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v-LINDELÖF SPACES

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The concept of v-Lindelöf space is introduced. It is shown that a space X is v-Lindelöf if and only if every Wallman realcompactification of X is Lindelöf and that if $X\times Y$ is a z-embedded v-Lindelöf subspace of $vX\times vY$, then $v(X\times Y)=vX\times vY$.

0. Introduction

All topological spaces discused in this paper are assumed to be Tychonoff. For any space X, $(\beta X, \beta_X)$ ((vX, v_X), resp.) denotes the Stone-Čech compactification (Hewitt realcompactification, resp.) of X.

An important open question in the theory of Hewitt realcompactifications of Tychonoff spaces concerns when the equality $vX\times vY=v(X\times Y)$ is valid ([9]). Glicksberg ([8]) showed that for any infinite spaces X and Y, $\beta X\times\beta Y=\beta(X\times Y)$ if and only if $X\times Y$ is pseudocompact. Comfort ([6]) showed that if $X\times Y$ is C^*-embedded in $vX\times vY$, then $vX\times vY=v(X\times Y)$ and that if $\operatorname{card}(X)$ or $\operatorname{card}(Y)$ is non-measurable and $X\times Y$ is C^*-embedded in $X\times\beta Y$, then $vX\times vY=v(X\times Y)$. In [10], it is shown that $X\times Y$ is C^*-embedded in $X\times\beta Y$ if and only if the projection $\pi_X\colon X\times Y\to X$ is z-closed.

In this paper we introduce the concept of v-Lindelöf spaces. We first show that a space X is v-Lindelöf if and only if every Wallman realcompactification of X is Lindelöf and show that for any v-Lindelöf space X, $|vX\setminus X|\le 1$ if and only if for any space T with $X\subset T$, $vX\subset vT$. Moreover, we will show that if $X\times Y$ is a z-embedded v-Lindelöf subspace of $vX\times vY$, then $v(X\times Y)=vX\times vY$ and that if $X\times Y$ is an v-Lindelöf space such that $\operatorname{card}(X)$ or $\operatorname{card}(Y)$ is non-measurable and X is a P-space, then $v(X\times Y)=vX\times vY$ if and only if the projection $\pi_X\colon X\times Y\to X$ iz z-closed. For the terminology, we refer to [7] and [11].

1. Realcompactifications

Recall that a space (Y,j) or simply Y is called a *compactification* (*realcompactification*, resp.) of a space X if $j\colon X\hookrightarrow Y$ is a dense embedding and Y is a compact (realcompact, resp.) space.

The ring of real-valued continuous functions on a space X is denoted by $C(X)$ and $C^*(X)$ denotes the subring of bounded functions. A subspace S of a space X is said to be *C-embedded* in X if every function in $C(S)$ extends to a function in $C(X)$. C^*-embedding is defined analogously. For a space X, βX is a unique

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realcompactification of X in which X is densely C^*-embedded and vX is a unique realcompactification of X in which X is densely C-embedded.

Definition 1.1 ([12]). Let X be a space and $\mathcal{F}$ a family of closed sets in X. Then $\mathcal{F}$ is called *a separating nest generated intersection ring* on X if
(i) for each closed set H in X and $x \notin H$, there are disjoint sets in $\mathcal{F}$, one containing H and the other containing x,
(ii) it is closed under finite unions and countable intersections, and
(iii) for any $F \in \mathcal{F}$, there are sequences (F_n) and (H_n) in $\mathcal{F}$ such that for any $n \in \mathbb{N}$, $X \setminus H_{n+1} \subset F_{n+1} \subset X \setminus H_n \subset F_n$ and $F = \bigcap F_n$.

For a space X, $Z(X)$ denotes the set of zero-sets in X, $\mathcal{L}(X)$ the set of separating nest generated intersection rings on X and for any subspaces S of X and $\mathcal{F} \subset 2^X$, let $\mathcal{F}_S = \{F \cap S : F \in \mathcal{F}\}$. For a subspace S of a space X and $\mathcal{F} \in \mathcal{L}(X)$, $Z(X) \in \mathcal{L}(X)$ and $\mathcal{F}_S \in \mathcal{L}(S)$ ([12]).

Let X be a space and $\mathcal{F} \in \mathcal{L}(X)$. Then $\mathcal{F}$ is a normal base on X ([1]). Let $\big(w(X,\mathcal{F}), w_X\big)$ be the Wallman compactification of X associated with $\mathcal{F}$ ([1]). Then $\mathcal{F} = Z\big(w(X,\mathcal{F})\big)_X$ and if (Y,j) is a compactification of X such that $\mathcal{F} = Z(Y)_X$, then there is a continuous map $f\colon w(X,\mathcal{F}) \to Y$ with $f \circ w_X = j$ ([12]).

Let $v(X,\mathcal{F}) = \{\alpha : \alpha$ is an $\mathcal{F}$-ultrafilter on X with the countable intersection property$\}$. Then the topology on $v(X,\mathcal{F})$, taking sets of the form $F^* = \{\alpha \in v(X,\mathcal{F}) : F \in \alpha\}$ as a base for the closed sets, coincides with the subspace topology on $v(X,\mathcal{F})$ of $w(X,\mathcal{F})$, $v(X,\mathcal{F})$ is a realcompactification of X (called *Wallman realcompactification*) ([12]), $v(X,\mathcal{F}) = v(X,\mathcal{F}^t)$ and $w(X,\mathcal{F}^t) = \beta\big(v(X,\mathcal{F}^t)\big)$, where $\mathcal{F}^t = Z\big(v(X,\mathcal{F})\big)_X$ ([3]).

In a space (X,τ), the family of G_δ-sets on X forms a base for a topology τ_δ on X and for $A \subset X$, $\aleph_1\text{-cl}_X(A)$ denotes the closure of A in (X,τ_δ).

Theorem 1.2. *A realcompactification (Y,j) of a space X is Wallman if and only if for non-empty zero-set Z in Y, $Z \cap X \neq \varnothing$. In this case, $Y = v(X,\mathcal{F})$ and $\mathcal{F} = Z(Y)_X$.*

Proof. ($\Leftarrow$) Let $\mathcal{F} = Z(Y)_X$, then $\mathcal{F} \in \mathcal{L}(X)$. Note that $Z(\beta Y)_X = Z(Y)_X = \mathcal{F}$. Hence, there is a continuous map $g\colon w(X,\mathcal{F}) \to \beta Y$ with $g \circ w_X = \beta_Y \circ j$. Let A and B be zero-sets in $w(X,\mathcal{F})$ with $A \cap B \cap X = \varnothing$, then $A \cap X$, $B \cap X \in \mathcal{F}$. Hence there are C, D in $Z(Y)$ with $A \cap X = C \cap X$ and $B \cap X = D \cap X$. Since $C \cap D \cap X = \varnothing$ and $C \cap D \in Z(Y)$, $C \cap D = \varnothing$ and hence $\text{cl}_{\beta Y}(C) \cap \text{cl}_{\beta Y}(D) = \varnothing$. So $\text{cl}_{\beta Y}(A \cap X) \cap \text{cl}_{\beta Y}(B \cap X) = \varnothing$. By Urysohn's extension theorem, there is a continuous map $h\colon \beta Y \to w(X,\mathcal{F})$ such that $w_X = h \circ \beta_Y \circ j$ and so h is a homeomorphism.

Note that $\aleph_1\text{-cl}_{\beta Y}(X) \subset \aleph_1\text{-}\text{cl}_{\beta Y}(Y)$. Let $x \notin \aleph_1\text{-}\text{cl}_{\beta Y}(X)$. Then there is a zero-set Z in βY such that $x \in Z$ and $Z \cap X = \varnothing$. Since $(S \cap Y) \cap X = \varnothing$, $Z \cap Y = \varnothing$. So $x \notin \aleph_1\text{-}\text{cl}_{\beta Y}(Y)$. Hence $\aleph_1\text{-}\text{cl}_{\beta Y}(X) = \aleph_1\text{-}\text{cl}_{\beta Y}(Y)$. It is well-known that $v(X,\mathcal{F}) = \aleph_1\text{-}\text{cl}_{w(X,\mathcal{F})}(X)$ ([1]). Since $w(X,\mathcal{F})$ and βY are homeomorphic, $\aleph_1\text{-}\text{cl}_{\beta Y}(Y) = v(X,\mathcal{F})$ and since Y is a realcompact space, $\aleph_1\text{-}\text{cl}_{\beta Y}(Y) = Y$. So $Y = v(X,\mathcal{F})$.

($\Rightarrow$) Since Y is a Wallman realcompactification of X, $Y = v(X,\mathcal{G})$ for some $\mathcal{G} \in \mathcal{L}(X)$. Then $v(X,\mathcal{G}) = v(X,\mathcal{G}^t)$ and $\beta\big(v(X,\mathcal{G}^t)\big) = w(X,\mathcal{G}^t)$, where $\mathcal{G}^t = Z\big(v(X,\mathcal{G})\big)_X$ ([3]). Hence there is a continuous map $f\colon w(X,\mathcal{G}^t) \to w(X,\mathcal{G})$ with $f \circ l = k \circ h$, where $h\colon v(X,\mathcal{G}^t) \to v(X,\mathcal{G})$ is a homeomorphism and $l\colon v(X,\mathcal{G}^t) \hookrightarrow w(X,\mathcal{G}^t)$ and $k\colon v(X,\mathcal{G}) \hookrightarrow w(X,\mathcal{G})$ are dense embeddings. Take any non-empty zero-set Z in Y. Since $h^{-1}(Z)$ is a zero-set in $v(X,\mathcal{G}^t)$, there is a zero-set A in

$\beta\big(v(X,\mathcal{G}^t)\big) = w(X,\mathcal{G}^t)$ with $h^{-1}(Z) = A \cap v(X,\mathcal{G}^t)$. Since $h^{-1}(Z) \neq 0$, pick $\alpha \in A \cap v(X,\mathcal{G}^t)$. Then there is a countable family $\{Z_n : n \in \mathbb{N}\}$ of zero-set neighborhoods of α in $w(X,\mathcal{G}^t)$ such that $A = \bigcap Z_n$. For any $n \in \mathbb{N}$, $Z_n \cap X \in \mathcal{G}^t$ and hence $Z_n \cap X \in \alpha$. Since α has the countable intersection property, $A \cap X = (\bigcap Z_n) \cap X \neq \varnothing$. Thus $h^{-1}(Z) \cap X = Z \cap X \neq \varnothing$.

2. v-Lindelöf spaces

Recall that a separating nest generated intersection ring $\mathcal{F}$ on a space X is called *complete* if $Z\big(v(X,\mathcal{F})\big)_X = \mathcal{F}$. For a space X, $Z(X)$ is complete and $v\big(X, Z(X)\big) = vX$. For a paracompact (or separable) space X, vX is Lindelöf if and only if every separating nest generated intersection ring on X is complete ([4], [5]).

Definition 2.1. A space X is called *v-Lindelöf* if vX is Lindelöf.

A z-ultrafilter on a space X is called *real* if it has the countable intersection porperty.

Proposition 2.2. *Let X be a space. Then the following are equivalent:*
(a) X is an v-Lindelöf space,
(b) every z-filter on X with the countable intersection property is contained in a real z-ultrafilter on X, and
(c) every Wallman realcompactification of X is Lindelöf.

Proof. (a)⇒(b) Let $\mathcal{F}$ be a z-filter on X with the countable intersection property, then $\mathcal{G} = \{Z \in Z(vX) : Z \cap X \in \mathcal{G}\}$ is a z-filter on vX with the countable intersection property. Since X is a v-Lindelöf space, $\bigcap \mathcal{G} \neq \varnothing$. Pick $\alpha \in \bigcap \mathcal{G}$. Then α is a real z-ultrafilter on X with $\mathcal{F} \subset \alpha$.

(b)⇒(c) Let (Y, j) be a Wallman realcompactification of X and $\mathcal{F}$ a z-filter on Y with the countable intersection property. Let $f\colon vX \to Y$ be the continuous map with $f \circ v_X = j$. Since Y is a Wallman realcompactification of X, by Theorem 1.2, for any $F \in \mathcal{F}$, $F \cap \mathcal{F} \neq \varnothing$. Hence $\mathcal{F}_X$ is a z-filter base on X with the countable intersection property. By (b), there is a real z-ultrafilter α on X with $\mathcal{F}_X \subset \alpha$. So for any $F \in \mathcal{F}$, $\alpha \in \mathrm{cl}_{vX}(F \cap X)$. Hence for any $F \in \mathcal{F}$, $f(\alpha) \in f\big(\mathrm{cl}_{vX}(F \cap X)\big) \subset \mathrm{cl}_Y\big(f(F \cap X)\big) = \mathrm{cl}_Y(F \cap X) \subset F$. So Y is Lindelöf.

(c)⇒(a) is trivial.

Every Lindelöf space is v-Lindelöf. If X is a pseudocompact space, then $vX = \beta X$ and hence X is a v-Lindelöf space. v-Lindelöf spaces are not productive and v-Lindelöf spaces are C-embedded hereditary.

Example 2.3. Let ω_1 be the first uncountable ordinal and $D(\omega_1)$ the discrete space of cardinality ω_1. Let $S = D(\omega_1) \cup \{p\}$, topologized as follows. Each point of $D(\omega_1)$ is isolated and a subset G of S that contains p is open in S if and only if $|S \backslash G| \leq \aleph_0$. Then S is a zerodimensional Hausdorff space and hence Tychonoff. Let $\mathbb{N}^* = \mathbb{N} \cup \{\omega\}$ denote the one-point compactification of $\mathbb{N}$ and $X = S \times \mathbb{N}^* \backslash \{(p, w)\}$. Then X is called *Dieudonneé plank* and $vX = S \times \mathbb{N}^*$ ([11]). Since S is Lindelöf, X is v-Lindelöf. But X is neither v-Lindelöf nor pseudocompact.

It is well-kmowm that for any $f \in C(X)$, $\mathrm{cl}_{vX}\big(Z(f)\big) = Z(f^v)$, where f^v is the extension of f to vX ([7]).

Proposition 2.4. *Let X be a v-Lindelöf space and A a zero-set in X. Then A is closed in vX if and only if A is Lindelöf.*

Proof. Suppose that A is Lindelöf. Let $p \in vX \setminus A$. If $p \in X$, then $p \notin \mathrm{cl}_{vX}(A)$. Suppose that $p \notin X$. For any $a \in A$, there is a cozero-sets neighborhood C_a of a in vX such that $p \notin C_a$. Since A is Lindelöf, there is a countable subfamily $\mathcal{U}$ of $\{C_a : a \in A\}$ with $A \subset \bigcup \mathcal{U}$. Let $C = \bigcup \mathcal{U}$ and $Z = vX \setminus C$. Then $p \in Z$, Z is a zero-set in vX and $A \cap Z = \varnothing$. Since X is C-embedded in vX, $\mathrm{cl}_{vX}(A) \cap \mathrm{cl}_{vX}(Z \cap X) = \varnothing$ and since $\mathrm{cl}_{vX}(Z \cap X) = Z$, $p \notin \mathrm{cl}_{vX}(A)$ and hence $A = \mathrm{cl}_{vX}(A)$. The converse is trivial.

Definition 2.5. Let X be a dense subspace of a space T, $\mathcal{F}$ a z-filter on X and $p \in T$. Then $\mathcal{F}$ converges to the limit p if every neighborhoods of p in T contains a member of $\mathcal{F}$.

Lemma 2.6 ([7]). *Let X be a dense subspace of T. Then X is C-embedded in T if and only if every point of T is the limit of a unique real z-ultrafilter on X.*

For any space X and $\mathcal{F} \subset 2^X$ let $\bigcap \mathrm{cl}_X(\mathcal{F}) = \bigcap \{\mathrm{cl}_X(F) : F \in \mathcal{F}\}$.

Theorem 2.7. *Let X be a v-Lindelöf space. Then the following are equivalent:*
(a) for any two disjoint zero-points in X, at least one of them is Lindelöf,
(b) $|vX \setminus X| \leq 1$, and
(c) for any space T with $X \subset T$, there is an embedding $f: vX \to vT$ such that $f(x) = x$ for all $x \in X$.

Proof. (a)⇒(b) Suppose that $2 \leq |vX \setminus X|$. Pick $p, q \in vX \setminus X$ with $p \neq q$. Since p and q are z-ultrafilters on X, there are disjoint zero-sets A, B in X such that $A \in p$ and $B \in q$. We may assume that A is Lindelöf. By Proposition 2.4, A is closed in vX. Note that $p \in \mathrm{cl}_{vX}(A) \setminus A$. This is a contradiction.

(b)⇒(a) Suppose that $vX \setminus X = \{p\}$. Take any disjoint zero-sets A, B in X. Then $\mathrm{cl}_{vX}(A) \cap \mathrm{cl}_{vX}(B) = \varnothing$ and hence $p \notin \mathrm{cl}_{vX}(A)$ or $p \notin \mathrm{cl}_{vX}(B)$. So $\mathrm{cl}_{vX}(A) = A$ or $\mathrm{cl}_{vX}(B) = B$. Hence A is Lindelöf or B is Lindelöf.

(b)⇒(c) Suppose that $vX \setminus X = \{p\}$. Take any space T with $X \subset T$. Then there is a continuous map $f: vX \to vT$ such that $f(x) = x$ for all $x \in X$ ([7]). Let $q = f(p)$ and $Y = X \cup \{q\}$. Then X is a dense subspace of Y. Let g be the corestriction of f to Y, then $g: vX \to Y$ is one-to-one, onto, and continuous.

We will show that g is a homeomorphism. Since vX is Lindelöf, Y is a Lindelöf space and hence Y is a realcompactification of X. Since X is C-embedded in vX, there is a unique real z-ultrafilter $\mathcal{A}^p$ on X such that p is a limit point of $\mathcal{A}^p$. Take any neighborhood V of q in Y. Then $g^{-1}(V)$ is a neighborhood of p in vX. Since p is a limit point of $\mathcal{A}^p$, there is $A \in \mathcal{A}^p$ with $A \subset g^{-1}(V)$ and so $g(A) = A \subset V$. Hence q is a limit point of $\mathcal{A}^p$. Suppose that $\mathcal{F}$ is a real z-ultrafilter on X such that q is a limit point of $\mathcal{F}$. If $\bigcap \mathcal{F} \neq \varnothing$, then $\bigcap \mathcal{F} = \{x\}$ for some $x \in X$. Since $x \neq q$, there are disjoint zero-set neighborhoods C and D of x and q in Y, respectively. Then $C \cap X \in \mathcal{F}$ and $C \cap D \cap X = \varnothing$. Hence q is not a limit point of $\mathcal{F}$ and so $\bigcap \mathcal{F} = \varnothing$. Since $\mathcal{F}$ is real, $\mathrm{cl}_{vX}(\mathcal{F}) = \{\mathrm{cl}_{vX}(F) : F \in \mathcal{F}\}$ is a z-filter on vX with countable intersection property and since vX is Lindelöf, $\bigcap \mathrm{cl}_{vX}(\mathcal{F}) \neq \varnothing$. Hence $\bigcap \mathrm{cl}_{vX}(\mathcal{F}) = \{p\}$ and so $\mathcal{F} = \mathcal{A}^p$. Thus every point of Y is the limit of a unique real z-ultrafilter on X. By Lemma 2.6, X is C-embedded in Y and therefore, g is a homeomorphism.

(c)⇒(b) Suppose that there are $p, q \in vX \setminus X$ with $p \neq q$. Let $Y = X \cup \{p, q\}$ and $R = \{(x, x) : x \in Y\} \cup \{(p, q), (q, p)\}$. Then R is an equivalence relation on Y. Let K be the quotient space Y/R and $\pi: Y \to K$ the quotient map. Clearly, K is a Tychonoff space and X is a dense subspace of K. By the assumption, there is an embedding $f: vX \to vK$ such that $f(x) = x$ for all $x \in X$. Since X is dense in Y

and $(v_K \circ \pi)|_X = f|_X$, $v_K \circ \pi = f|_Y$. Since f is one-to-one and $p \neq q$, $f(p) \neq f(q)$ but $v_K\big(\pi(p)\big) = \pi(p) = [p] = [q] = \pi(q) = v_K\big(\pi(q)\big)$. This is a contradiction.

A subspace Y of a space X is z-embedded in X if for any zero-set A in Y, there is a zero-set Z in X with $A = Z \cap Y$. It is known that a space X is z-embedded in each of its compactifications if and only if for any two disjoint zero-sets in X, one of them is Lindelöf ([2]). Using this, we have the following:

Corollary 2.8. *Let X be a v-Lindelöf space. Then $|vX \setminus X| \leq 1$ if and only if X is z-embedded in each of its compactifications.*

3. Hewitt realcompactification of product spaces

The equality $v(X \times Y) = vX \times vY$ is to be interpreted to mean that $X \times Y$ is C-embedded in $vX \times vY$.

Lemma 3.1 ([6]). *Let X and Y be spaces. Then $v(X \times Y) = vX \times vY$ if and only if $X \times Y$ is C^*-embedded in $vX \times vY$.*

Theorem 3.2. *Let X and Y be spaces such that $X \times Y$ is a v-Lindelöf spaces. Then $X \times Y$ is z-embedded in $vX \times vY$ if and only if $v(X \times Y) = vX \times vY$.*

Proof. Suppose that $X \times Y$ is z-embedded in $vX \times vY$. Since $vX \times vY$ is a realcompact space, there is a continuous map $f\colon v(X \times Y) \to vX \times vY$ such that $f\big((x,y)\big) = (x,y)$ for all $(x,y) \in X \times Y$. Take any $(p,q) \in (vX \times vY) \setminus (X \times Y)$. Then $\{(p,q)\} = \big(\bigcap \mathrm{cl}_{vX}(p)\big) \times \big(\bigcap \mathrm{cl}_{vX}(q)\big)$. Let $\mathcal{F}$ be the z-filter on $X \times Y$ generated by $\{A \times B : A \in p, B \in q\}$. Then $\mathcal{F}$ has the countable intersection property and $\bigcap \mathcal{F} = \varnothing$. Since $X \times Y$ is v-Lindelöf, $\bigcap \mathrm{cl}_{v(X\times Y)}(\mathcal{F}) \neq \varnothing$. Pick $x \in \bigcap \mathrm{cl}_{v(X\times Y)}(\mathcal{F})$. Then for any $A \in p$ and $B \in q$, $f(x) \in f\big(\mathrm{cl}_{v(X\times Y)}(A \times B)\big) \subset \mathrm{cl}_{(vX\times vY)}\big(f(A \times B)\big) = \mathrm{cl}_{(vX\times vY)}(A \times B) = \mathrm{cl}_{vX}(A) \times \mathrm{cl}_{vY}(B)$. Hence $f(x) \in \big(\bigcap \mathrm{cl}_{vX}(p)\big) \times \big(\bigcap \mathrm{cl}_{vX}(q)\big)$. So $f(x) = (p,q)$. Thus f is onto.

Take any zero-sets E, F in $X \times Y$ with $E \cap F = \varnothing$. Since $X \times Y$ is z-embedded in $vX \times vY$, there are zero-sets C, D in $vX \times vY$ with $E = C \cap (X \times Y)$ and $F = D \cap (X \times Y)$. Since $f^{-1}(C \cap D) \cap (X \times Y) = \varnothing$ and $f^{-1}(C \cap D)$ is a zero-set in $v(X \times Y)$, $f^{-1}(C \cap D) = \varnothing$ and since f is onto, $C \cap D = \varnothing$. So $\mathrm{cl}_{(vX\times vY)}(E) \cap \mathrm{cl}_{(vX\times vY)}(F) = \varnothing$. By Urysohn's extension theorem, $X \times Y$ is C^*-embedded in $vX \times vY$. By Lemma 3.1, $v(X \times Y) = vX \times vY$. The converse is trivial.

Definition 3.3. Let X and Y be spaces. Then $f\colon X \to Y$ is called *z-closed* if for any zero-set Z in X, $f(Z)$ is closed in Y.

Recall that a space X is called a *P-space* if every G_δ-set in X is open in X.

Remark 3.4. (1) If the projection $\pi_X\colon X \times Y \to X$ is z-closed, then X is a P-space or Y is a pseudocompact space ([11]). (2) The projection $\pi_X\colon X \times Y \to X$ is z-closed if and only if $X \times Y$ is C^*-embedded in $X \times \beta Y$ ([6]). (3) If $\mathrm{card}(X)$ or $\mathrm{card}(Y)$ is non-measurable and $X \times Y$ is C^*-embedded in $X \times \beta Y$, then $v(X \times Y) = vX \times vY$ ([6]).

Theorem 3.5. *Let X be a P-space and $X \times Y$ a v-Lindelöf space. If $v(X \times Y) = vX \times vY$, then the projection $\pi_X\colon X \times Y \to X$ is z-closed.*

Proof. Take any zero-set A in $X \times Y$ and $x \notin \pi_X(A)$. Then $(\{x\} \times Y) \cap A = \varnothing$. We will show that $\{x\} \times Y$ is C-embedded in $X \times Y$. Take any continuous map $f\colon \{x\} \times Y \to \mathbb{R}$. Note that the map $h\colon Y \to \{x\} \times Y$, defined by $h(y) = (x,y)$, is a homeomorphism. Let $k = f \circ h$ and define a map $\underline{0}\colon X \to \mathbb{R}$ by $\underline{0}(x) = 0$

for all $x \in X$. Then the map $l\colon X \times Y \to \mathbb{R}$, defined $l\big((z,y)\big) = \underline{0}(z) + k(y)$, is continuous and $l|_{\{x\}\times Y} = f$. Hence $\{x\} \times Y$ is C-embedded in $X \times Y$. Thus $\{x\}\times Y$ and A are completely separated in $X\times Y$ ([7]). Since $v(X\times Y) = vX\times vY$, $(\{x\}\times vY)\cap \mathrm{cl}_{vX\times vY}(A) = \varnothing$. For any $y \in vY$, there are open neighborhoods C_y, D_y of x, y in X, Y, respectively such that $(C_y \times D_y) \cap A = \varnothing$. Since vY is Lindelöf, there is a sequence (y_n) in vY with $\{x\} \times vY \subset \bigcup\{C_{y_n} \times D_{y_n} : n \in \mathbb{N}\}$. Let $Z = \bigcap\{C_{y_n} : n \in \mathbb{N}\}$. Since X is a P-space, Z is open in X and $\{x\} \times vY \subset Z \times \big(\bigcup(D_{y_n} : n \in \mathbb{N}\}\big)$. Moreover, $(Z \times vY) \cap A = \varnothing$. Thus $Z \cap \pi_X(A) = \varnothing$ and so $x \notin \mathrm{cl}_X\big(\pi_X(A)\big)$. Therefore $\pi_X(A)$ is closed in X.

Corollary 3.6. *Suppose that $X \times Y$ is a v-Lindelöf space such that* card(X) *or* card(Y) *is non-measurable and X is a P-space. Then π_X is z-closed if and only if $v(X \times Y) = vX \times vY$.*

REFERENCES

[1] R.A. Aló and H.L. Shapiro, *Z-realcompactifications and normal bases*, J. Austral. Math. Soc. **9** (1966), 489–495.

[2] R. Blair, *Spaces in which special sets are Z-embedded*, Canadian J. Math. **28** (1976), 673–690.

[3] J.L. Blasko, *Complete bases and Wallman realcompactifications*, Proc. Amer. Math. Soc. **75** (1979), 114–118.

[4] J.L. Blasko, *Complete bases in topological spaces*, Studia Sci. Math. Hung. **20** (1985), 49–54.

[5] J.L. Blasko, *Complete bases in topological spaces, II*, Studia Sci. Math. Hung. **24** (1989), 447–452.

[6] W.W. Comfort, *On Hewitt realcompactification of a product space*, Trans. Amer. Soc. **181** (1968), 107–118.

[7] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, New York, 1960.

[8] Irving Glicksberg, *Stone-Čech compactification of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382.

[9] H. Herrlich and M. Hušek, *Some open categorical problems in* **TOP**, Applied Categorical Structure **1** (1993), 1–19.

[10] W.G. Mcarthur, *Hewitt realcompactifications of products*, Canadian J. Math. **22** (1970), no. 3, 645–656.

[11] J. Porter and R.G. Woods, *Extensions and absolutes of Hausdorff spaces*, Springer, Berlin, 1988.

[12] A.K. Steiner and E.F. Steiner, *Nested generated intersection rings in Tychonoff spaces*, Trans. Amer. Math. Soc. **303** (1970), 779-804.

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