le 1$ by inserting any real number y instead of $n\in\mathbb{Z}$. It means that

Definition 1.1. The generalized Čebyšev polynomial (gCp) is a real function defined over $\langle -1,\infty)\times\mathbb{R}$ by the formula

$$P_y(x)=\begin{cases}\cos(y\arccos x), & x\in\langle -1,1\rangle,\\ \cosh(y\operatorname{areacosh} x), & x\in(1,\infty);\end{cases}$$

The aim of the paper is studying basic properties of the gCp functions considered as a special function.

1. Involute equation

Assume first that $\mathcal{P}$ is the family of all gCp functions considered over $\langle -1,1\rangle$, and that $\mathcal{P}^\star$ the family of all involutions of $\langle -1,1\rangle$ such that

$$P_L^\star\circ P_y=P_{\frac{1}{y}}\circ P_L^\star \tag{1.1}$$

holds for every $y\in\mathbb{R}$ and $L\in\mathbb{R}$. This identity is said to be the *involute identity*, see [Z].

Theorem 1.1. *Function $P^\star$ is an element of the class $\mathcal{P}^\star$ if and only if there is a real number L, such that*

$$P_L^\star(x)=\cos\left(\frac{L}{\arccos x}\right) \tag{1.2}$$

holds for every $-1 < x < 1$*. Moreover, if* $(\xi, \eta) \in Q := (-1, 1) \times \mathbb{R}$ *is an arbitrary point, then there is a number* $L_{(\xi,\eta)}$ *such that* $P^\star(\xi) = \eta$*.*

Proof. Assume that $h \in \mathcal{P}^*$. Introducing the following function $M = \arccos \circ P^\star \circ \cos$, we see that (1.1) may be written as

$$M(xy) = \frac{1}{y} M(x) \tag{1.3}$$

for $-1 < x < 1$ and $y \in \mathbb{R}\backslash\{0\}$. By the definition of $\mathcal{P}^*$ and the regularity of function arccos, it follows that the function M is differentiable over $(-1, 1)$. Hence, the classical *Euler identity for homogenous functions*, see [Z], implies that all the solutions of (1.3) can be written as

$$M(x) = \frac{L}{x} \tag{1.4}$$

for $0 < x < \infty$ and $L \in \mathbb{R}$. Thus,

$$h(x) = \cos\left(\frac{L}{\arccos x}\right). \tag{1.5}$$

Let $(\xi, \eta) \in Q$. Setting $L_{(\xi,\eta)} = \arccos(\xi) \arccos(\eta)$ we see that $\cos\left(\frac{L_{(\xi,\eta)}}{\arccos \xi}\right) = \eta$, so the second part of our theorem follows.

On the other hand, it is evident that each function of the form (1.5) belongs to $\mathcal{P}^*$ which ends the proof.

The function

$$P^\star_L(x) = \cos\left(\frac{L}{\arccos x}\right). \tag{1.6}$$

is called a *conjugate generalized Čebyšev polynomial.*

The basic properties of the conjugate gCp are stated in

Theorem 1.2. *For each* $K, L \in \mathbb{R}$ *we have*

- (i) *for every fixed* $L \in \mathbb{R}$, $P^\star_L$ *is a decreasing authomorphism of* $(-1, 1)$ *and for each* $x \in (-1, 1)$, $P^\star_L$ *is a decreasing diffeomorphism of* $\mathbb{R}$ *onto* $(-1, 1)$*;*
- (ii) $P^\star_L \circ P^\star_K = P_{L/K}$*;*
- (iii) $P^\star_L \circ P_K = P^\star_{L/K}$*;*
- (iv) $P_K \circ P^\star_L = P^\star_{LK}$*;*
- (v) $P^\star_L \circ P_K \circ P^\star_L = P_{1/K}$*;*

Proof. Property (i) follows from the definition of $P_L^\star$. Properties (ii)–(v) may be simply checked in the following way:

$$(ii)\quad P_L^\star \circ P_K^\star(x) = \cos\left(\frac{L}{\arccos\left(\cos\left(\frac{K}{\arccos x}\right)\right)}\right) = \cos\left(\frac{L}{K}\arccos x\right) = P_{L/K}(x),$$

$$(iii)\quad P_L^\star \circ P_K(x) = \cos\left(\frac{L}{\arccos\left(\cos\left(K\arccos x\right)\right)}\right) = \cos\left(\frac{\frac{L}{K}}{\arccos x}\right) = P_{L/K}^\star(x),$$

$$(iv)\quad P_K \circ P_L^\star(x) = \cos\left(K\arccos\left(\cos\left(\frac{L}{\arccos x}\right)\right)\right) = \cos\left(\frac{KL}{\arccos x}\right) = P_{KL}^\star(x),$$

$$(v)\quad P_L^\star \circ P_K \circ P_L^\star(x) = \cos\left(\frac{L}{\arccos\left(\cos\left(K\arccos\left(\cos\left(\frac{L}{\arccos x}\right)\right)\right)\right)}\right) =$$

$$= \cos\left(\frac{1}{K}\arccos x\right) = P_{1/K}(x).$$

Theorem 1.3. *The union of the families $\mathcal{P}$ and $\mathcal{P}^*$ is a group under the composition as the group action.*

Proof. By the properties stated in Theorem 1.2 it is rather obvious that the union of the families $\mathcal{P}$ and $\mathcal{P}^*$ is a group under composition, where the role of the neutral element plays function $P_1(x) = x$, and the inverse element to every function $P_K(x) = \cos(K\arccos x)$, $K \in \mathbb{R}$ is the function $(P_K)^{-1} = P_{1/K}$ whereas the inverse element to every function $P_L^\star(x) = \cos(\frac{L}{\arccos x})$, $L \in \mathbb{R}$ is the same function $P_L^\star$.

One may also generalize Čebyšev polynomials for every $x > 1$. Then the (gCp) function has the following form

$$P_K(x) = \cosh\left(K \operatorname{areacosh} x\right), \quad x > 1. \tag{1.7}$$

Now assume that $\mathcal{Q}$ denote the family of all (gCp) functions considered over $(1, \infty)$, and that $\mathcal{Q}^*$ is the family of all involutions of $(1, \infty)$ such that

$$P_L^\star \circ P_K = P_{1/K} \circ P_L^\star \tag{1.8}$$

holds for every $K \in \mathbb{R}$.

Theorem 1.4. *Function $P_L^\star \in \mathcal{Q}^*$ if and only if there is a number L such that*

$$P_L^\star(x) = \cosh\left(\frac{L}{\operatorname{areacosh} x}\right) \tag{1.9}$$

holds for every $x \in (1, \infty)$. Moreover, if $(\xi, \eta) \in (1, \infty) \times \mathbb{R}$ is an arbitrary point, then there is a number $L_{(\xi,\eta)}$ such that $P_L^\star(\xi) = \eta$.

Proof. The proof of this theorem is based on the same rules as that of Theorem 1.1.

Theorem 1.5. *The union of the families $\mathcal{Q}$ and $\mathcal{Q}^*$ is a group under the composition as the group action.*

Basic properties of a (gCp) functions and functions $P_L^\star \in \mathcal{Q}^*$ for $x \in (1, \infty)$ may be written in

Theorem 1.6. *For each $K, L \in \mathbb{R}$ we have*

(i) *for every fixed $L \in \mathbb{R}$, $P_L^\star$ is a decreasing authomorphism of $(1, \infty)$ and for each $x \in (1, \infty)$, $P_L^\star$ is a decreasing diffeomorphism of $\mathbb{R}$ onto $(1, \infty)$;*
(ii) $P_L^\star \circ P_K^\star = P_{L/K}$;
(iii) $P_L^\star \circ P_K = P_{L/K}^\star$;
(iv) $P_K \circ P_L^\star = P_{LK}^\star$;
(v) $P_L^\star \circ P_K \circ P_L^\star = P_{1/K}$;

Proof. The proof of this theorem is the same as that of Theorem 1.3.

It is not very difficult to show that functions $P_K(x)$ and $P_L^\star(x)$ for every $x > 1$ has the following form

$$P_K\left(x\right) = \frac{\left(x + \sqrt{x^2 - 1}\right)^k + \left(x - \sqrt{x^2 - 1}\right)^k}{2}, \tag{1.10}$$

and

$$P_L^\star(x) = \frac{1}{2}\left(e^{\frac{L}{\ln(x + \sqrt{x^2-1})}} + e^{\frac{L}{\ln(x - \sqrt{x^2-1})}}\right). \tag{1.11}$$

It is also not very difficult to show that the following differential equations for functions $P_L^\star$ for $x \in (-1, 1)$ hold:

$$\left(P_L^\star\left(x\right)\right)'' + \left(\frac{2}{\sqrt{1 - x^2}\arccos x} - x\right)\left(P_L^\star(x)\right)' + \frac{L^2}{\left(1 - x^2\right)\arccos^4 x}P_L^\star(x) = 0, \tag{1.12}$$

and for $x > 1$

$$\left(P_L^\star\left(x\right)\right)'' - \left(\frac{2}{\sqrt{x^2 - 1}\,\mathrm{areacosh}\, x} + x\right)\left(P_L^\star(x)\right)' - \frac{L^2}{\left(x^2 - 1\right)\mathrm{areacosh}^4\, x}P_L^\star(x) = 0. \tag{1.12'}$$

2. A dynamical base and the dynamical dimension of the group of generalized Čebyšev polynomials

Assume that $(G, \circ, \rho)$ is a topological group, where "$\circ$" is the group multiplication and ρ is a metric in G. Let $\left(G_{(a_1,a_2,\ldots,a_n)}, \circ\right)$ be a subgroup of the group $(G, \circ)$, generated by elements $a_1, a_2, \ldots, a_n \in G$, and such that the set $G_{(a_1,a_2,\ldots,a_n)}$ is a dense subset of G in the topology induced by the metric ρ.

Definition 2.1. A minimal set of generators such that the set $G_{(a_1,a_2,\ldots,a_n)}$ is a dense subset of G is said to be a dynamical base of the given topological group $(G,\circ,\rho)$, whereas the number of elements of the dynamical base is said to be its dynamical dimension, denoted by $\mathrm{Dyn}(G)$.

One may consider the topological group $(\mathcal{P},\circ,\rho)$ with composition as the group action, where $\mathcal{P}$ denotes the family of all generalized Čebyšev polynomials defined on $\langle -1,1\rangle$ and ρ is the metric given by the following formula

$$\rho(P_K,P_L) = \max_{x\in\langle -1,1\rangle} |P_K(x)-P_L(x)|, \quad K,L\in\mathbb{R}. \tag{2.1}$$

One may show, after not very sophisticated steps, that the dynamical dimension of the topological group $(\mathcal{P},\circ,\rho)$ is equal to 2.

First, one has to introduce theorem as below.

Theorem 2.1. *Suppose that $(\mathbb{R}^+,\cdot,d)$ is the topological group formed by positive real numbers $\mathbb{R}^+$, with the multiplication " $\cdot$ " as the group multiplication and the euclidean distance d. The dynamical dimension of $(\mathbb{R}^+,\cdot,d)$ is 2 and its dynamical base is formed by any set $\{a,b\}$, where a,b are positive integers and relatively prime ones.*

Proof. Consider the set $P_{a,b}=\{x=a^pb^q : p,q\in\mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers, and prove that it is a dense subset of $\mathbb{R}^+$. First, one will show that the set

$$\{[n\alpha] := n\alpha - \mathrm{Ent}\,(n\alpha) : n\in\mathbb{N}\}, \tag{2.2}$$

where $\mathbb{N}$ is the set of natural numbers and α is an irrational number, is a dense subset of the interval $\langle 0,1\rangle$. In order to prove the density of the set given by the formula (2.2), it is enough to show that there exists an element $[k\alpha]\,,k\in\mathbb{N}$ in every given interval $I\subset\langle 0,1\rangle$.

Assume that m is a given natural number. Divide the interval $\langle 0,1\rangle$ into m equal parts and consider the following elements $[\alpha]\,,[2\alpha]\,,[3\alpha]\,,\ldots,[(m+1)\,\alpha]$. Hence, one obtains m intervals $\langle\frac{i}{m},\frac{i+1}{m}\rangle$, $i=0,1,\ldots,m-1$, that make a partition of $\langle 0,1\rangle$. There exists $i_0\in\{0,1,\ldots,m-1\}$ such that at least two elements $[k\alpha]$ and $[l\alpha]$ belong to the interval $\langle\frac{i_0}{m},\frac{i_0+1}{m}\rangle$. It means that

$$[k\alpha]\in\langle\frac{i_0}{m},\frac{i_0+1}{m}\rangle, \quad \text{and} \quad [l\alpha]\in\langle\frac{i_0}{m},\frac{i_0+1}{m}\rangle. \tag{2.3}$$

Without any loss of comprehension one may assume that $[k\alpha]>[l\alpha]$ and then, by (2.3), one gets the following inequality

$$0<[k\alpha]-[l\alpha]<\frac{1}{m}\,, \tag{2.4}$$

equivalent to

$$0 < [(k-l)\,\alpha] < \frac{1}{m}\ . \tag{2.5}$$

The last inequality reads that there exists an element $[(k-l)\,\alpha]$ of the interval $\langle 0, \frac{1}{m}\rangle$ such that $[(k-l)\,\alpha] \in \{[n\alpha] : n \in \mathbb{N}\}$, and then, after a finite number n_1 of steps, the element $[n_1\,(k-l)\,\alpha]$ belongs to the interval $\langle \frac{1}{m}, \frac{2}{m}\rangle$. In the same way one may get numbers

$$[n_i\,(k-l)\,\alpha] = n_i\,(k-l)\,\alpha - \mathrm{Ent}\,(n_i\,(k-l)\,\alpha)\,, \quad n_i \in \mathbb{N}, \quad i = 0, 1, \ldots, m-1,$$

such that the inclusion

$$[n_i\,(k-l)\,\alpha] \in \langle \frac{i}{m}, \frac{i+1}{m}\rangle$$

holds. Summing up, one has proved that for an arbitrary $m \in \mathbb{N}$, there exists an element $[k\alpha]\,, k \in \mathbb{N}$, in every interval $\langle \frac{i}{m}, \frac{i+1}{m}\rangle, i = 0, 1, \ldots, m-1$. Thus, the set given by the formula (2.2) is a dense subset of $\langle 0, 1\rangle$.

Substituting $\alpha = \log_a b$ one sees that the set $\{[m \log_a b] : m \in \mathbb{N}\}$ is a dense subset of $\langle 0, 1\rangle$. Moreover, the set $\{n + m \log_a b : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$. It follows that the set $P_{a,b} = \{x = a^p b^q : p, q \in \mathbb{Z}\}$, as a homeomorphic set with the set $\{n + m \log_a b : m, n \in \mathbb{Z}\}$, is a dense subset of $\mathbb{R}^+$.

Extending the already obtained result, one may prove

Theorem 2.2. *Given the topological group $(\mathbb{R}\backslash\{0\}, \cdot, d)$ with the multiplication "$\cdot$" as the group action, and the same metric as before. Then its dynamical dimension is 2 and the set $\{-2, 3\}$ can be considered as its dynamical base.*

Proof. By the proof of Theorem 2.1 one concludes that the set $P_{4,3}$ is a dense subset of $\mathbb{R}^+$. It is not difficult to show that the set $\{-2x : x \in P_{4,3}\}$ is a dense subset of $\mathbb{R}^- := \{x \in \mathbb{R} : x < 0\}$. One can see that $P_{4,3} \cup \{-2x : x \in P_{4,3}\} = P_{-2,3}$. Thus $P_{-2,3}$ is a dense subset of $\mathbb{R}\backslash\{0\}$.

By the properties of (gCp) functions and using Theorem 1.2 and Theorem 2.1 one may prove the following theorem.

Theorem 2.3. *The dynamical base of the topological group $(\mathcal{P}, \circ, \rho)$ is equal to 2 and a dynamical base of this group can be formed by the functions P_2, P_3, where*

$$P_2(x) = 2x^2 - 1, \quad \textit{and} \quad P_3(x) = 4x^3 - 3x.$$

Proof. One may notice that the (gCp) functions have got the following properties

$$P_K \circ P_L = P_{KL} \tag{2.6}$$

and

$$P_K(x) = P_{-K}(x). \tag{2.7}$$

By Theorem 2.1, every positive real number K can be approximated by numbers of the form $(-2)^n 3^m$. It means that for an arbitrary $\varepsilon > 0$ there exists such n, m integers that

$$|K - (-2)^n 3^m| < \varepsilon. \tag{2.8}$$

By formulae (2.6) and (2.7) one can approximate every given generalized Čebyšev polynomial by the function

$$P_{2^n 3^m}(x) = \cos(2^n 3^m \arccos x), \tag{2.9}$$

where n, m are integers.

It is not very difficult to show that the following inequality holds

$$|K - 2^n 3^m| < \frac{\varepsilon}{\pi}$$

and then we obtain

$$|P_K(x) - P_{2^n 3^m}(x)| < \varepsilon. \tag{2.10}$$

It means that every (gCp) function can be approximated with an arbitrary precision by the function given by formula (2.9). Thus the set $\{P_2, P_3\}$ can be considered as a dynamical base of the topological group $(\mathcal{P}, \circ, \rho)$.

Thus, every generalized Čebyšev polynomial P_K, $K \in \mathbb{R}$ can be approximated with an arbitrary precision by functions obtained only by composition of the Čebyšev polynomials P_2 and P_3.

REFERENCES

[KZ1] W. Kawa and J. Zając (Jr), *Algebraic approach to the approximation of the distortion function Φ_K*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **20**.

[KZ2] W. Kawa and J. Zając (Jr), *Dynamical approximation of special functions in quasiconformal theory*, Folia Sci. Univ. Tech. Resoviensis Math. **17** (1995).

[P] S. Paszkowski, *Zastosowania numeryczne wielomianów i szeregów Czebyszewa*, Warszawa, PWN, 1975.

[Z] J. Zając, *Quasihomographies in the theory of Teichmüller spaces*, Disserationes Mathematicae (1996).

Department of Mathematics, The Catholic University of Lublin

wkawa@kul.lublin.pl

Received 25.05.98