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THE SUMS OF STATIONARY VARIABLES BETWEEN TWO CURVILINEAR BOUNDARIES

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The convergence speed of invariant principle of the boundary problem is investigated. The dependence of a convergence speed on the mixing coefficient of the stationary sequence random variables is formed.

Let $\{\xi_i\}_{i\geq 1}$ be a sequence of stationary random variables and $s_n = \sum_1^n \xi_i$ a sequence of increasing sums. It is well-known that the stay probability of the sums s_n into the domains may approximate by the stay probability of Wiener process into the corresponding domains [2]. In the article [2] Levi-Prochorov estimation of the approximation error is obtained. Levi-Prochorov estimation is defined for all measurable domains. The purpose of this paper is to obtain such estimation in tube domains. This restriction of the class of domains leads to the essential weaken of conditions on the mixing coefficient.

Under the analysis of the increasing sums in tube domains we must consider the boundary effects. For this purpose we add to the boundaries the small constants $\pm\varepsilon$. Further, we must estimate this perturbation.

Later on the following fact is necessary. Given two functions $g_1(t)$, $g_2(t)$ such that $g_1(t) < g_2(t)$, $t \geq 0$, consider two problems with the common operator part: $u_t = 2u_{xx}$.

The boundary conditions and the domains of the first and second problem have the following forms

$$u\big(g_i(t),t\big) = 0,\ u(x,0) = 1,\ x \in [g_1(0), g_2(0)];$$
$$(x,t) \in D := \big\{(x,t) : g_1(t) \leq x \leq g_2(t),\ t \in [0,T]\big\},$$
$$u\big(g_1(t)-\varepsilon,t\big) = 0,\ u\big(g_2(t)+\varepsilon,t\big) = 0,\ u(x,0) = 1,\ x \in [g_1(0)-\varepsilon, g_2(0)+\varepsilon];$$
$$(x,t) \in D_\varepsilon := \big\{(x,t) : g_1(t)-\varepsilon \leq x \leq g_2(t)+\varepsilon,\ t \in [0,T]\big\},$$

respectively.

Thus in the second problem we have the perturbed domain. Denote the solutions of the first and the second problems by the $u(x,t)$, $u_\varepsilon(x,t)$ respectively. Extend

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$u(x,t)$ on D_ε by the following way: if $(x,t) \in D_\varepsilon$ and $x > g_2(t)$ then $u(x,t) = u\big(g_2(t),t\big)$ and $u(x,t) = u\big(g_1(t),t\big)$ if $(x,t) \in D_\varepsilon$ and $x < g_1(t)$. Now, if $g_i(t) \in C^2$, $i = 1,2$ then the following estimate holds

$$|u_\varepsilon(x,t) - u(x,t)| \le c\varepsilon, \quad \forall\ (x,t) \in D_\varepsilon \times [0,T],\ c \le \infty.$$

Let $\{\xi_i\}_{i\ge1}$ be a stationary sequence of the random variables with the zero average and the bounded third absolute moment. Find real twice differentiable functions $g^+(t)$, $g^-(t)$ satisfying the relations $g^+(t) > g^-(t)$, $g^-(0) < 0 < g^+(0)$, $t \in [0,1]$.

Define the sigma-algebra $F_a^b = \sigma(\xi_i, a \le i \le b)$ and the mixing coefficient

$$\varphi(n) = \sup_{1\le k\le\infty} \sup_{A\in F_1^k, B\in F_{k+n}^\infty} |P(A/B) - P(A)|, \quad \varphi(0) = 0.$$

The following Lemma can be proved by using the theorem of total probability.

Lemma 1. *Let $\{\eta_i\}_{1\le i\le m}$, $\{\hat\eta_i\}_{1\le i\le m}$ be two set of random variables and $\{l(i),\ 1 \le i \le m\}$ the set of positive integers such that the following conditions hold:*

1. $l(i) < l(i+1)$,

2. $\{\eta_i \in A\} \in F_{l(i)}^{l(i+1)}$, where A is any set of Borel sigma-algebra B_R on $R = (-\infty,\infty)$.

3. The random variables $\hat\eta_i$, $1 \le i \le m$, are mutually independent and independent of η_i, $1 \le i \le m$.

4. The random variables η_i ma $\hat\eta_i$, $1 \le i \le m$, have the identical distribution functions.

Denote $r = \min_{1\le i\le m-1} |l(i+1) - l(i)|$, then the following inequality

$$\Big|P\big(a_i \le \sum_{k=1}^{i} \eta_k \le b_i,\ i = \overline{1,m}\big) - P\big(a_i \le \sum_{k=1}^{i} \hat\eta_k \le b_i,\ i = \overline{1,m}\big)\Big| \le \varphi(r)m,$$

is valid.

Put $\|f(t)\| = \max_{0\le t\le1} |f(t)|$, $f(t) \in C_{[0,1]}$, by $w(t)$ is a Wiener process, $w(0) = 0$;

$$s_k = \sum_{i=1}^{k} \xi_i, \quad t_k = k/n, \quad s_0 = t_0 = 0; \quad \sigma = M\xi_1^2 + 2\sum_{i=2}^{\infty} M\xi_1 M\xi_i.$$

Denote the broken line with the knots in the points $(t_k, (n\sigma)^{-1/2}s_k)$ by $s_n(t)$: $s_n(t_k) = (n\sigma)^{-1/2}s_k$. Consider the probabilities

$$J_n = P\big(g^-(t_k) < s_n(t_k) < g^+(t_k),\ k = \overline{0,n}\big),$$
$$J = P\big(g^-(t) < w(t) < g^+(t);\ t \in [0,1]\big).$$

Theorem 1. *Let* $\sigma > 0$ *and* $\varphi(n) = n^{-\varphi}$, $h = (1+\varphi)^{-1}$, *now if* $\varphi \geq 20$, *then the following inequality* $|J_n - J| \leq cn^{-1/9+h}$, $c < \infty$ *is valid.*

Proof. Describe the plan of the proof. At first we decompose the sums s_k on the alternating blocs (Bernshtein method). Futher, we'll construct the new broken lines $s_n^{(1)}(t)$, $s_n^{(2)}(t)$ by these blocks and prove that these broken lines are approximations of $s_n(t)$ in the uniform metric by the method of the article [2]. After this we'll pass from the dependent blocks to the independent blocks. The evaluation of such change is obtained by the proved Lemma 1. Finally, the invariant principle is proved by the modified Kolmogorov-Petrovsky method from the article [1]. The idea of this method consists of using the heat equation. It is well-known [3, p.180] that $J = u(0,0)$, where $u(t,x)$ is the solution of the following initial-boundary problem

$$\begin{gathered} u_t = -\tfrac{1}{2}u_{xx}, \quad g^-(t) < x < g^+(t),\, t \in [0,1] \\ u(1,x) = I_x\big([g^-(1), g^+(1)]\big), \quad u\big(t, g^\pm(t)\big) = 0, \end{gathered}$$

where $I_x(A)$ is the indicator of a set A.

Further we'll prove that the probability J_n is the approximation of the $u(0,0)$.

We'll divide on the blocks by analogy with the article [2]. Assume that

$$n = ml + r, \quad l = b + h.$$

We drop the dependence of the indices l, m, r, b, h on the parameter h. As usual, under using of the Bernshtein method, the concrete forms of the $l(n), m(n), r(n), b(n), h(n)$ will be defined later. Now, it is sufficient to know that

$$l(n) = o(n), \quad h(n) = o\big(l(n)\big), \quad r(n) \geq 0.$$

Put

$$\eta_i = \sum_{k=1}^{b} \xi_{(i-1)l+k}, \quad \psi_i = \sum_{k=1}^{h} \xi_{(i-1)l+b+k}.$$

Denote the random broken line with the knots $\big(t_{kl}, s_{kl}(n\sigma)^{-1/2}\big)$, $k = \overline{0,m}$ and if $r > 0$ with the last knot $\big(1, s_n(n\sigma)^{-1/2}\big)$, by $s_n(t)$.

It follows from the results of [2] that the following inequalities hold

$$\begin{aligned} P\big(\|s_n(t) - s_n^{(1)}(t)\| \geq \varepsilon,\ t \in [0,1]\big) &\leq \sum_{k=1}^{m} P\Big(\max_{1\leq i\leq l} |s_{i+kl} - s_{kl}| \geq \varepsilon(n\sigma)^{1/2}\Big) = \\ &= \sum_{k=1}^{m} P\Big(\max_{1\leq i\leq l} |s_i| \geq \varepsilon(n\sigma)^{1/2}\Big) \leq m\sum_{i=1}^{l} M|\xi_i|^3 \varepsilon^{-3}(n\sigma)^{-3/2}. \quad (1) \end{aligned}$$

Prove that the following inequalities

$$\begin{gathered} P\big(g^-(t_k) + \varepsilon < s_n^{(1)}(t) < g^+(t_k - \varepsilon,\ k = \overline{0,n}\big) - P\big(\|s_n(t) - s_n^{(1)}(t)\| \geq \varepsilon\big) \leq \\ \leq P\big(g^-(t_k) < s_n(t) < g^+(t_k),\ k = \overline{0,n}\big) \leq \\ \leq P\big(g^-(t_k) - \varepsilon < s_n^{(1)}(t) < g^+(t_k) + \varepsilon,\ k = \overline{0,n}\big) + P\big(\|s_n(t) - s_n^{(1)}(t)\| \geq \varepsilon\big) \quad (2) \end{gathered}$$

are valid. We drop the indices t, k:

$$P(g^- < s_n < g^+) = P\big(g^- < s_n < g^+, \|s_n - s_n^{(1)}\| < \varepsilon\big) + P\big(g^- < s_n < g^+,\ \|s_n - s_n^{(1)}\| \ge \varepsilon\big).$$

Now, the right part of inequality (2) follows from the following inclusion

$$\big\{g^- < s_n < g^+,\ \|s_n - s_n^{(1)}\| < \varepsilon\big\} \subseteq \{g^- - \varepsilon < s_n^{(1)} < g^+ + \varepsilon\}.$$

Further, by analogy with the previous reasoning,

$$\big\{g_- + \varepsilon < s_n^{(1)} < g^+ - \varepsilon,\ \|s_n - s_n^{(1)}\| < \varepsilon\big\} \subseteq \{g^- < s_n < g^+\},$$

and

$$\begin{aligned} &P(g^- < s_n < g^+) \ge \\ &\ge P\big(g- + \varepsilon < s_n^{(1)} < g^+ - \varepsilon\big) - P\big(g^- + \varepsilon < s_n^{(1)} < g^+ - \varepsilon,\ \|s_n - s_n^{(1)}\| > \varepsilon\big) \ge \\ &\qquad \ge P\big(g^- + \varepsilon < s_n^{(1)} < g^+ - \varepsilon\big) - P\big(\|s_n - s_n^{(1)}\| \ge \varepsilon\big), \end{aligned}$$

which is what had to be proved.

Now, replace the calculation of the $P(g^- < s_n < g^+)$ by the calculation of $P(g^- - \varepsilon < s_n^{(1)} < g^+ + \varepsilon)$.

Denote the random broken line with the knots $\big(t_{kl}, \sum_{i=1}^{k} \eta_i (n\sigma)^{-1/2}\big)$, $k = 0, \dots, m-1$ by $s_n^{(2)}(t)$. Further,

$$P\big(\|s_n^{(1)}(t) - s_n^{(2)}(t)\| \ge \varepsilon\big) \le P\big(\max_{1 \le s \le m} \sum_{k=1}^{s} \psi_k \ge \varepsilon (n\sigma)^{-1/2}\big) \le \sum_{i=1}^{m} M|\psi_i|^3 \varepsilon^{-3} (n\sigma)^{-1/2} \tag{3}$$

The following inequalities hold

$$\begin{aligned} &P\big(g^-(t_{kl}) + 2\varepsilon < s_n^{(2)}(t) < g^+(t_{kl}) - 2\varepsilon,\ k = \overline{0, m+1}\big) - P\big(\|s_n^{(2)}(t) - s_n^{(1)}(t)\| > \varepsilon\big) \le \\ &\qquad \le P\big(g^-(t_{kl}) - \varepsilon < s_n^{(1)}(t) < g^+(t_{kl}) + \varepsilon,\ k = \overline{0, m+1}\big) \le \\ &\le P\big(g^-(t_{kl}) - 2\varepsilon < s_n^{(2)}(t) < g^+(t_{kl}) + 2\varepsilon,\ k = \overline{0, m+1}\big) + P\big(\|s_n^{(1)}(t) - s_n^{(2)}(t)\| \ge \varepsilon\big). \end{aligned}$$

Replace the calculation of $P(g^- - \varepsilon < s_n^{(1)} < g^+ + \varepsilon)$ by the calculation of $P(g^- - 2\varepsilon < s_n^{(2)} < g^+ + 2\varepsilon)$.

Denote the broken line with knots $\big(t_{kl}, \sum_{i=1}^{k} \hat{\eta}_i (n\sigma)^{-1/2}\big)$, $k = \overline{0, m+1}$, by $s_n^{(3)}(t)$. Here $\hat{\eta}_i$ are mutually independent and identically distributed with η_i, $i = \overline{0, m+1}$ random variables. From Lemma 1 we imply that

$$\big|P\big(g^- - 2\varepsilon < s_n^{(2)} < g^+ + 2\varepsilon\big) - P\big(g^- - 2\varepsilon < s_n^{(3)} < g^+ + 2\varepsilon\big)\big| \le \varphi(h) m.$$

Let $v(t_{kl}, x)$ be the probability of the following event: at the time moment t_{kl} the broken line $s_n^{(3)}(t)$ was in the point x and during the time $(t_{kl}, 1]$ it will not exit from the intervals $\big[g^-(t_{sl}) - 2\varepsilon, g^+(t_{sl}) + 2\varepsilon\big]$, $s \ge k+1$. Assume that $v_{kl}(t_{kl}, x) = 0$ if $x \in \big[g^-(t_{kl}) - 2\varepsilon, g^+(t_{kl}) + 2\varepsilon\big]$.

Put $F_{il}(x) := P\big(\hat{\eta}_1(n\sigma)^{-1/2} < x\big)$, $i = \overline{1, m+1}$. Define the operators F_{il}:

$$F_{(i+l)}z(x) = \begin{cases} 0, & \text{if } \ x \notin \big[g^-(t_{il}) - 2\varepsilon, g^+(t_{kl}) + 2\varepsilon\big], \\ \int z(u+x)dF_{(i+1)l}(u), & \text{if } \ x \in \big[g^-(t_{il}) - 2\varepsilon, g^+(t_{il}) + 2\varepsilon\big]. \end{cases}$$

Then, it is obvious that the following equalities hold

$$v(t_{kl}, x) = F_{(k+1)l}v(t_{(k+1)l}, \cdot)(x), \quad k = 0, \dots, m,$$
$$v_\varepsilon(1, x) = \begin{cases} 1, & \text{if } x \in \big(g^-(1) - 2\varepsilon, g^+(1) + 2\varepsilon\big), \\ 0, & \text{otherwise;} \end{cases}$$
$$v_\varepsilon(0, x) = F_l \cdot \dots \cdot \dot{F}_{(m+1)l}v_\varepsilon(1, \cdot)(x).$$

Put $K(x, y) = \frac{1}{\sqrt{\pi}} \int_x^y e^{-z^2} dz$; $\alpha \geq 0$,

$$Z^+_{\varepsilon,\alpha}(x) = \begin{cases} 1, & x \in \big[g^-(1) - 2\varepsilon, g^+(1) + 2\varepsilon\big]; \\ K\big(\frac{1}{g^+(1)+2\varepsilon-x)} + \frac{1}{g^+(1)+2\varepsilon+\alpha-x}, \infty\big), & x \in \big[g^+(1) + 2\varepsilon, g^+(1) + 2\varepsilon + \alpha\big]; \\ K\big(-\infty, \frac{1}{g^-(1)-2\varepsilon-x} + \frac{1}{g^-(1)-2\varepsilon-\alpha-x}\big), & x \in \big[g^-(1) - 2\varepsilon - \alpha, g^-(1) - 2\varepsilon\big]; \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to prove that

$$v_\varepsilon(1, x) \leq z^+_{\varepsilon,\alpha}(x), \quad v_\varepsilon(0, x) \leq F_l \cdot \dots \cdot F_{(m+1)l}Z^+_{\varepsilon,\alpha}(x).$$

The function $Z^+_{\varepsilon,\alpha}(x)$ has the bell-shaped form. We'll define it as the "upper" function of the $v_\varepsilon(1, x)$.

Now, consider the solution $u_{\varepsilon,\delta,\alpha}(t, x)$ of the following heat conductivity problem:

$$u_t = -\frac{1}{2}u_{xx}, \quad (x, t) \in D_{2\varepsilon+\delta},$$
$$u(1, x) = Z^+_{\varepsilon,\alpha}(x), \quad u(t, g^-(t) - 2\varepsilon - \delta) = 0, \quad u(t, g^+(t) + 2\varepsilon + \delta) = 0,$$

where

$$D_{2\varepsilon+\delta} := \big\{(x, t) : x \in \big(g^-(t) - 2\varepsilon - \delta, g^+(t) + 2\varepsilon + \delta\big), \quad t \in [0, 1]\big\}; \quad \varepsilon, \delta \sim 0_+.$$

As $Z^+_{\varepsilon,\alpha}(x)$ has the fourth continuous derivatives and the functions $g^\pm(t)$ are twice continuously differentiable, using the results of the monograph [4] we obtain that the following inequalities hold

$$\sup\big|\tfrac{\partial^l}{\partial x^l}u(t, x)\big| \leq c_1 < \infty, \quad l = \overline{0, 3}, \quad \sup\big|\tfrac{\partial^2}{\partial t^2}u(t, x)\big| \leq c_2 < \infty,$$

where supremum is calculated on the domain $D_{2\varepsilon+\delta}$.

Keeping in mind the remark at the beginning, we can assert that for all δ and α the following inequalities hold

$$|u_{\varepsilon,\delta,\alpha}(t, x) - u_{\varepsilon,0,\alpha}(t, x)| < a_1\delta,$$
$$|u_{\varepsilon,\delta,\alpha}(t, x) - u_{\varepsilon,0,0}(t, x)| < a_2(\alpha + \delta), \quad \max(a_1, a_2) < \infty, \quad \alpha, \delta \to 0.$$

Setting $\tilde{u}(t,x) = u_{\varepsilon,\delta,\alpha}(t,x) + a_1\delta$, we obtain $\tilde{u}(t,x) \le u_{\varepsilon,\delta,\alpha}$. Consider the inner points: t_{kl}, $k \le m$. We'll obtain by the Taylor series

$$\int \tilde{u}(t_{kl}, x-z)dF_{kl}(z) = \int\limits_{|z|\le\delta} + \int\limits_{|z|>\delta} = \tilde{u}(t_{kl},x) + \frac{\partial}{\partial x}\tilde{u}(t_{kl},x)\int\limits_{|z|\ge\delta} zdF_{kl}(z)+$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{u}(t_{kl},x)\Big(\int z^2dF_{kl}(z) - \int\limits_{|z|\ge\delta} z^2dF_{kl}(z)\Big)-$$
$$-\frac{1}{6}\int\limits_{|z|\le\delta}\frac{\partial^3}{\partial x^3}\tilde{u}(t_{kl},x_z)z^3dF_{kl}(z) + \int\limits_{|z|\ge\delta}\tilde{u}(t_{kl},x+z)dF_{kl}(z) + \int\limits_{|z|\ge\delta} dF_{kl}(z),$$

where $x_z \in (x, x-z)$. Note, that the integration domain has a more complicated form. However, such representation is sufficient for the asymptotical analysis.

Now we use that $\tilde{u}(t,x)$ is the solution of the heat conductivity equation. The Taylor-series expansion about $t \in [t_{(k-1)}l, t_{kl}]$ has the following form

$$\tilde{u}(t,x) = \tilde{u}(t_{kl},x) + \frac{\partial}{\partial t}\tilde{u}(t_{kl},x)(t-t_{kl}) + \frac{\partial^2}{\partial t^2}\tilde{u}(\tilde{t},x)(t-t_{kl})^2,$$

where t is some point of the interval $[t, t_{kl}]$.

Setting $t = t_{(k-1)l}$, from the latter we obtain

$$\tilde{u}(t_{(k-1)l},x) = \tilde{u}(t_{kl},x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{u}(t_{kl},x)\frac{l}{n} + \frac{\partial^2}{\partial t^2}\tilde{u}(\tilde{t},x)\Big(\frac{l}{n}\Big)^2$$

Thus,

$$F_{kl}\tilde{u}(t_{kl},x) = \tilde{u}(t_{(k-1)l},x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{u}(t_{kl},x)\Big(\frac{1}{n\sigma}M\eta_1^2 - \frac{l}{n}\Big) - \frac{\partial^2}{\partial t^2}\tilde{u}(\tilde{t},x)\Big(\frac{l}{n}\Big)^2 + a_n,$$

where

$$a_n = \sum_{i=1}^{2}\frac{(-1)^{i+1}}{i!}\frac{\partial^i}{\partial x^i}\tilde{u}(t_{kl},x)\int\limits_{|z|\ge\delta} z^i dF_{kl}(z) - \frac{1}{6}\int\limits_{|z|\le\delta}\frac{\partial^3}{\partial x^3}\tilde{u}(t_{kl},x_z)dF_{kl}(z)-$$
$$-\int\limits_{|z|\ge\delta}\tilde{u}(t_{kl},x+z)dF_{kl}(z).$$

Using the inequality

$$\int\limits_{|z|>\delta}|z^i|dF_{kl}(z) \le \frac{M|\eta_1|^3}{\delta^{3-r}n^{3/2}\sigma^{3/2}}, \quad r = 0,1,2,$$

we obtain the following inequality $|a_n| \le 4c_1\frac{m|\eta_1|^3}{n^{3/2}\sigma^{3/2}\delta^3}$. Keeping in mind that the sequence ξ_i, $i \ge 1$, is stationary we obtain (see [5, sec.4])

$$M\eta_1^2 = M\Big(\sum_{i=1}^{b}\xi_i\Big)^2 = bM\xi_1^2 + 2\sum_{k=2}^{b-1}(b-k)M\xi_1\xi_k.$$

Further,

$$\Big|\frac{1}{n\sigma}M\eta_1^2 - \frac{l}{n}\Big| = \Big|\frac{1}{n\sigma}(M\eta_1^2 - b\sigma) + \frac{b-l}{n}\Big| \le \frac{b}{n\sigma}\sum_{k\ge b}|M\xi_1\xi_k| + 2\frac{1}{n\sigma}\sum_{k\ge 2}k|M\xi_1\xi_k| + \frac{h}{n} \le$$
$$\le 2M\xi_1^2\Big(\frac{b}{n\sigma}\sum_{k\ge b}\varphi^{1/2}(k) + \frac{2}{n\sigma}\sum_{k\ge 1}k\varphi^{1/2}(k)\Big) + \frac{h}{n}.$$

Thus,

$$F_{kl}\tilde u(t_{kl},x) \le \tilde u(t_{(k-1)l},x) + r_n, \quad x\in (g^-(t_{(k-1)l}) - 2\varepsilon, g_+(t_{(k-1)l}) + 2\varepsilon).$$
$$r_n \le \frac{c_2 l^2}{2n^2} + 2M\xi_1^2\Big(\frac{b}{n\sigma}\sum_{k\ge b}\varphi^{1/2}(k) + \frac{2}{n\sigma}\sum_{k\ge 1}k\varphi^{1/2}(k)\Big) + \frac{h}{n} + 4c_1\frac{M|\eta_1|^3}{\delta^3 n^{3/2}\sigma^{3/2}}; \tag{4}$$

$c_i < \infty$, $i = 1,2$.

Note that if $x \in \big(g^-(t) - 2\varepsilon, g^+(t) + 2\varepsilon\big)$, and the solution $\tilde u(t,x)$ is defined in the interval $\big(g^-(t) - 2\varepsilon - \delta, g^+(t) + 2\varepsilon + \delta\big)$, then we can expand it into a series under the following condition

$$|z| \le \delta, \quad \text{where} \quad \big(t_{(k+1)l} - t_{kl}\big)g = g\tfrac{l}{n} < \delta, \tag{5}$$

The constant g is a maximal Lipschitz constant of the functions $g^{\pm}(t)$. Thus, for all x we have $F_{kl}\tilde u(t_{kl},x) \le \tilde u(t_{(k-1)l},x) + r_n$.

Applying procedure (4) m times, we obtain under $\alpha = \delta$:

$$v_\varepsilon(0,x) \le F_l\cdot\dots\cdot F_{ml}u_{\varepsilon,0,\alpha}(1,\cdot)(x) \le \tilde u(0,x) + mr_n \le u_{\varepsilon,0,0}(0,x) + 2\delta a_2 + mr_n.$$

Now, from (1), (3), (4) and $|u_{\varepsilon,0,0,}(x,t) - u_{0,0,0}(0,0)| \le a\varepsilon$, $a<\infty$, if $\delta = \varepsilon$ we obtain the following inequality

$$P\big(g^-(t_k < s_n(t) < g^+(t_k),\ k = \overline{0,n}\big) \le u_{0,0,0}(0,0) + mr_n + m\varphi(h)m +$$
$$+ P\big(\|s_n^{(1)}(t) - s_n^{(2)}(t)\| \ge \varepsilon\big) + P\big(\|s_n(t) - s_n^{(1)}(t)\| \ge \varepsilon\big) + \mathrm{const}\cdot\varepsilon. \tag{6}$$

The inverse inequality can be proved by the determination of "narrowed" domain with respect to the domain $\big(g^-(t), g^+(t),\, t\in[0,1]\big)$ in the left part of inequalies (2) and the replacement of the "upper" function $Z^+_{\varepsilon,\alpha}(x)$ by the "lower" $Z^-_{\varepsilon,\alpha}(x)$:

$$Z^-_{\varepsilon,\delta}(x) = \begin{cases} 1, & x\in\big[g^-(1) - 2\varepsilon + \alpha, g^+(1) + 2\varepsilon - \alpha\big],\\ K\big(\frac{1}{g^+(1)+2\varepsilon-\alpha-x} + \frac{1}{g^+(1)+2\varepsilon-x}, \infty\big), & x\in\big[g^+(1) + 2\varepsilon - \alpha, g^+91) + 2\varepsilon\big],\\ K\big(-\infty, \frac{1}{g^-(1)-2\varepsilon-x} + \frac{1}{g^-(1)-2\varepsilon+\alpha-x}\big), & x\in\big[g^-(1) - 2\varepsilon, g^-(1) - 2\varepsilon + \alpha\big],\\ 0, & x \ge g^+(1) + 2\varepsilon \vee x \le g^-(1) - 2\varepsilon. \end{cases}$$

Thus, $v_\varepsilon(1,x) \ge Z^-_{\varepsilon,\alpha}(x)$ and $v_\varepsilon(0,x) \ge F_l\cdot\dots\cdot F_{(m+1)l}Z^-_{\varepsilon,\alpha}(x)$.

Finally, define the functions $l, h, \varepsilon, \varphi, b$.

Assume that these functions have the following form

$$l = n^l, \quad b = n^b, \quad h = n^h, \quad \varphi = n^{-\varphi}, \quad \varepsilon = n^{-\varepsilon}.$$

The right part of (6) gives the form of convergence speed by the according values. Now, we obtain minimization of the right part of (6) by the choice of constants $l, b, h, \varphi, \varepsilon$.

Define of the asymptotics of estimations

$$mr_n \sim \frac{n}{l}\Big((\frac{l}{n})^2 + \frac{b}{n}\sum_{k\geq b}\varphi^{1/2}(k) + \frac{1}{n}\sum_{j\geq 1} j\varphi^{1/2}(j) + \frac{h}{n} + \frac{b^{3/2}}{\varepsilon^3 n^{3/2}}\Big);$$

here we used the estimation of [2]: $M\big(\sum_{i=1}^n |\xi_i|\big)^t \leq \text{const}\cdot n^{t/2}$. Thus,

$$mr_n \sim \max\{n^{l-1}, n^{b-l-\frac{b\varphi}{2}+1}, n^{-l}, n^{h-l}, n^{-\frac{3}{2}b-\frac{1}{2}-l+3\varepsilon}\}, \quad \varphi(h)m \sim \frac{n}{l}n^{-\varphi h} = n^{-1-l-\varphi h};$$

$$P(\|s_n - s_n^{(1)}\| \geq \varepsilon) \leq m\sum_{i=1}^{l} M|\xi_i|^3\varepsilon^{-3}(\sigma n)^{-3/2} \sim \frac{n}{l}\varepsilon^{-3}n^{-3/2} = n^{1+3\varepsilon-\frac{3}{2}},$$

$$P(\|s_n^{(1)} - s_n^{(2)}\| \geq \varepsilon) \leq \sum_{i=1}^{m} M|\psi_i|^3\varepsilon^3(n\sigma)^{-3/2} \sim h^{3/2}\frac{n}{l}\varepsilon^{-3}n^{-3/2} = n^{\frac{3}{2}h+1-l+3\varepsilon-\frac{3}{2}},$$

$$\text{const}\cdot\varepsilon \sim n^{-\varepsilon},$$

From (5) we have that $\frac{l}{n} < \varepsilon$. Further, $n^{l-1} = o(n^{-\varepsilon})$, that is $\varepsilon > 1 - l$.

Setting $b = l$, $\varepsilon = \frac{1}{9} - h$, for determination of the necessary constants we must solve the following system of inequalities

$$l - 1 \leq -\tfrac{1}{9} + h,\ 1 - \tfrac{\varphi l}{2} \leq -\tfrac{1}{9} + h,\ -l \leq -\tfrac{1}{9} + h,$$
$$h - l \leq -\tfrac{1}{9} + h,\ \tfrac{l}{2} - \tfrac{1}{2} + \tfrac{1}{3} - 3h \leq -\tfrac{1}{9} + h,$$
$$1 - l - \varphi h \leq -\tfrac{1}{9} + h,\ -\tfrac{1}{2} + \tfrac{3h}{2} - l + \tfrac{1}{3} - 3h \leq -\tfrac{1}{9} = h,\ -\tfrac{1}{2} + \tfrac{1}{3} - 3h \leq -\tfrac{1}{9} + h. \tag{7}$$

Considering h $(0 < h < l)$ as argument, we obtain the dependence of φ on h. The minimal φ which satisfies system (7) is defined by the following conditions:

$$l = \tfrac{1}{9}, \quad h \leq \tfrac{1}{16}, \quad \varphi = \max\{20, \tfrac{l}{h} - 1\}.$$

This completes the proof.

Finally, note that in [2] the mixing coefficient must satisfy the following condition $\varphi(k) \leq Ak^{-(110+\varepsilon)}$, $\varepsilon > 0$, where A is a bounded constant.

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