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ON APPROXIMATION OF SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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In this paper we investigate an algorithm of approximation of defferential-difference equations by systems of ordinary differential equations. We investigate conditions for convergence of solutions of approximating systems to solutions of initial problem in a finite interval. The value of precision of approximation for different classes of function is obtained.

Introduction

There has been a great development of the theory of differential-difference equations (DDE) the last years due to the increasing number of applications of DDE in various fields of science and technology. Of special interest are the differential equations with deviating argument.

Since the solutions of the DDE in general are not found explicitly, the methods for their approximate solutions are of great importance. The approximation algorithm of differential-difference equations by system of ordinary differential equations has been considered by many authors [1,2] in the researches of control and stability problems in systems with delay. The most popular is N.N. Krasovsky and Yu.M. Repin [1] approximation system plan. This approximation system plan was applied to neutral type equations [3] and to differential functional equations.

The aim of the present paper is to improve precision of Krasovsky-Repin approximation plan of differential equation with delay by system of ordinary differential equations.

1. Approximation plan

Consider the initial problem

$$x'(t) = f\big(t, x(t), x(t-\tau)\big), \quad t \in [0,T], \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0], \tag{2}$$

where $\tau > 0$ is constant, $\varphi(t)$ is a continuous function; $f(t,u,v)$ is a continuous function which satisfies the Lipschitz condition by u and v with constant L_1 and L_2.

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The segment $[-\tau, 0]$ is divided into m parts by the points $t_j = -\frac{j\tau}{m}$, $j = 0, \dots, m$, $m \in \mathbb{N}$ and the functions $y_j(t) = x\big(t - \frac{\tau j}{m}\big)$, $j = 0, \dots, m$ are introduced. Initial problem (1)–(2) in [1] is assigned to the system of ordinary differential equations

$$\begin{aligned} z_0' &= f\big(t, z_0(t), z_m(t)\big), \\ z_j'(t) &= \frac{m}{\tau}\big(z_{j-1}(t) - z_j(t)\big), \quad j = 0, \dots, m, \end{aligned} \tag{3}$$

with the initial conditions

$$z_j(0) = \varphi\Big(-\frac{j\tau}{m}\Big), \quad j = 0, \dots, m. \tag{4}$$

N.N. Krasovsky shows [1] that uniformly for all bounded functions $\varphi(t)$

$$\max_{[0,T]}\big|x(t) - z_0(t)\big| = \alpha(m) \to 0, \text{ for } m \to \infty.$$

If the solution of problem (1)–(2) satisfies the Lipschitz condition, then

$$|\alpha(m)| \le \frac{K}{\sqrt{m}}, \quad K > 0. \tag{5}$$

So, the replacing of delay equation (1) by system (3) is correct on fixed interval $[0, T]$ provided m is taken large enough. We can consider system (3) as series of seccisive jointed delay elements [1]. We defined functions $z_j(t)$ ($j = 0, \dots, m$, $m \in \mathbb{N}$), as solutions of system of differential equations

$$\begin{aligned} \big[z_0(t) - A(t)z_m(t)\big]' &= f\big(t, z_0(t), z_m(t)\big), \\ z_{j+1}'(t) &= \frac{m}{\tau}\big(z_{j-1}(t) - z_j(t)\big), \quad j = 0, \dots, m, \end{aligned} \tag{6}$$

$$z_j(0) = \varphi\big(-\frac{j\tau}{m}\big), \quad j = 0, \dots, m. \tag{7}$$

We assume that system of ordinary differential equations (6) approximates differential-difference equation (1) on the interval $[0, T]$, if the following condition

$$\big|x\big(t - j\frac{\tau}{m}\big) - y_j(t)\big| \to 0, \tag{8}$$

holds for $m \to \infty$, $t \in [0, T]$, $j = 0, \dots, m$.

Further we show that Cauchy problem (7)–(8) approximates problem (1)–(2) for delay equations and states the value of precision approximation.

2. Delay element approximation.

Lemma 1. *Consider the system of linear differential equations*

$$\begin{aligned} \frac{1}{2}\big(\frac{\tau}{m}\big)^2 z_1'' + \frac{\tau}{m} z_1' + z_1 &= x(t), \\ \frac{1}{2}\big(\frac{\tau}{m}\big)^2 z_j'' + \frac{\tau}{m} z_j' + z_j &= z_{j-1}, \quad j = 2, \dots, m, \end{aligned} \tag{9}$$

with the initial conditions

$$z_j(0) = x\big(-\frac{j\tau}{m}\big), \quad z_j'(0) = x'\big(-\frac{j\tau}{m}\big), \quad j = 1, \dots, m, \tag{10}$$

where $x(t) \in C^l[-\tau, T]$, $x'(t)$ satisfies the Lipschitz condition, $\tau, T > 0$ are constants. Then

$$\Big|z_j(t) - x\big(t - \frac{j\tau}{m}\big)\Big| \leq \frac{A}{m}, \quad j = 1, \dots, m, \tag{11}$$

is valid, where $A > 0$ is constant which doesn't depend on j and m.

Proof. We assume that $x(t) \in C^3[\tau, T]$ and consider the problem

$$\frac{\tau^2}{2} z'' + \tau z + z = x(t), \quad z(0) = x(-\tau), \quad z'(0) = x'(-\tau). \tag{12}$$

Denote $y(t) = x(t-\tau)$ and estimate the value of difference $\varepsilon(t) = z(t) - y(t)$, which is the solution of the problem

$$\varepsilon''(t) = \frac{2}{\tau}\varepsilon'(t) + \frac{2}{\tau^2}\varepsilon(t) = \varphi(t), \quad \varepsilon(t) = \varphi(t), \quad \varepsilon(0) = 0, \quad \varepsilon'(0) = 0, \tag{13}$$

where $\varphi(t) = \frac{2}{\tau^2}\big[x(t) - x(t-\tau) - x'(t-\tau)\big] - x''(t-\tau)$.

If $x''(t)$ satisfies the Lipschitz conditions with constant K_2, then $|\varphi(t)| \leq K_2\tau$. If $x'''(t)$ exists and is bounded by M_2, then $|\varphi(t) \leq \frac{1}{6}M_2\tau$. For the solution of (13) we get

$$\varepsilon(t) = \int_0^t K(t,s)\varphi(s)\, ds, \tag{14}$$

where $K(t,s) = \tau e^{(-t-s)/\tau} \sin\big(\frac{t-s}{\tau}\big)$.

Using the following property of function $\varphi(t)$ and $K(t,s)$ from (14) we obtain

$$|\varepsilon(t)| \leq C\tau^3, \tag{15}$$

where $C = K_2$, or $C = M_2/6$.

Now we consider system of equations (9). Denote $y_j(t) = x\big(t - \frac{\tau j}{m}\big)$ and consider the difference $\varepsilon_j(t) = z_j(t) - y_j(t)$. For $\varepsilon_1(t)$ and according to (15) we get $|\varepsilon_1(t)| = |z_1(t) - y_1(t)| \leq C\big(\frac{\tau}{m}\big)^3$.

Continuing similarly, we receive

$$|\varepsilon_j(t)| \leq jC\Big(\frac{\tau}{m}\Big)^3 \leq C\frac{\tau^3}{m^2}. \tag{16}$$

Now we impose more weak conditions on $x(t)$. We assume that $x'(t)$ satisfies the Lipschitz conditions with constant K_1 and $|x'(t)| < M_1$. Consider the smoothing function

$$x_1(t) = \frac{1}{h}\int_t^{t+h} x(s)\, ds, \quad t \in [-\tau, T],$$

the second derivative of the function satisfies the Lipschitz condition with constant $\frac{2K_1}{h}$.

Let's estimate the value of function $x_2(t) = x(t) - x_1(t)$ and it derivative

$$|x_2(t)| = \Big|x(t) - \frac{1}{h}\int\limits_t^{t+h} x(s)\,ds\Big| = \Big|x(t) - \frac{1}{h}\int\limits_t^{t+h} \big[x(t) - (s-z)x'(\vartheta s)\big]ds\Big| \le \frac{hM_1}{2}, \quad (17)$$

$$|x_2'(t)| = \Big|x'(t) - \frac{1}{h}\big(x(t+h) - x(t)\big)\Big| \le \frac{h}{2}K_1. \quad (18)$$

Consider problem (12), where $x(t) = x_1(t) + x_2(t)$. Let $z = z_1 + z_2$, where z_1 and z_2 are the solutions of the problems

$$\frac{\tau^2}{2}z_1'' + \tau z_1' + z_1 = x_1(t), \quad z_1(0) = x_1(-\tau), \quad z_1'(0) = x_1'(-\tau),$$
$$\frac{\tau^2}{2}z_2'' + \tau z_2' + z_2 = x_2(t), \quad z_2(0) = x_2(-\tau), \quad z_2'(0) = x_2'(-\tau),$$

Estimate the difference $z(t) - x(t-\tau)$. We have

$$|z(t) - x(t-\tau)| \le |z_1(t) + z_2(t) - x_1(t-\tau) - x_2(t-\tau)| \le |z_1(t) - x_1(t-\tau)| + |z_2(t)| + |x_2(t-\tau)|.$$

As function $x_1(t)$ is sufficiently smooth, then according to (15)

$$\big|z_1(t) - x_1(t-\tau)\big| \le \frac{2K_1\tau^3}{h}. \quad (19)$$

For function $z_2(t)$, taking into account (17), (18), we can receive estimation $|z_2(t)| \le hB$, where

$$B = \frac{\tau K_1}{2} + M_1\Big(1 + \frac{\tau^2}{2}\Big). \quad (20)$$

For $x_2(t-\tau)$ estimation (17) holds. Therefore,

$$\big|z(t) - x(t-\tau)\big| \le \frac{2K_1\tau^3}{h} + 2Bh.$$

If we consider system of equations (9), where $x(t) = x_1(t) + x_2(t)$, and estimate similarly, we receive

$$\Big|z_j(t) - x\Big(t - \frac{j\tau}{m}\Big)\Big| \le j\frac{2K_1\tau^3}{hm^3} + 2Bh \le \frac{2K_1\tau^3}{hm^2} + 2Bh.$$

Putting $h = \frac{\tau^{3/2}}{m}$, we have $\big|z_j(t) - x\big(t - \frac{j\tau}{m}\big)\big| \le \frac{2\tau^{3/2}(K_1+B)}{m} \equiv \frac{A}{m}$. Lemma 1 is proved.

3. Delay equation approximation

Consider the question about the closeness of the solutions of (1)–(2) and (7)–(8).

Theorem 1. *Let's assume that the initial function $\varphi(t) \in C^1[-\tau;0]$ satisfies the "matching" condition*

$$\lim_{s\to 0-} \varphi'(s) = f\big(0, \varphi(0), \varphi(-\tau)\big) \tag{21}$$

function $f(t,u,v)$ is continuous and satisfies the Lipschitz condition by t, u and v with constant L_0, L_1, L_2. Then we have

$$\max_{s\in[0,T]} |x(s) - z_0(s)| = \alpha\big(\tfrac{1}{m}\big),$$

where $\lim\limits_{r\to 0} \alpha(r) = 0$.

Proof. If condition (21) is satisfied, then the solution of (1), (2) $x(t)$ is in $C^1[-\tau, T]$ and $x'(t)$ satisfies the Lipschitz cindition. Let $z_j(t)$, $j = 0, \dots, m$ be the solution of differential equation system (7), (8). Let us denote

$$R_j(t) = \max_{0\le s\le t} \Big| z_j(s) - x\big(s - \frac{j\tau}{m}\big)\Big|, \quad j = 0, \dots, m. \tag{22}$$

Putting $z_j = z_j^{(1)} + z_j^{(2)}$, where $z_j^{(1)}, z_j^{(2)}$ are solutions of problems

$$\begin{cases} \frac{\tau}{m} {z'_1}^{(1)} + z_1^{(1)} = x(t), \quad z_1^{(1)}(0) = z_1^0 = x\big(\frac{-\tau}{m}\big), \\ \frac{\tau}{m} {z'_2}^{(1)} + z_2^{(1)} = z_1^{(1)}(t), \quad z_2^{(1)}(0) = z_2^0 = x\big(\frac{-2\tau}{m}\big), \\ \cdots\cdots\cdots\cdots \\ \frac{\tau}{m} {z'_m}^{(1)} + z_m^{(1)} = z_{m-1}^{(1)}(t), \quad z_m^{(1)}(0) = z_m^0 = x\big(\frac{-m\tau}{m}\big), \end{cases} \tag{23}$$

$$\begin{cases} \frac{\tau}{m} {z'_1}^{(2)} + z_1^{(2)} = z_0(t) - x(t), \quad z_1^{(2)}(0) = 0, \\ \frac{\tau}{m} {z'_2}^{(2)} + z_2^{(2)} = z_2^{(2)}(t), \quad z_2^{(2)}(0) = 0, \\ \cdots\cdots\cdots\cdots \\ \frac{\tau}{m} {z'_m}^{(2)} + z_m^{(2)} = z_{m-1}^{(2)}(t), \quad z_m^{(2)}(0) = 0. \end{cases} \tag{24}$$

For system (23) the conditions of Lemma 1 are fultifilled, then

$$\Big| z_{j1} - x\big(t - \frac{j\tau}{m}\big)\Big| \le \frac{A}{m}, \quad j = 1, \dots, m.$$

The solution of system (24) satisfies inequality $|z_{j2}(t)| \le R_0(t)$, $j = 1, \dots, m$. Therefore, we obtain

$$R_j(t) \le \frac{A}{m} + R_0(t), \quad j = 1, \dots, m. \tag{25}$$

Let's transform (1) and (4) in integral form

$$x(t) = x(0) + \int_0^t f\big(s, x(s), x(s-\tau)\big)\, ds,$$
$$z_0(t) = x(0) + \int_0^t f\big(s, z_0(s), z_m(s)\big)\, ds.$$

Using the property of function $f(t,u,v)$ and inequality (25) we obtain

$$|x(t)-z_0(t)| \le \int_0^t \big[L_1 R_0(s) + L_2 R_m(s)\big]\, ds \le \int_0^t \Big[(L_1+L_2)R_0(s) + \frac{AL_2}{m}\Big]\, ds.$$

Using Growall's Lemma [4] we obtain

$$R(t) = \max_{0\le s\le t} |x(s)-z_0(s)| \le \frac{AL_2}{(L_1+L_2)m}\big(e^{(L_1+L_2)t}-1\big).$$

From the last inequality it follows that the solution $x(t)$ of the initial problem (1),(2) is uniformly approximated by the function $z_0(t)$ that can be defined from approximate system (7),(8) on any bounded segment $[0,T]$. Theorem 1 is proved.

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