УДК 517.537.72

ON THE GROWTH OF ENTIRE DIRICHLET SERIES

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M.M. Sheremeta. *On the growth of entire Dirichlet series*, Matematychni Studii, **9**(1998) 165–170.

The Kiselman theorem on the order of entire function is extended on entire Dirichlet series of arbitrary growth.

1°. Ch. Kiselman [1] showed that the order of an entire function f does not exceed $\varrho = 1$ iff there exists an entire function H of two complex variables such that $H(z,e) = f(z)$ and $H(z,w) \le \exp\{|z|\}$ for all $z \in \mathbb{C}$ and $|w| \le 1$. In this assertion the condition $\varrho = 1$ may be replaced by the condition $\varrho \in [1,+\infty)$, but then holomorphicity of H in $\mathbb{C}$ must be replaced by holomorphicity of H in the cylinder $\left\{(z,w) : |w| < \exp\left(\frac{\varrho}{\varrho-1}\right)\right\}$.

In the present note we extend the Kiselman result, on one hand, on more general scale of growth and, on the other hand, on entire Dirichlet series.

Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of nonnegative numbers and $S(\Lambda)$ be a class of entire Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it. \tag{1}$$

For $F \in S(\Lambda)$ we put $M(\sigma, F) = \sup\{|F(\sigma+it)| : t \in \mathbb{R}\}$, and let

$$\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \ge 0\}$$

be the maximal term of series (1).

By Ω we denote the class of positive on $(-\infty,+\infty)$ functions Φ such that the derivative Φ' is continuous, positive and increasing to $+\infty$ on $(-\infty,+\infty)$. For $\Phi \in \Omega$ let φ be the inverse to Φ' and $\Psi(\sigma) = \sigma - \Phi(\sigma)/\Phi'(s)$ be the function associated by Newton to Φ. Then [2, p. 18] Φ is continuous and increasing to $+\infty$ on $(-\infty,+\infty)$.

Our aim is to prove the following

1991 *Mathematics Subject Classification.* 30B50, 30D15.

Theorem 1. *Let $\Phi \in \Omega$ be such that $\sigma\Phi'(\sigma)/\Phi(\sigma) \geq q > 1$ for all $\sigma \geq \sigma_0$, the sequence Λ satisfies the condition*

$$\ln n = o(\lambda_n \Psi(\varphi(\lambda_n))), \quad n \to \infty, \tag{2}$$

and $F \in S(\Lambda)$. In order that

$$\ln M(\sigma, F) \leq \Phi((1 + (1))\sigma), \quad \sigma \to +\infty,$$

it is necessary and sufficient that there exists an absolutely convergent in $\mathbb{C}^2$ double Dirichlet series

$$H(s, w) = \sum_{n,m=0}^{\infty} b_{nm} \exp\{s\lambda_n + w\mu_m\}, \quad \mu_m \geq 0, \tag{3}$$

such that

$$H(s, 1) = F(s), \quad s \in \mathbb{C}, \tag{4}$$

and

$$|H(s, w)| \leq F_\Phi(\sigma), \quad s \in \mathbb{C}, \operatorname{Re} w \leq 0, \tag{5}$$

where F_Φ is continuous on $(-\infty, +\infty)$ function such that $\ln F_\Phi(\sigma) = \Phi\big((1+(1))\sigma\big)$, as $\sigma \to +\infty$.

By L we denote the class continuous, positive and increasing to $+\infty$ on $(x_0, +\infty)$ functions, and if $\alpha \in L$ then the value

$$\varrho_\alpha = \varlimsup_{\sigma \to +\infty} \frac{\alpha(\ln M(\sigma, F))}{\sigma}$$

is called the α-order of entire Dirichlet series (1). If $\alpha(x) = x$ then α-order coincides with R-order (Ritt order).

Theorem 2. *Let $\alpha \in L$ be a slowly increasing function (that is $x\alpha'(x)/\alpha(x) \to 0$, $x \to +\infty$) such that $\alpha^{-1}(x) = \Phi(x) \in \Omega$, a sequence Λ satisfies the condition (2) and $F \in S(\Lambda)$. In order that α-order of the function (1) does not exceed $\varrho \in [1, +\infty)$ it is necessary and sufficient that there exists an absolutely convergent in $\mathbb{C}^2$ double Dirichlet series (3) such that the relations (4) and (5) hold and α-order of $F_\Phi(\sigma)$ is at most 1.*

2°. We need some lemmas.

Lemma 1 [2, p. 18]. *Let $\Phi \in \Omega$ and $F \in S(\Lambda)$. In order that $\ln \mu(\sigma, F) \leq \Phi(\sigma)$, $\sigma \in \mathbb{R}$, it is necessary and sufficient that $\ln |a_n| \leq -\lambda_n \Psi\big(\varphi(\lambda_n)\big)$, $n \geq 0$.*

Lemma 2. *Let $\Phi \in \Omega$, Λ satisfies (2), $F \in S(\Lambda)$ and $0 < \varrho < +\infty$. If*

$$\ln M(\sigma, F) \leq \frac{1}{\varrho}\Phi(\varrho\sigma) + \gamma(\varrho), \quad \gamma(\varrho) = \text{const}, \tag{6}$$

for all $\sigma \in \mathbb{R}$, *then there exists an absolutely convergent in* $Q = \mathbb{C} \times \{w : \operatorname{Re} w < \varrho^*/(\varrho^* - 1)\}$, $\varrho^* = \max\{\varrho, 1\}$, *double Dirichlet series (3) such that relations (4) and (5) hold with*

$$F_\Phi(\sigma) = \sum_{n=0}^{\infty} \exp\{-\lambda_n \Psi(\varphi(\lambda_n)) + \sigma\lambda_n\}. \tag{7}$$

Proof. Convergence of series (7) for all $\sigma \in \mathbb{R}$ follows from (2). We put

$$\mu_n = \big(\ln|a_n| + \lambda_n \Psi(\varphi(\lambda_n))\big)^+,$$

where $a^+ = \max\{a, 0\}$, and consider the double Dirichlet series

$$H(s,w) = \sum_{n=0}^{\infty} a_n e^{-\mu_n} \exp\{s\lambda_n + w\mu_n\}, \tag{8}$$

that is series (3) with $b_{nn} = a_n e^{-\mu_n}\,(n \ge 0)$ and $b_{nm} = 0\,(m \ne n)$. It is clear that for this series $H(s,1) = F(s)$ for all $s \in \mathbb{C}$, and if $\operatorname{Re} w \le 0$ then in view of inequality $\ln|a_n| + \lambda_n \Psi\big(\varphi(\lambda_n)\big) \le \mu_n$ and Lemma 1 we have

$$|H(s,w)| \le \sum_{n=0}^{\infty} |a_n| e^{-\mu_n} \exp\{\sigma\lambda_n\} \le F_\Phi(\sigma),$$

where F_Φ is defined by formula (7).

Hence for series (8) relations (4) and (5) hold, and we have to prove the absolute convergence of this series in Q. In view of (5) it is sufficient to show that

$$\sum_{n=0}^{\infty} |a_n| e^{-\mu_n} \exp\{\sigma\lambda_n + \omega\mu_n\} < +\infty \tag{9}$$

for all $\sigma \in \mathbb{R}$ and $0 < \omega < \varrho^*/(\varrho^* - 1)$. We put $\Phi_\varrho(\sigma) = \frac{1}{\varrho}\Phi(\varrho\sigma) + \gamma(\varrho)$. Then $\Phi'_\varrho(\sigma) = \Phi'(\varrho\sigma)$, the inverse to Φ'_ϱ is the function $\varphi_\varrho(t) = \frac{1}{\varrho}\varphi(t)$, and the function associated by Newton to Φ_ϱ is $\Psi_\varrho(\sigma) = \frac{1}{\varrho}\Psi(\varrho\sigma) - \gamma(\varrho)/\Phi'(\varrho\sigma)$. Thus,

$$\Psi_\varrho\big(\varphi_\varrho(t)\big) = \frac{1}{\varrho}\Psi\big(\varphi(t)\big) - \frac{1}{t}\gamma(\varrho)$$

and in view of Lemma 1 from (6) we have

$$\ln|a_n| \le -\frac{1}{\varrho}\lambda_n \Psi\big(\varphi(\lambda_n)\big) + \gamma(\varrho), \quad n \ge 0.$$

If $\mu_n > 0$ then it follows that

$$\begin{aligned}
|a_n| \exp\big\{\sigma\lambda_n + (\omega - 1)\mu_n\big\} &= \exp\big\{\sigma\lambda_n + \ln|a_n| + (\omega - 1)\big(\ln|a_n| + \lambda_n\Psi(\varphi(\lambda_n))\big)\big\} = \\
= \exp\big\{\sigma\lambda_n + \omega\ln|a_n| + (\omega - 1)\lambda_n\Psi(\varphi(\lambda_n))\big\} &\le \exp\Big\{\sigma\lambda_n - \Big(\frac{\omega}{\varrho} - \omega + 1\Big)\lambda_n\Psi(\varphi(\lambda_n))\Big\} = \\
= \exp\Big\{-(1 + o(1))\Big(\frac{\omega}{\varrho} - \omega + 1\Big)\lambda_n\Psi(\varphi(\lambda_n))\Big\}&, \quad n \to \infty,
\end{aligned} \tag{10}$$

provided $\frac{\omega}{\varrho}-\omega+1>0$. If $\varrho\in(0,1]$ then the last inequality is obvious, and if $\varrho>1$ then it is equivalent to the inequality $\omega<\varrho/(\varrho-1)=\varrho^*/(\varrho^*-1)$.

Using (10) and (2) we have for all $\sigma\in\mathbb{R}$ and $0<\omega<\varrho^*/(\varrho^*-1)$

$$\sum_{n=0}^{\infty}a_ne^{-\mu_n}\exp\{\sigma\lambda_n+\omega\mu_n\}=\sum_{\mu_n=0}a_n\exp\{\sigma\lambda_n\}+\sum_{\mu_n>0}a_n\exp\left\{\sigma\lambda_n+(\omega-1)\mu_n\right\}\le$$
$$\le\sum_{n=0}^{\infty}a_n\exp\{\sigma\lambda_n\}+\sum_{n=0}^{\infty}\exp\Big\{-(1+o(1))\Big(\frac{\omega}{\varrho}-\omega+1\Big)\lambda_n\Psi(\varphi(\lambda_n))\Big\}<+\infty,$$

that is we have (9), and Lemma 2 is proved.

The following lemma is in slightly different form than in [1], but we give its proof for convenience of the reader.

Lemma 3. *Let $\Phi\in\Omega$ and $F\in S(\Lambda)$. If there exists an absolutely convergent in $\mathbb{C}\times\{w:\,\mathrm{Re}\,w<a\}$, $a>1$, double Dirichlet series (3) such that relations (4) and*

$$|H(s,w)|\le\exp\{\Phi(\sigma)\},\quad s\in\mathbb{C},\,\mathrm{Re}\,w\le0,\tag{11}$$

hold then for $\varrho>a/(a-1)$ inequality (6) holds.

Proof. We put

$$M(\sigma,\omega,H)=\sup\big\{|H(\sigma+it,\omega+i\tau)|:\,t\in\mathbb{R},\tau\in\mathbb{R}\big\},\quad\sigma\in\mathbb{R},\,\omega<a.$$

Then the function $h(\sigma,\omega)=\ln M(\sigma,\omega,H)$ is convex in $\mathbb{R}\times(-\infty,a)$. We take in $\mathbb{R}\times(-\infty,a)$ three points $z_1=(0,\alpha)$, $z_2=(\sigma,1)$ and $z_3=\big(\frac{\alpha\sigma}{\alpha-1},0\big)$, where $\alpha\in(1,a)$ is an arbitrary number. It is clear that $z_2=\frac{1}{\alpha}z_1+\big(1-\frac{1}{\alpha}\big)z_3$. Thus from convexity of h we have $h(z_2)=\frac{1}{\alpha}h(z_1)+\big(1-\frac{1}{\alpha}\big)h(z_3)$, that is

$$h(\sigma,1)=\frac{1}{\alpha}h(0,\alpha)+\Big(1-\frac{1}{\alpha}\Big)h\Big(\frac{\alpha\sigma}{\alpha-1},0\Big).\tag{12}$$

Inequality (4) implies $M(\sigma,F)\le M(\sigma,1,H)$, that is $M(\sigma,F)\le h(\sigma,1)$, and (11) implies the inequality $M(\sigma,0,H)\le\Phi(\sigma)$, that is $h\big(\frac{\alpha\sigma}{\alpha-1}\big)\le\Phi\big(\frac{\alpha\sigma}{\alpha-1}\big)$. Thus from (12) we have

$$\ln M(\sigma,F)\le\Big(1-\frac{1}{\alpha}\Big)\Phi\Big(\frac{\alpha\sigma}{\alpha-1}\Big)=\gamma_0(\alpha)\tag{13}$$

for every $\alpha\in(1,a)$ and all $\sigma\in\mathbb{R}$, where $\gamma_0(\alpha)=\frac{1}{\alpha}\ln M(0,\alpha,H)$. We put $\varrho=\alpha/(\alpha-1)$ and $\gamma(\varrho)=\gamma_0(\varrho/(\varrho-1)$. Then (13) implies (6) for all $\varrho>a/(a-1)$, and Lemma 3 is proved.

Lemma 3. *Suppose that the exponents of absolutely convergent in $\mathbb{C}$ Dirichlet series (1) can be nonincreasing and nonnegative, but the sequence (λ_n) contains an infinite number of positive terms. If $1>|a_n|\downarrow0$ and $\ln n=o\big(\ln\frac{1}{|a_n|}\big)$ as $n\to\infty$, then*

$$M(\sigma,F)\le\mu\big((1+o(1))\sigma,F\big),\quad\sigma\to+\infty.$$

Proof. If $\lambda_n > 0$ we put $r_n = \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}$. Then for every $\varepsilon > 0$

$$M(\sigma, F) \le \sum_{n=0}^{\infty} |a_n| \exp\{\sigma\lambda_n\} = \Big(\sum_{\lambda_n=0} + \sum_{r_n \le (1+\varepsilon)\sigma} + \sum_{r_n > (1+\varepsilon)\sigma}\Big) |a_n| \exp\{\sigma\lambda_n\} \le$$
$$\le K_1 + \sum_{r_n \le (1+\varepsilon)\sigma} |a_n| \exp\big\{(1+\varepsilon)\sigma\lambda_n - \varepsilon\sigma\lambda_n\big\} + \sum_{r_n > (1+\varepsilon)\sigma} |a_n| \exp\{\sigma\lambda_n\} \le$$
$$\le K_1 + \mu((1+\varepsilon)\sigma, F) \sum_{r_n \le (1+\varepsilon)\sigma} \exp\big\{-\varepsilon\lambda_n r_n/(1+\varepsilon)\big\} +$$
$$+ \sum_{r_n > (1+\varepsilon)\sigma} |a_n| \exp\big\{\lambda_n r_n/(1+\varepsilon)\big\} \le K_1 + \mu((1+\varepsilon)\sigma, F) \sum_{n=0}^{\infty} |a_n|^{\varepsilon/(1+\varepsilon)}, \quad (14)$$

where $K_1 = \text{const}$. From the condition $\ln n = o\big(\ln \frac{1}{|a_n|}\big)$, $n \to \infty$, it follows that

$$\sum_{n=0}^{\infty} |a_n|^{\varepsilon/(1+\varepsilon)} = K_2(\varepsilon) < +\infty.$$

Hence from (14) we have $M(\sigma, F) \le K_3(\varepsilon)\mu\big((1+\varepsilon)\sigma, F\big)$, where $K_3(\varepsilon) = \text{const}$. But the function $\ln \mu(\sigma, F)$ is convex. Thus, $M(\sigma, F) \le \mu\big((1+2\varepsilon)\sigma, F\big)$ for $\sigma \ge \sigma_0$, and Lemma 4 is proved.

3°. We prove Theorem 1. Let $\varepsilon > 0$ be such that $\frac{1}{\varepsilon}(1+\varepsilon)\ln(1+\varepsilon) < q$, and let $\eta = \varepsilon - \frac{1}{q}(1+\varepsilon)\ln(1+\varepsilon)$. Then $0 < \eta < \varepsilon$ and from the inequality $\ln M(\sigma, F) \le \Phi\big((1+(1))\sigma\big)$, $\sigma \to +\infty$, it follows that $\ln M(\sigma, F) \le \Phi\big((1+\eta)\sigma\big)$, $\sigma \ge \sigma_1(\eta)$. But

$$\ln \frac{\Phi((1+\varepsilon)\sigma)}{\Phi((1+\eta)\sigma)} = \frac{\Phi'((1+\xi)\sigma)}{\Phi'((1+\xi)\sigma)}(\varepsilon - \eta)\sigma \ge \frac{q(\varepsilon-\eta)}{1+\varepsilon} = \ln(1+\varepsilon),$$

where $\eta \le \xi \le \varepsilon$, for all $\sigma \ge \sigma_2 \ge \max\{\sigma_0, \sigma_1\}$. Thus we have

$$\ln M(\sigma, F) \le \frac{1}{1+\varepsilon}\Phi\big((1+\varepsilon)\sigma\big), \quad \sigma \ge \sigma_2,$$

whence we obtain there inequality (6) with $\varrho = 1+\varepsilon$ and some $\gamma(1+\varepsilon)$. By Lemma 2 exists absolutely convergent in $\mathbb{C} \times \{w : \operatorname{Re} w < (1+\varepsilon)/\varepsilon\}$ double Dirichlet series (3) such that relations (4) and (5) hold with the function F_Φ defined in (7). In view of arbitrariness of ε this series is absolutely convergent in $\mathbb{C}$, and we need to estimate the function F_Φ.

The coefficients $a_n = \exp\big\{-\lambda_n\Psi(\varphi(\lambda_n))\big\} \downarrow 0$ $(n \to \infty)$ and in view of (2) the relation $\ln n = o\big(\ln \frac{1}{|a_n|}\big)$, $n \to \infty$, holds. Thus by Lemma 4

$$F_\Phi(\sigma) \le \mu\big((1+o(1))\sigma, F_\Phi\big), \quad \sigma \to +\infty,$$

and by Lemma 1 $\ln \mu(\sigma, F_\Phi) \le \Phi(\sigma)$, $\sigma \in \mathbb{R}$. Hence we have

$$\ln F_\Phi(\sigma) \le \Phi\big((1+o(1))\sigma\big), \quad \sigma \to \infty,$$

and so necessarity is proved.

If now series (3) is absolutely convergent in $\mathbb{C}$ and satisfies (4) and (5) with $\ln F_\Phi(\sigma) \le \Phi\big((1+o(1))\sigma\big)$, $\sigma \to \infty$, that is for every $\varepsilon > 0$ and all $\sigma \in \mathbb{R}$ the inequality $\ln F_\Phi(\sigma) \le \Phi\big((1+\varepsilon)\sigma\big) + \gamma_1(\varepsilon)$ holds, then by Lemma 3

$$\ln M(\sigma, F) \le \frac{1}{\varrho}\Phi\big((1+\varepsilon)\sigma\big) + \gamma_1(\varepsilon) + \gamma(\varrho), \quad \gamma(\varrho) = \text{const},$$

whence in view of arbitrariness of $\varepsilon > 0$ and $\varrho > 1$ we have $\ln M(\sigma, F) \le \Phi\big((1 + o(1))\sigma\big)$, $\sigma \to \infty$. Theorem 1 is completely proved.

Finally, we prove Theorem 2. Since $\Phi(x) = \alpha^{-1}(x)$ and $x\alpha'(x)/\alpha(x) \to 0$ as $x \to +\infty$, we have $\sigma\Phi'(\sigma)/\Phi(\sigma) \to +\infty$ as $\sigma \to +\infty$. Thus if $0 < a < b < +\infty$ then $\ln \Phi(b\sigma) - \ln \Phi(a\sigma) \to +\infty$ as $\sigma \to +\infty$, and if $\ln M(\sigma, F) \le \Phi(a\sigma)$, $\sigma \ge \sigma_0$, then $\ln M(\sigma, F) \le \frac{1}{b}\Phi(b\sigma)$, $\sigma \ge \sigma_1 \ge \sigma_0$, for every $b > a$.

Since the α-order of F is at most ϱ then for every $\varepsilon > 0$ and all $\sigma \ge \sigma^*(\varepsilon)$ we have

$$\ln M(\sigma, F) \le \alpha^{-1}((\varrho+\varepsilon)\sigma) = \Phi((\varrho+\varepsilon)\sigma)$$

and therefore for all $\sigma \in \mathbb{R}$

$$\ln M(\sigma, F) \le \frac{1}{1+2\varepsilon}\Phi\big((\varrho+2\varepsilon)\sigma\big) + \gamma(\varepsilon), \quad \gamma(\varepsilon) = \text{const},$$

that is inequality (6) with $\varrho + 2\varepsilon$ instead of ϱ holds. Thus by Lemma 2 there exists absolutely convergent in $\mathbb{C} \times \{w : \operatorname{Re} w < (\varrho+2\varepsilon)/(\varrho+2\varepsilon-1)\}$ double Dirichlet series (3) such that the relations (4) and (5) hold with the function F_Φ defined in (7). In view of arbitrariness of ε this series is absolutely convergent in $\mathbb{C} \times \{w : \operatorname{Re} w < \varrho/(\varrho-1)\}$ and as in the proof of Theorem 1 we have

$$\ln F_\Phi(\sigma) \le \Phi\big((1+o(1))\sigma\big) = \alpha^{-1}\big((1+o(1))\sigma\big), \quad \sigma \to +\infty,$$

that is the α-order of F is at most 1. Necessarity is proved. The proof of sufficiency is analogous to that of sufficiency in Theorem 1.

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Received 15.06.1997