УДК 512.12

ON EXTENSION OF (PSEUDO-)METRICS FROM A SUBGROUP OF A TOPOLOGICAL GROUP ONTO THE GROUP

O.V. Ravsky

O.V. Ravsky. *On extension of (pseudo-)metrics from a subgroup of a topological group onto the group*, Matematychni Studii, **11**(1999) 31–40.

In this paper we obtain necessary and in some cases sufficient conditions for existence of an extension of a two-side invariant metric from a subgroup of a SIN-group onto the group. In particular, it is found a simple criterion for existence of extension of a bounded metric from a normal subgroup onto the group. It is also proved that every metric from an open subgroup of an abelian group can be extended onto the group.

О.В. Равский. *О продолжении (псевдо-) метрик с подгруппы топологической группы на группу* // Математичні Студії. – 1999. – Т.11, № 1. – С.31–40.

Получены необходимые, а в некоторых случаях достаточные условия существования продолжения двусторонне инвариантной метрики с подгруппы SIN-группы на группу. В частности, найден простой критерий продолжаемости ограниченной метрики с нормальной подгруппы. Доказано также, что каждая метрика продолжаема с открытой подгруппы абелевой группы.

1. Introduction

The problem of extension of functions from subobjects to objects in various categories was considered by many authors. The classic Tietze-Urysohn theorem on extensions of functions from a closed subspace of a topological space and its generalizations belong to the known results. Hausdorff [H] showed that every metric from a closed subspace of a metrizable space can be extended onto the space. Isbell [Is] showed that every bounded uniform metric from a subspace of a uniform space can be extended onto the space. The linear operators extending metrics from a closed subspace of a metrizable space onto the space were considered e.g., in [B], [Z].

In the present paper we consider a partial case of the general problem for topological SIN-groups and two-side invariant metrics. We obtain necessary and in some cases sufficient conditions for existing an extension of a two-side invariant metric from a subgroup of a SIN-group onto the group. In particular, we find a simple criterion for existence of extension of a bounded metric from a normal subgroup onto the group. Besides, we prove that every metric from an open subgroup of an abelian group can be extended onto the group.

1991 *Mathematics Subject Classification.* 22A05, 54C20.

2. Notation and conventions

Let (G,τ) be a topological group and $\mathcal{B}\tau$ be the filter of neighborhoods of the unity e of G. The group G is called a *SIN-group*, if it satisfies one of the following equivalent conditions
1. The left and right group uniformities on the group G coinside.
2. $(\forall U \in \mathcal{B}\tau)(\exists V \in \mathcal{B}\tau)(\forall x \in G) : V^x \subset U$.
3. $(\forall U \in \mathcal{B}\tau) : \bigcap_{x\in G} U^x \in \mathcal{B}\tau$.
4. $(\forall U \in \mathcal{B}\tau)(\exists V \in \mathcal{B}\tau)(\forall x \in G) : V^x = V \subset U$.[1]

A (pseudo-)metric $d\colon G \times G \to \mathbb{R}_+$ on the group G is *two-side invariant,* if for arbitrary elements x, y, a, b of the group G we have $d(axb, ayb) = d(x, y)$. A group is metrizable by a two-side invariant metric iff it is a first countable SIN-group (see [GZ, p. 46], [R, p. 139]). All groups considered in this paper are topological first countable SIN-groups, a subgroup inherits the topology from a group. All (pseudo-)metrics are two-side invariant.

Let H be a group and e the unit of H. A function $f\colon H \to \mathbb{R}_+$ is called *a norm*, if it satisfies the following conditions:
1. $(\forall x \in H) : f(x) \geq 0$ and $f(x) = 0$ iff $x = e$.
2. $(\forall x \in H) : f(x^{-1}) = f(x)$.
3. $(\forall x, y \in H) : f(xy) \leq f(x) + f(y)$.
4. $(\forall x, y \in H) : f(x^y) = f(x)$.[2]

If instead of conditon 1 it satisfies condition $1'$: $(\forall x \in H) : f(x) \geq 0$ and $f(e) = 0$, then the function f is called *a pseudonorm.*

If the group H is a subgroup of a group G and instead of conditon 4 the (pseudo-)norm f satisfies condition $4'$: $(\forall x \in H, y \in G$ with $x^y \in H) : f(x^y) = f(x)$, then the (pseudo-)norm f is called *G-invariant.*

It is easy to see that if $d(x, y)$ is a two-side invariant (pseudo-)metric, then $f_d(x) = d(x, e)$ is a (pseudo-)norm, and vise versa.

Let f be a norm on a group G. The group topology on the group G generated by the norm f will be denoted τf. The norm f is *dominating* if the topology τf is stronger than the topology of the group G. The norm f is *admissible* if the topology τf and the topology of the group G coinside. It's easy to see that a norm is admissible iff it is dominating and continuous. The set of elements x of the group G such that $f(x) < \varepsilon$ will be denoted (f, ε).

We shall say that a norm f on a subgroup H of the group G *can be extended* onto the group G, if f is admissible and there exists an admissible norm f^* on the group G such that $f^*|H = f$. We shall say that the norm f^* is *an extension* of the norm f.

Thus the problem of extension of metrics is equivalent to the problem of extension of norms. It's easy to see that if a norm f can be extended onto the group G, then f is a G-invariant norm.

Remark 1. We give an example of a metrizable SIN-group G, a normal subgroup H in G and an admissible norm f on the group H such that f is not a G-invariant norm. Let $\mathbb{R}$ be the group of real numbers with the standard topology, the group $\mathbb{Z}_2$ acts on the group $\mathbb{R} \times \mathbb{R} = H$ as follows: $\sigma_0(x, y) = (x, y), \sigma_1(x, y) = (y, x)$.

[1] Where $V^x = x^{-1}Vx$.

[2] Where $x^y = y^{-1}xy$.

Then for arbitrary elements a, b of the group $\mathbb{Z}_2$ we have $\sigma_{ab} = \sigma_a \sigma_b$, and therefore there exists the topological semidirect product (see [RD, p. 120]) $H \times_\sigma \mathbb{Z}_2 = G$. For every element (x, y) of the group H put $f((x, y)) = \sqrt{x^2 + 4y^2}$. It is easy to see that f is an admissible norm on H, but f is not a G-invariant norm.

Let H be a subgroup of a group G. Then $\langle\langle H \rangle\rangle$ denotes the normal closure of H, that is the smallest normal subgroup of the group G containing the subgroup H.

As usual, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{R}, \mathbb{R}_+$ stand for the sets of natural, integer, rational, complex, real and real nonnegative numbers, respectively.

3. The extensions of norms on the closure

Proposition 1. *Let H be a dense subgroup of a group G. Then every admissible norm f on the group H can be uniquely extended onto the group G.*

Proof. Let d be an invariant admissible metric on the SIN-group H. Consider the Weil's completions $\hat{H}$ and $\hat{G}$ of the groups H and G respectively. Since H is dense in G these two completions are naturally isomorphic [E, 8.3.11]. We have the diagram

$$\begin{array}{ccc} H & \xrightarrow{i_H} & \hat{H} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\ G & \xrightarrow[i_G]{} & \hat{G} \end{array}$$

where i, i_H, i_G are natural embeddings. The composition $i_G \circ i$ is a dense embedding and $i_G \circ i = h \circ i_H$, where h is an isomorphism. Hence, we may assume that $G \subset \hat{H}$, so that we may extend the metric d onto the group $\hat{H}$, and thus onto the group G. The uniqueness of the extension follows from the continuity of a metric and the fact that $\overline{H} = G$.

4. Majorants and minorants

Definition. Let H be a subgroup of a group G. For every pseudonorm f on the group H define a function $f' \colon \langle\langle H \rangle\rangle \to \mathbb{R}_+$ letting

$$f'(x) = \inf\Big\{\sum_{i=1}^{n} f(x_i) \mid n \in \mathbb{N}, x_1, \dots, x_n \in H, y_1, \dots, y_n \in G : \prod_{i=1}^{n} x_i^{y_i} = x\Big\}.$$

It is easy to see that f' is a pseudonorm on the group $\langle\langle H \rangle\rangle$, and if H is a normal subgroup of the group G, then $f' = f$.

Lemma 1. *If a norm f can be extended from a subgroup H of a group G onto the group G, then the following conditions are satisfied: 1) $f'|H = f$, 2) f' is dominating for the group $\langle\langle H \rangle\rangle$.*

Proof. Let f^* be an extension of the norm f on the group G. Then from the triangle's inequality for every element x of the group $\langle\langle H \rangle\rangle$ we have $f^*(x) \leq \sum_{i=1}^{n} f(x_i)$, where $n \in \mathbb{N}$, $x_1, \dots, x_n \in H$, $y_1, ..., y_n \in G$ such that $\prod_{i=1}^{n} x_i^{y_i} = x$.

Therefore $f^*(x) \leq f'(x)$ and since $f^*|H = f$, we have $f'|H = f$ and $f^*|\langle\langle H \rangle\rangle \leq f'$, and hence f' is dominating on the group $\langle\langle H \rangle\rangle$.

Corollary. *If a norm f can be extended from a subgroup H of a group G onto the group G, then f' is a norm.*

Proposition 2. *Let H be a subgroup of a group G. If every nontrivial normal subgroup H' of a group $\langle\langle H\rangle\rangle$ intersects the group H in more than one element and $f'|H = f$, then f' is a norm.*

Proof. Put $N_f = \{x \in \langle\langle H\rangle\rangle : f'(x) = 0\}$. Then N_f is a normal subgroup of the group G, and $N_f \cap H = e$. Therefore the subgroup N_f is trivial and hence f' is a norm.

Remark 2. We give an example showing that the condition of the proposition 2 isn't always satisfied. Let the group $\mathbb{Z}_2$ acts on the group $\mathbb{C}$ by conjunction: $\sigma_0(z) = z, \sigma_1(z) = \overline{z}$. Then for arbitrary elements a, b of the group $\mathbb{Z}_2$ we have $\sigma_{ab} = \sigma_a\sigma_b$, and therefore there exists the topological semidirect product (see [5, p. 120]) $\mathbb{C} \times_\sigma \mathbb{Z}_2 = G$. Put $H = \{(x, x) \in \mathbb{C} : x \in \mathbb{R}\}$, $H' = \mathbb{R}$. Then $\langle\langle H\rangle\rangle = \mathbb{C}$, H' is a normal subgroup of the group $\langle\langle H\rangle\rangle$, but $H \cap H' = \{0\}$.

Remark 3. Lemma 1 can be strengthened using the continuity of the extension f^* of the norm f. Put for every element x of the group $\langle\langle H\rangle\rangle$ $f''(x) = \lim_{\overline{y \to x}} f'(y)$, where limit is taken by the neighborhood filter of the element x. Then for every extension f^* of the norm f on the group (G, τ), we have $f^*|\overline{\langle\langle H\rangle\rangle} \leq f''$ and, in particular, the following condition is necessary for existence of an extension of the norm f onto the group G :

$$f''|H = f. \tag{4.1}$$

Definition. Let F, H be subgroups of a group G, such that $F \leq H$. Let f and h be pseudonorms on the groups F and H respectively. Determine the function $f_h{:}H \to \mathbb{R}_+$ by $f_h(x) = \inf\{f(y) + h(z) : y \in F, z \in H, yz = x\}$.

Lemma 2. *Let F, H be normal subgroups of a group G, and $F \leq H$. Let f and h be G-invariant pseudonorms on the groups F and H, respectively. Then f_h is a G-invariant pseudonorm on the group H and $\tau f_h \leq \tau h$. If $\tau f \geq \tau h|F$, then $\tau f_h = \tau h$. If $h|F \geq f$, then $f_h|F = f$.*

Proof. Verify the properties of a pseudonorm for the function f_h:
1. Clearly that $f_h \geq 0$ and $f_h(e) = 0$.
2. $(\forall x \in H) : f_h(x^{-1}) = \inf\{f(y) + h(z) : y \in F, z \in H, yz = x^{-1}\} = \inf\{f(y^z) + h(z^{-1}) : y \in F, z \in H, y^z z^{-1} = x\} = f_h(x)$.
3. $(\forall y_1, y_2 \in F)(\forall x_1, x_2 \in G) : f(y_1) + f(y_2) + h(z_1) + h(z_2) \geq f(y_1y_2) + h(z_1^{y_2} z_2)$, and therefore $f_h(y_1z_1) + f_h(y_2z_2) \geq f_h(y_1y_2z_1^{y_2} z_2) = f_h(y_1z_1y_2z_2)$.
4. $(\forall x \in H)(\forall t \in G) : f_h(x^t) = \inf\{f(y) + h(z) : y \in F, z \in H, yz = x^t\} = \inf\{f(y^{t^{-1}}) + h(z^{t^{-1}}) : y \in F, z \in H, y^{t^{-1}} z^{t^{-1}} = x\} = f_h(x)$.

Let $\tau f \geq \tau h|F$ and $\varepsilon > 0$. There exists $\delta > 0, \delta < \varepsilon$ such that $(f, \delta) \subset (h, \varepsilon)$. Then $(f_h, \delta) \subset (h, 2\varepsilon)$. Hence $\tau f_h \geq \tau h$, and then $\tau f_h = \tau h$.

Clearly that $f_h|F \leq f$. On the other hand, if $h|F \geq f$, then for every element x of the group H we have $f_h(x) = \inf\{f(y) + h(z) : y \in F, z \in H, yz = x\} = \inf\{f(y) + h(z) : y, z \in F, yz = x\} \geq \inf\{f(y) + f(z) : y, z \in F, yz = x\} \geq \inf\{f(x) : y, z \in F, yz = x\} = f(x)$. Hence, $f_h|F = f$.

Theorem 1. *Let H be a normal subgroup of a group G. An admissible G-invariant norm f on the group H can be extended onto the group G iff there exists an admissible norm h on the group G such that $h|H \geq f$.*

Proof. Sufficiency. It follows from Lemma 2 that the norm $f^* = f_h$ is an extension of the norm f.
Necessity. As the norm h we may get an extension f^* of the norm f.

Proposition 3. *Let H be a subgroup of a group G, f be a G-invariant norm on the group H. Then f' is dominating on the group $\langle\langle H\rangle\rangle$ iff there exists an admissible norm h on the group $\langle\langle H\rangle\rangle$ such that $h|H \leq f$.*

Proof. Sufficiency. Let h be an admissible norm on the group $\langle\langle H\rangle\rangle$ such that $h|H \leq f$. Then $f' \geq h$ and therefore $\tau f' \geq \tau h$. Hence f' is dominating on the group $\langle\langle H\rangle\rangle$.
Necessity. Let g^* be an arbitrary admissible norm on the group G. Put $h = (f')_{g^*|H}$. Then Lemma 2 implies that $\tau h = \tau g^*$ and $h \leq f'$, therefore h is an admissible norm on the group $\langle\langle H\rangle\rangle$ and $h|H \leq f'|H \leq f$.

Proposition 4. *Let H be a subgroup of a group G, f be an admissible norm on the group H. Suppose the following conditions are satisfied: 1) There exists an admissible norm g on the group G such that $g|\langle\langle H\rangle\rangle \geq f'$; 2) $f'|H = f$; 3) f' is dominating on the group $\langle\langle H\rangle\rangle$. Then the norm f can be extended onto the group G.*

Proof. Put $f^* = (f')_g$. It follows from condition 1 and Lemma 2 that f^* is a norm, $f^*|\langle\langle H\rangle\rangle = f'$ and by the condition 2 $f^*|H = f$. It follows from condition 3 and Lemma 2 that f^* is an admissible norm on the group G.

The next lemma is known (see [GZ, Theorem 4.9]):

Lemma 3. *Let $\{U_k|k \in \mathbb{N}\}$ be a sequence of symmetrical neighborhoods of unit e of a group G such that $U_{k+1}^2 \subset U_k, k \in \mathbb{N}$, and let $H = \bigcap\{U_k|k \in \mathbb{N}\}$. Then there exists a function g on the group G such that 1) g is a uniformly continuous function with respect to the left uniform structure on the group G; 2) $g(x) = 0 \iff x \in H$; 3) $g(x) \leq 2^{-k+2}$, if $x \in U_k$; 4) $g(x) \geq 2^{-k}$, if $x \notin U_k$; 5) $(\forall x \in G) : g(x^{-1}) = g(x)$; 6) Moreover, if $(\forall x \in G)(\forall k \in \mathbb{N}) : U_k^x = U_k$, then g is a pseudonorm.*

Lemma 4. *Let H be a subgroup of the group G, f be a bounded admissible norm on the group H. Then there exists a bounded admissible norm g on the group G such that $g|H \geq f$.*

Proof. Put $K = \sup\{f(x) : x \in H\}$. For every number $k \in \mathbb{N}$ put $W_k = (f, K2^{-k})$. Then there exists a filter $\{V_k\}$ of neighborhoods of the unit e of the group G such that for every $k \in \mathbb{N}$ the following conditions are satisfied: 1) $(\forall y \in G) : V_k^y = V_k$; 2) $V_{k+1}^2 \subset V_k$; 3) $H \cap V_k \subset W_k$; 4) $\bigcap_{k\in\mathbb{N}} V_k = e$. Then the sequence $\{V_k\}$ satisfies the conditions of Lemma 3, and therefore there exists an admissible norm g_1 on the group G satisfying the assertion of Lemma 3. Let x be an arbitrary element of the group H. There exists a natural number k such that $K2^{-k-1} \leq f(x) < K2^{-k}$. Then $x \in W_k \backslash W_{k+1}$, hence $x \notin V_{k+1}$. Therefore $g_1(x) \geq 2^{-k-1}$, and thus $2Kg_1(x) \geq f(x)$, and we may put $g = 2Kg_1$.

Theorem 2. *Every bounded admissible G-invariant norm from a normal subgroup of the group G can be extended onto the group G.*

Proof. It follows from Lemma 4 and Theorem 1.

Proposition 5. *Let H be a subgroup of a group G, f be a bounded admissible norm on the group H. If the following conditions are satisfied: 1) $f'|H = f$, 2) f' is admissible on the group $\langle\langle H\rangle\rangle$; then the norm f can be extended onto the group G.*

Proof. The bounded norm f' is a G-invariant extension of the norm f on the normal group $\langle\langle H\rangle\rangle$. Then it follows from Theorem 2 that norm f' can be extended onto group G, and therefore the norm f can be extended onto the group G.

5. Some necessary conditions for existence of an extension

We deduce some necessary conditions in order that an admissible norm f on a subgroup H of a group G has an extension f^* onto the group G.

Suppose x is an element of the group G and there exist $y_1, \dots, y_n \in G, \delta_1, \dots, \delta_n \in \{-1, 1\}$ such that $\prod_{i=1}^{n} (x^{\delta_i})^{y_i} = x' \in H$. Then $f(x') \le nf^*(x)$. And if we put

$$S(x) = \sup\Big\{\frac{f(x')}{n} : x' = \prod_{i=1,\delta_i=\pm 1}^{n} (x^{\delta_i})^{y_i} \in H\Big\},$$

then we get $f^*(x) \ge S(x)$. Therefore if a norm f can be extended from the subgroup H onto the group G, then the following conditions are satisfied:

$$(\forall x \in G) : S(x) < \infty. \tag{5.1}$$

$$(\forall x \in H) : S(x) = f(x). \tag{5.2}$$

Remark 4. If f is a G-invariant norm on a normal subgroup H of a group G, then f satisfies the condition (5.2) (it follows from the triangle's inequality).

Similarly for every subset $K \subset G$ put

$$S(K) = \sup\Big\{\frac{f(x')}{n} : H \ni x' = \prod_{i=1,\delta_i=\pm 1}^{n} (x_i^{\delta_i})^{g_i}, x_i \in K, g_i \in G\Big\}.$$

Therefore if a norm f can be extended from the subgroup H onto the group G, then the following condition is satisfied:

$$\text{for every compact } K \subset G \text{ we have } S(K) < \infty. \tag{5.3}$$

If $y \in H, x \in G$ and f^* is an extension of the norm f, then $f^*(x) \ge f^*(xy) - f(y)$. Define a sequence $\{S_n\}$ setting $S_1(x) = S(x), S_{n+1}(x) = \sup_{y\in H}\{S_n(xy) - f(y)\}$. It is easy to see that for every number n we have $S_n \le f^*$, $S_n \le S_{n+1}$.

Put $R(x) = \lim_{n\to\infty} S_n(x)$. Therefore, if a norm f can be extended from the subgroup H onto the group G, then the following conditions are satisfied:

$$(\forall x \in G) : R(x) < \infty. \tag{5.4}$$

$$\lim_{x_n \to e} R(x_n) = 0. \tag{5.5}$$

Remark 5. Some properties of the function $R(x)$:
1. $(\forall x \in G)(\forall y \in H) : R(x) + f(y) \geq R(xy)$ (it is proved by passing to the limit in an inequality $S_{n+1}(x) + f(y) \geq S_n(xy)$).
2. If $(\forall x \in H) : S(x) = f(x)$, then $(\forall x \in H) : R(x) = f(x)$.
3. $(\forall x \in H) : R(x^{-1}) = R(x)$.
4. If $(\forall y \in G, \forall x \in H$ with $x^y \in H) : f(x^y) = f(x)$, then $(\forall y \in G) : R(x^y) = R(x)$.

6. Some particular cases

Proposition 6. *Let H be an open subgroup of a group G. Then a norm f can be extended from the group H to a G-invariant norm on the group $\langle\langle H\rangle\rangle$ iff $f'|H = f$.*

Proof. Sufficiency. Since H is an open subgroup of the group G, and G is a SIN-group, H is a neighborhood of the unit of the group G. Therefore there exists a neighborhood U of the unit such that $\bigcup_{g\in G} U^g \subset H$. Then there exists $\varepsilon > 0$ such that $(f, \varepsilon) \subset U$ and hence $\bigcup_{g\in G}(f, \varepsilon)^g \in H$, and then $\bigcup_{g\in G}(f, \varepsilon)^g = (f, \varepsilon)$. It's easy to see that for every $\delta \leq \varepsilon$, we have $(f, \delta) = (f', \delta)$. Therefore the admissible norm f' is a desired extension.
Necessity. The proof is analogous to that of Lemma 1.

Theorem 3. *Every admissible norm can be extended from an open subgroup H of an abelian group G onto the group G.*

Proof. Let f be an admissible norm on the group H. Then there exists $\varepsilon > 0$ such that (f, ε) is open in the group G, and $\bigcup_{g\in G}(f, \varepsilon)^g = (f, \varepsilon)$. Put $\Omega_f = \{(H^*, f^*) : H \leq H^* \leq G, f^*$ is an extension of f onto $H^*, (f^*, \varepsilon) = (f, \varepsilon)\}$. Define a relation $\prec$ on Ω_f by the following: $(H_1, f_1) \prec (H_2, f_2)$ iff $H_1 \subset H_2$ and $f_2|H_1 = f_1$.

$(\Omega_f, \prec)$ is an inductive set. Actually, to prove this it suffices to note that for every chain $\{(H_\alpha, f_\alpha)\} \subset \Omega_f$ we have[3] $(\bigcup H_\alpha, \bigcup f_\alpha) \subset \Omega_f$, since $\bigcup f_\alpha$ is an admissible norm.

Let (H^*, f^*) be a maximal element of Ω_f. Suppose that $H^* \neq G$. Then there exists $x \in G\backslash H^*$. Define a function $f_1{:}\langle H^*, x\rangle \to \mathbb{R}_+$ as follows.

There are two cases:
1. $|\langle H^*, x\rangle : H| = \infty$. Then $\langle H^*, x\rangle/H \simeq \mathbb{Z}$. Let y be an element of the group $\langle H^*, x\rangle$. Then y has a unique representation in the form $y = k_y x + x_y$, where $k_y \in \mathbb{Z}, x_y \in H^*$. Put $f_1(y) = f^*(x_y) + |k_y|\varepsilon$.
2. $|\langle H^*, x\rangle : H| = n < \infty$. Let y be an element of the group $\langle H^*, x\rangle$. Then y has a unique representation in the form $y = k_y x + x_y$, where $k_y \in \{0, \dots, n-1\}$, $x_y \in H^*$. Put

$$f_1(y) = \begin{cases} \varepsilon + \frac{1}{2}f^*(x_y) + \frac{1}{2}f^*(nx) + \frac{1}{2}f^*(x_y + nx), & k_y > 0 \\ f^*(x_y), & k_y = 0. \end{cases}$$

Clearly that in the case 1 $(\langle H^*, x\rangle, f_1) \in \Omega_f$. In the case 2 it suffices to prove that f_1 is a norm on the group $\langle H^*, x\rangle$.

If $k > 0$, then $-(kx + x_y) = (n-k)x + (-nx - x_y)$, and therefore $f_1(-(kx + x_y)) = f_1(kx + x_y)$.

[3] $\bigcup f_\alpha$ is the function $f^*{:}\bigcup H_\alpha \to \mathbb{R} : \{\forall x_\alpha \in H_\alpha : f^*(x_\alpha) = f_\alpha(x_\alpha)\}$.

If $k_1, k_2 > 0, k_1 + k_2 > n$, then $f_1(x_1 + k_1 x) + f_1(x_2 + k_2 x) - f_1(x_1 + x_2 + nx + (k_1 + k_2 - n)x) = (f^*(x_1) + f^*(x_1 + nx) + f^*(x_2) + f^*(x_2 + nx) + f^*(nx) + 2\varepsilon - f^*(x_1 + x_2 + nx) - f^*(x_1 + x_2 + 2nx))/2 \geq 0$.

If $k_1, k_2 > 0, k_1 + k_2 < n$, then $f_1(x_1 + k_1 x) + f_1(x_2 + k_2 x) - f_1(x_1 + x_2 + (k_1 + k_2)x) = (f^*(x_1) + f^*(x_1 + nx) + f^*(x_2) + f^*(x_2 + nx) + f^*(nx) + 2\varepsilon - f^*(x_1 + x_2) - f^*(x_1 + x_2 + nx))/2 \geq 0$.

If $k_1, k_2 > 0, k_1 + k_2 = n$, then $f_1(x_1 + k_1 x) + f_1(x_2 + k_2 x) - f_1(x_1 + x_2 + nx) = (f^*(x_1) + f^*(x_1 + nx) + f^*(x_2) + f^*(x_2 + nx))/2 + f^*(nx) + \varepsilon - f^*(x_1 + x_2 + nx) \geq 0$.

If $k > 0$, then $f_1(x_1) + f_1(x_2 + kx) - f_1(x_1 + x_2 + kx) = (2f^*(x_1) + f^*(x_2) + f^*(x_2 + nx) - f^*(x_1 + x_2) - f^*(x_1 + x_2 + nx))/2 \geq 0$.

The rest of the norm's properties can be trivially verified.

Hence $(\langle H^*, x\rangle, f_1) \in \Omega_f$, which is a contradiction. Therefore $H^* = G$.

Theorem 4. *Let H be a subgroup of an abelian group G. If $G/\overline{H}$ is finitely generated, then every admissible norm can be extended from the group H onto the group G.*

Proof. It suffices to consider case when $G/\overline{H}$ is a cyclic group. It follows from Proposition 1 that without loss of generality we may suppose that the group H is a closed subgroup of the group G.

If $|G : H| \leq \aleph_0$, then H is an open subgroup of the group G, and the theorem's assertion follows from Theorem 3.

If $|G : H| = \aleph_0$, then $G/H \simeq (\mathbb{Z}, \sigma)$, where σ is a metrizable group SIN-topology on the group $\mathbb{Z}$. Therefore, there exists a norm g on the group $\mathbb{Z}$, generating the topology σ. Let $\pi{:}G \to \mathbb{Z}$ be the quotient-map. Fix an arbitrary element a of the group G such that $\pi(a) = 1$. Then every element x of the group G has a unique representation in the form $x = na + h$, where $n \in \mathbb{Z}, h \in H$. Put $f^*(x) = f(h) + g(n)$. It's easy to see that f^* is a norm on the group G.

Let $\varepsilon > 0$. Then there exists $U, V \subset G$ such that U, V are open in G, $\pi(U) \subset (g, \varepsilon)$, $V \cap H \subset (f, \varepsilon)$. Then for every element $x \in U \cap V$ we have $f^*(x) < 2\varepsilon$, therefore f^* is a continuous norm on the group G.

Let U be a neighborhood of the unit of the group G. Then there exists $\varepsilon_1 > 0$ such that $(f, \varepsilon_1) \subset U$. It follows from the openness of the map π that there exists $\varepsilon_2 > 0$ such that $\pi(U) \supset (g, \varepsilon_2)$. Put $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then for every element $x \in (f^*, \varepsilon)$ we have $x \in U + U$. Therefore f^* is a dominating norm on the group G. Hence f^* is an admissible norm and therefore an extension of the norm f.

7. Counterexamples

Example 1. We give an example of a countable metrizable SIN-group G, an open normal subgroup H of index 2 in G and an admissible norm g on the group H such that g can not be extended onto the group G.

Let the group $\mathbb{Z}_2$ acts on the group $\mathbb{Q}$ as follows: $\sigma_0(x) = x, \sigma_1(x) = -x$. Then for arbitrary elements a, b of the group $\mathbb{Z}_2$ we have $\sigma_{ab} = \sigma_a \sigma_b$, and therefore there exists the topological semidirect product (see [RD], [GZ p.15]) $\mathbb{Q} \times_\sigma \mathbb{Z}_2 = G$.

The table of multiplication on the group G has the following form: $(x, 0)(y, 0) = (x + y, 0)$, $(x, 1)(y, 0) = (x - y, 1)$, $(x, 0)(y, 1) = (x + y, 1)$, $(x, 1)(y, 1) = (x - y, 0)$. In addition, $(x, 0)^{-1} = (-x, 0)$, $(x, 1)^{-1} = (x, 1)$, $(y, 1)(x, 1)(y, 1) = (-x, 1)$, $(y, 0)(x, 1)(-y, 0) = (x + 2y, 1)$, $(y, 1)(x, 0)(y, 1) = (-x, 0)$.

Put $H = (\mathbb{Q}, 0)$. Define a function $f{:}G \to \mathbb{R}_+$ setting $f(x,0) = \operatorname{arctg}|x|$, $f(x,1) = \frac{\pi}{4}$. We claim that the function f is a norm on G. Evidently, f satisfies the properties (1),(2),(4) of norms.

To show that f satisfies the triangle inequality, conside the following cases:

1. $x, y \in H$. The inequality is satisfied, because $\operatorname{arctg}|x| + \operatorname{arctg}|y| \geq \operatorname{arctg}|x+y|$.
2. $x \in H, y \notin H$. The inequality holds, because $\pi/4 + \operatorname{arctg}|x| \geq \pi/4$.
3. $x, y \notin H$. The inequality is satisfied, because $\pi/2 \geq \operatorname{arctg}|x-y|$.

Hence, $(G, \tau f)$ is a metrizable SIN-group, H is an open normal subgroup of the group G. Define a function $g{:}H \to \mathbb{R}_+$, $g(x) = |x|$. Clearly that g is an admissible norm on the group H. Assuming that g^* is an extension of the norm g onto the group G, we would get that $g^*(x,1) = g^*((y,0)(x,1)(-y,0)) = g^*(x+2y,1)$ for every $x, y \in \mathbb{Q}$. Therefore $g^*|(\mathbb{Q},1) = C = \text{const}$. But then $g^*(x,0) = g^*((x+1,1)(1,1)) \leq 2C$, contrary to $g^*(x,0) = |x|$.

Remark 6. The norm g cannot be extended onto the group G, because for an arbitrary $x \in (\mathbb{Q}, 1)$ we have $S(x) = \infty$.

Example 2. We give an example of a countable metrizable abelian SIN-group G, a closed subgroup H in G and an admissible norm f on the group H such that g cannot be extended onto the group G.

Put $G = \langle \mathbb{Q}, x_1, \ldots, x_n, \ldots; nx_n = n \rangle$. Define a function $f{:}G \to \mathbb{R}_+$ as the follows. Let y be an element of the group G. Then y has a unique representation in the form $y = x(y) + \sum a_{yn} x_n$, where $x(y) \in \mathbb{Q}, a_{yn} \in \{0, ..., n-1\}$. Put

$$f(y) = \operatorname{arctg}|x(y)| + \frac{\pi}{2} \sum \frac{a_{yn}}{n}.$$

Then the function f satisfies the following conditions:

1. $(\forall y \in G) : f(y) \geq 0$ and $f(y) = 0$ iff $y = e$.
2. Let y, z be arbitrary elements of the group G. Then $f(y) + f(z) = f(x(y) + \sum a_{yn} x_n) + f(x(z) + \sum a_{zn} x_n) = \operatorname{arctg}|x(y)| + \operatorname{arctg}|x(z)| + \frac{\pi}{2} \sum (a_{yn} + a_{zn})/n$, but $f(y+z) = f(x(y) + x(z) + \sum a_{(y+z)n} x_n + t)$, where $t = \sum t_n$,

$$a_{(y+z)n} = \begin{cases} a_{yn} + a_{zn}, & a_{yn} + a_{zn} < n \\ a_{yn} + a_{zn} - n, & a_{yn} + a_{zn} \geq n \end{cases}; \quad t_n = \begin{cases} 0, & a_{yn} + a_{zn} < n \\ n, & a_{yn} + a_{zn} \geq n \end{cases},$$

therefore

$$f(y+z) = \operatorname{arctg}|y+z+t| + \frac{\pi}{2} \Big(\sum_{a_{yn}+a_{zn}<n} \frac{a_{yn} + a_{zn}}{n} + \sum_{a_{yn}+a_{zn} \geq n} \frac{a_{yn} + a_{zn} - n}{n} \Big).$$

The following cases are possible:

1. For every n we have $a_{yn} + a_{zn} < n$. Then $t = 0$, $f(y) + f(z) - f(y+z) = \operatorname{arctg}|y| + \operatorname{arctg}|z| - \operatorname{arctg}|y+z| \geq 0$.
2. There exists n such that $a_{yn} + a_{zn} \geq n$. Then $f(y) + f(z) - f(y+z) \geq \pi/2 - \operatorname{arctg}|y+z+t| \geq 0$.

Hence for arbitrary elements y, z of the group G we have $f(y) + f(z) \geq f(y+z)$.

Set a function $f_1{:}G \to \mathbb{R}_+$, $f_1(y) = f(y) + f(-y)$. Then $f_1(y) = f_1(-y)$, for every $y \in G$. Next, for arbitrary elements y, z of the group G we have $f_1(y) + f_1(z) - f_1(y+z) = f(y) + f(z) - f(y+z) + f(-y) + f(-z) - f(-y-z) \geq 0$.

Hence, f_1 is a norm on the group G and $(G, \tau f_1)$ is a topological group.

Now put $H = \mathbb{Q}$. Then H is a closed subgroup of the group G. Define a function $g{:}H \to \mathbb{R}_+$, $g(x) = |x|$. Clearly that g is an admissible norm on the group H.

If there exists an extension g^* of the norm g onto the group G, then $g^*(x_n) \geq n/n = 1$. On the other hand, $x_n \to 0$ with $n \to \infty$. Then $g^*(x_n) \to 0$ with $n \to \infty$, which is a contradiction.

Remark 7. The norm g cannot be extended onto the group G, because the condition (5.5) is not satisfied.

REFERENCES

[GZ]. Гуран І.Й., Зарічний М.М. Елементи теорії топологічних груп. – Київ: НМК ВО, 1991.– 76 с.

[E]. Энгелькинг Р., Общая топология – М.,: Мир, 1981.

[B]. Bessaga C. *Functional analytic aspects of geometry. Linear extending of metrics and related problems,* in: *Progress of Functional Analysis,* Proc. Peniscola Meeting 1990 on the 60th birthday of Professor M. Valdivia, North-Holland, Amsterdam, 1992.– P.247-257.

[H]. Hausdorff F. *Erweiterung einer Homömorpie* Fund. Math. – 1930.– V.16.– P.353–360.

[I]. Isbell J.R. *On finite-dimensional uniform spaces* Pacific J. of Math.– 1959.– V.9.– P.107–121.

[RD]. Roelcke W., Dierolf S. Uniform structures on topological groups and their quotiens, Mc Graw Hill, 1981.– 276 p.

[Z]. Zarichnyi M. *Regular linear operators extending metrics: a short proof* Bull. Pol. Ac.:Math.– 1996.– V.44.– P.267–269.

Department of Mechanics and Mathematics, Lviv State University

Universytetska str. 1, 290602, Lviv, Ukraine

Received 25.08.98