УДК 515.12

ON TOPOLOGICAL GROUPS CONTAINING A FRÉCHET-URYSOHN FAN

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T. Banakh. *On topological groups containing a Fréchet-Urysohn fan*, Matematychni Studii, **9**(1998) 149–154.

Suppose G is a topological group containing a (closed) topological copy of the Fréchet-Urysohn fan. If G is a perfectly normal sequential space (a normal k-space) then every closed metrizable subset in G is locally compact. Applying this result to topological groups whose underlying topological space can be written as the direct limit of a sequence of closed metrizable subsets, we get that every such a group either is metrizable or is homeomorphic to the product of a k_ω-space and a discrete space.

The present investigation was stimulated by the paper [Pe] of E. Pentsak who studied the topology of the direct limit $X^\infty = \varinjlim X^n$ of the sequence

$$X \subset X \times X \subset X \times X \times X \subset \dots,$$

where $(X,*)$ was a "nice" pointed space and X^n was identified with the subspace $X^n \times \{*\}$ of X^{n+1}. In particular, in [Pe] the topology of the direct limit l_2^∞, where l_2 is the separable Hilbert space, was characterized. To characterize the space l_2^∞, it was necessary to glue together maps into l_2^∞ and at this point it turned out that the equiconnected function generated by the natural convex structure on l_2^∞ was discontinuous. The same concerned the addition operation on l_2^∞ — it was discontinuous.

So, the following question arose: is l_2^∞ homeomorphic to a topological group or a convex set in a linear topological space?

We pose this problem more generally: find simple conditions on a topological space X under which X does not support certain algebraic structure.

In order to answer this question, we will define spaces K, V, and W, called test spaces, and will prove that existence in X (closed) subspaces homeomorphic to one (or several) of the spaces K, V, W forbids X to carry certain algebraic structures.

Now we define two of three test spaces.

1) *The space K.* Let

$$K = \big\{(0,0)\big\} \cup \big\{(\tfrac{1}{n}, \tfrac{1}{n\cdot m}) \mid n,m \in \mathbb{N}\big\} \subset \mathbb{R}^2.$$

The space K is metrizable and not locally compact. Moreover, K is a minimal space with these properties in the sense that each metrizable non-locally compact space

1991 *Mathematics Subject Classification.* 22A05, 22A15, 54H11.

contains a closed copy of K. For the sake of simplicity of denotations in the sequel, put $x_0=(0,0)$ and $x_{n,m}=(\frac{1}{n},\frac{1}{nm})$, $n,m\in\mathbb{N}$. Thus $K=\{x_0,x_{n,m}\mid n,m\in\mathbb{N}\}$.

2) *The space V (the Fréchet-Urysohn fan).* Let $S_0=\{0\}\cup\{\frac{1}{n}\mid n\in\mathbb{N}\}$ denote the convergent sequence and let

$$V=\mathbb{N}\times S_0/\mathbb{N}\times\{0\}.$$

Denote by $\pi_V\colon\mathbb{N}\times S_0\to V$ the quotient map. Let $y_{n,m}=\pi_V(n,\frac{1}{m})$, $n,m\in\mathbb{N}$, and y_0 be the (unique) non-isolated point of V. So $V=\{y_0,y_{n,m}\mid n,m\in\mathbb{N}\}$. Evidently, for every $n\in\mathbb{N}$ the sequence $\{y_{n,m}\}_{m=1}^{\infty}$ converges to y_0. For each $k\in\mathbb{N}$ let $V_k=\{y_0,y_{n,m}\mid n\le k, m\in\mathbb{N}\}$. It is easy to see that V has the direct limit topology with respect to the sequence $V_1\subset V_2\subset\dots$ (that is a set $U\subset V$ is open if and only if the intersection $U\cap V_n$ is open in V_n for every $n\in\mathbb{N}$).

A space X contains a closed copy of V, provided X can be written as the direct limit of a sequence

$$X_1\subset X_2\subset\dots,$$

where each X_n is a closed metrizable subset of X, nowhere dense in X_{n+1}. In particular, the space l_2^∞ contains a topological copy of V.

We call a subset A of a topological group G *multiplicative* if for every $a,b\in A$ we have $a*b\in A$ (here $*$ stands for the group operation on G). The following theorem implies that l_2^∞ carries no topological group structure.

Theorem 1. *A normal k-space X containing closed copies of the test spaces K and V is homeomorphic to a) no closed multiplicative subset of a topological group and b) no closed convex set in a linear topological space.*

Proof. Assume the converse and let $f\colon K\times V\to X$ be the map defined for $(x,y)\in K\times V\subset X\times X$ by $f(x,y)=x*y$ if X is a closed multiplicative subset of a topological group with $*$ standing for the group operation, or by $f(x,y)=\frac{1}{2}x+\frac{1}{2}y$ if X is a closed convex set in a linear topological space. It is easily verified that the map f has the following properties:
1) the map $(\mathrm{pr}_K,f)\colon K\times V\to K\times X$ is a closed embedding;
2) the map $f_{y_0}\colon K\to X$ defined by $f_{y_0}\colon x\mapsto f(x,y_0)$, $x\in K$, is a closed embedding.

Denote by $\mathrm{conv}(K)=\{(0,0)\}\cup\{(x,y)\mid 0<y\le x\le 1\}$ the convex hull of K in $\mathbb{R}^2$ and let $h\colon X\to\mathrm{conv}(K)$ be a continuous extension of the map $f_{y_0}^{-1}\colon f_{y_0}(K)\to K$. (This extension h can be constructed as follows. Observe that $\mathrm{conv}(K)$ being a convex G_δ-subset of the plane, admits a closed embedding $i\colon\mathrm{conv}(K)\to\mathbb{R}^\omega$ and a retraction $r\colon\mathbb{R}^\omega\to\mathrm{conv}(K)$, $r\circ i=\mathrm{id}$. By the Tietze Extension Theorem, the map $i\circ f_{y_0}^{-1}\colon f_{y_0}(K)\to\mathbb{R}^\omega$ can be extended to a map $g\colon X\to\mathbb{R}^\omega$. Then the map $h=r\circ g\colon X\to\mathrm{conv}(K)$ extends $f_{y_0}^{-1}$).

For $n,m\in\mathbb{N}$ let $\varepsilon_{n,m}=\frac{1}{2nm(m+1)}$ and set

$$O_{n,m}=\mathrm{conv}(K)\cap\big((\tfrac{1}{n}-\varepsilon_{n,m},\tfrac{1}{n}+\varepsilon_{n,m})\times(\tfrac{1}{nm}-\varepsilon_{n,m},\tfrac{1}{nm}+\varepsilon_{n,m})\big).$$

One can check that $O_{n,m}$, $n,m\in\mathbb{N}$, is a collection of pairwise disjoint neighborhoods of the points $x_{n,m}$ in $\mathrm{conv}(K)$. Since $\lim_{m\to\infty}y_{n,m}=y_0$ and $h\circ f(x_{n,m},y_0)=x_{n,m}=(\frac{1}{n},\frac{1}{nm})$, for every $n,m\in\mathbb{N}$, we may find a number $k(n,m)\in\mathbb{N}$ such that $h\circ f(x_{n,m},y_{n,k(n,m)})\in O_{n,m}$. Moreover, these numbers can be chosen so that $k(n,m+1)>k(n,m)$ for $n,m\in\mathbb{N}$. Let

$$Z=\{f(x_{n,m},y_{n,k(n,m)})\mid n,m\in\mathbb{N}\}.$$

Since $h\circ f(x_0,y_0)\notin O_{n,m}$ for all n,m, we get $f(x_0,y_0)\notin Z$. We claim that Z is closed in X. Since X is a k-space, it suffices to show that for every compact subset $C\subset X$ the intersection $C\cap Z$ is closed in C. Observe that if $f(x_{n,m},y_{n,k(n,m)})\in C$ then $h(C)\cap O_{n,m}\neq\varnothing$. Since $h(C)\subset \operatorname{conv}(K)$ is compact, we get the set $C_1=\{x_0\}\cup\{x_{n,m}\mid h(C)\cap O_{n,m}\neq\varnothing\}$ is compact too. Because the map $(\mathrm{pr}_K,f)\colon K\times V\to K\times X$ is a closed embedding, the set $(\mathrm{pr}_K,f)^{-1}(C_1\times C)\subset K\times V$ is compact and its projection C_2 onto V is compact too. Since $V=\varinjlim V_n$, we get $C_2\subset V_{n_0}$ for some n_0. Observe that $C\cap Z\subset f(C_1\times C_2)$ and thus $C\cap Z\subset\{f(x_{n,m},y_{n,k(n,m)})\mid n\le n_0, x_{n,m}\in C_1\}$. Because of compactness of C_1 the latter set is finite. Then $C\cap Z$ is finite and hence closed in C.

Therefore $Z\not\ni f(x_0,y_0)$ is a closed set in X. Using continuity of f, find neighborhoods $U(x_0)\subset K$ and $V(y_0)\subset V$ of x_0 and y_0 such that $f(U(x_0)\times V(y_0))\cap Z=\varnothing$. Fix n such that $x_{n,m}\in U(x_0)$ for every m. Since the sequence $\{y_{n,m}\}_{m=1}^\infty$ converges to y_0 and the sequence $\{k(n,m)\}_{m=1}^\infty$ is increasing we may find m such that $y_{n,k(n,m)}\in V(y_0)$. Then $f(U(x_0)\times V(y_0))\cap Z\ni f(x_{n,m},y_{n,k(n,m)})$ is not empty, a contradiction. □

In light of Theorem 1 the following Question arises.

Question. Is l_2^∞ homeomorphic a) to a multiplicative subset of a topological group, b) to a convex set in a linear topological space?

We will give a negative answer to the question a) under the additional assumption that the multiplicative subset contains an idempotent (the unity of the group). This follows from topological homogeneity of l_2^∞ and the following theorem.

Theorem 2. *A perfectly normal sequential space X containing a closed copy of K and a copy of V is homeomorphic to a) no closed convex set in a linear topological space, b) no closed multiplicative subset of a topological group, c) no multiplicative subset of a topological group such that the nonisolated point y_0 of $V\subset X$ is an idempotent.*

Proof. Assume the converse and similarly as in the previous proof define the map $f\colon K\times V\to X$. Observe that the map f has the following properties:

1′) the map $(\mathrm{pr}_k,f)\colon K\to V\to K\times X$ is an embedding;

2′) the map $f_{y_0}\colon K\to X$ is a closed embedding (in the case (c) f_{y_0} is the identity embedding because y_0 is the unity of the group).

Let $h\colon X\to\operatorname{conv}(K)$ be a continuous extension of the map $f_{y_0}^{-1}\colon f_{y_0}(K)\to K$ such that $h^{-1}(x_0)=f(x_0,y_0)$. (The map h can be constructed as follows. Using the perfect normality of X, fix a map $\lambda\colon X\to[0,1]$ such that $\lambda^{-1}(0)=f_{y_0}(K)$. Let $\tilde h\colon X\to\operatorname{conv}(K)$ be any extension of the map $f_{y_0}^{-1}\colon f_{y_0}(K)\to K$ and define a map $h\colon X\to\operatorname{conv}(K)$ letting $h(x)=\lambda(x)\cdot x_{1,1}+(1-\lambda(x))\tilde h(x)$ for $x\in K$.)

Similarly as in the previous proof define neighborhoods $O_{n,m}$ and the set $Z\not\ni f(x_0,y_0)$. As in the proof of Theorem 1, to get a contradiction, it suffices to show that the set Z is closed in X. Since the space X is sequential, it is enough to verify that for every convergent sequence $S\subset X$ the intersection $S\cap Z$ is closed in S. We shall show that $S\cap Z$ is always finite. Assume on the contrary, that $S\cap Z$ is infinite.

If $f(x_0,y_0)$ is not a limit point of S then $S\cap Z$ is finite because the collection $\{h^{-1}(O_{n,m})\}_{n,m\in\mathbb N}$ is discrete in $X\setminus f(x_0,y_0)$ and $S\setminus f(x_0,y_0)$ is compact. So assume $f(x_0,y_0)$ is a limit point of S. Enumerate $S\cap Z=\{z_i\}_{i=1}^\infty$. Evidently, the sequence $\{z_i\}_{i=1}^\infty$ converges to $f(x_0,y_0)$. For every $i\in\mathbb N$ find (unique) n_i,m_i such that $z_i=f(x_{n_i,m_i},y_{n_i,k(n_i,m_i)})$. Observe that the sequence $\{x_{n_i,m_i}\}_{i=1}^\infty$ converges to x_0. Then the sequence $\{(x_{n_i,m_i},z_i)\}_{i=1}^\infty$ converges to $(x_0,f(x_0,y_0))$ and lies (together with its limit) in $(\mathrm{pr}_K, f)(K, V)$. Since (pr_K, f) is an embedding and the projection $\mathrm{pr}_V: K \times V \to V$ is continuous, we get the set

$$C_2 = \{y_0\} \cup \{y_{n_i,k(n_i,m_i)} \mid i \in \mathbb{N}\} = \\ \mathrm{pr}_V \circ (\mathrm{pr}_K, f)^{-1}\big(\{(x_0, f(x_0,y_0))\} \cup \{(x_{n_i,m_i}, z_i) \mid i \in \mathbb{N}\}\big) \subset V$$

is compact. Then $C_2 \subset V_{n_0}$ for some n_0 and thus the sequence $\{n_i\}$ is bounded, a contradiction with the convergence of the sequence $\{x_{n_i,m_i}\}$. $\square$

Therefore both Theorems 1 and 2 give us that l_2^∞ is homeomorphic to no topological group. And what about its powers $(l_2^\infty)^n$? Do they admit a compatible group structure? It turns out that the answer here is negative too. Observe that Theorems 1 or 2 are not applicable because the powers of l_2^∞ are not k-spaces.

3) *The test space W.* We let W be the direct limit of a sequence $W_0 \subset W_1 \subset \dots$, where the spaces $W_n \subset K \times V$ are defined as follows. In $K \times V$ let us consider the points: $z_0 = (x_0, y_0)$, $z_{n,m} = (x_{n,m}, y_0)$, and $z_{n,m,p,q} = (x_{n,m}, y_{p,n+q})$, $n, m, p, q \in \mathbb{N}$. Let $W_0 = \{z_0, z_{n,m} \mid n, m \in \mathbb{N}\}$ and $W_p = W_{p-1} \cup \{z_{n,m,p,q} \mid n, m, q \in \mathbb{N}\}$ for $p \geq 1$. It is easy to see that for every $p \geq 1$ W_p is a closed subspace of $K \times V_p$ and W_0 is a nowhere dense closed copy of K in $W_0 \cup (W_p \setminus W_{p-1})$. On the union $W = \bigcup_{p=0}^\infty W_p$ consider the topology of the direct limit $\varinjlim W_p$ allowing a subset $U \subset W$ to be open if and only if $U \cap W_p$ is open in W_p for every p. Observe that a space X contains a closed copy of W, provided X can be written as the direct dimit of a sequence $X_0 \subset X_1 \subset \dots$, where each X_n is a closed metrizable subset of X, X_n is nowhere dense in X_{n+1}, and X_0 is not locally compact. Since the space l_2^∞ admits such a representation, it contains a copy of W.

Remark that each direct limit X of a sequence of metrizable spaces satisfies the following property:

($\mathcal{M}$) for every map $f: Y \to X$ of a metrizable space Y, every point $y \in Y$ has a neighborhood $U \subset Y$ such that $f(U)$ admits a countable neighborhood base at $f(y)$.

Observe that a finite product of spaces with the property ($\mathcal{M}$) satisfies this property too. Since the space l_2^∞ has the property ($\mathcal{M}$) and contains a copy of the test space W, the following theorem implies that for every $1 \leq n \leq \omega$ the power $(l_2^\infty)^n$ does not admit a compatible group operation.

Theorem 3. *A topological group containing a copy of the test space W can not be embedded into a countable product of spaces satisfying the property ($\mathcal{M}$).*

Proof. Suppose $W \subset X \subset \prod_{n=1}^\infty X_n$, where each X_n has the property ($\mathcal{M}$). Suppose X is a topological group and denote by $*$ the group operation and by e the unity of X. For each $k \in \mathbb{N}$ let $Y_k = \prod_{i=1}^k X_i$ and denote by $\mathrm{pr}_k: X \to Y_k$ the projection onto the initial k coordinates. By induction on k we shall construct increasing number sequences $\{n(k)\}_{k=1}^\infty$, $\{q(k,m)\}_{m=1}^\infty$, $k \in \mathbb{N}$, such that for every k the sequence

$$\big\{\mathrm{pr}_k(z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)})\big\}_{m=1}^\infty$$

converges in Y_k.

Let $n(0) = 0$ and suppose that for $k-1$, the number $n(k-1)$ is known. Define a map $f_k: W_0 \times W_k \to Y_k$ letting $f_k(x,y) = \mathrm{pr}_k(x^{-1} * y)$ for $(x,y) \in W_0 \times W_k$. Since the space $W_0 \times W_k$ is metrizable and the space Y_k, being a finite product of the spaces X_i's, has the property ($\mathcal{M}$), the point (z_0, z_0) has a neighborhood $U_1 \times U_2 \subset W_0 \times W_k$ such that $f_k(U_1 \times U_2)$ has a countable neighborhood base $\{O_m\}_{m=1}^\infty$ at $f_k(z_0, z_0) = \mathrm{pr}_k(e)$. Pick $n(k) > n(k-1)$ so that $z_{n(k),m} \in U_1$ for every m. Since for every m the sequence $\{z_{n(k),m,k,q}\}_{q=1}^{\infty}$ converges to $z_{n(k),m}$, we see that the sequence $\{\mathrm{pr}_k(z_{n(k),m}^{-1} * z_{n(k),m,k,q})\}_{q=1}^{\infty}$ converges to $\mathrm{pr}_k(e)$. Thus, inductively, for every m we can find a number $q(k,m) > q(k,m-1)$ such that $\mathrm{pr}_k(z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)}) \in O_m$. The iductive step is complete.

Consider the set

$$Z = \{z_{n(k),m,k,q(k,m)} \mid k, m \in \mathbb{N}\} \subset W$$

and notice that Z is closed in W. Since $z_0 \notin Z$, we may find neighborhoods $U(z_0), U(e) \subset X$ of z_0 and e such that $(U(z_0) * U(e)) \cap Z = \varnothing$. Let k be such that $U(e) \supset \mathrm{pr}_k^{-1}(O)$ for some neighborhood $O \subset Y_k$ of $\mathrm{pr}_k(e)$. We may assume the number k to be so large that $z_{n(k),m} \in U(z_0)$ for every m. Find finally m such that

$$\mathrm{pr}_k\big(z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)}\big) \in O.$$

Then $z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)} \in U(e)$ and hence the intersection $(U(z_0) * U(e)) \cap Z \ni z_{n(k),m}^{-1} * \big(z_{n(k),m}^{-1} * z_{n(k),m,k,q(k,m)}\big) = z_{n(k),m,k,q(k,m)}$ is not empty, a contradiction. □

Now let us consider some applications of the obtained results.

Structure of topological groups that are $\mathcal{M}_\omega$-spaces

Recall that a topological space X is called a *k_ω-space* if X contains a countable collection $\mathcal{K}$ of compact subsets of X such that a subset U of X is open in X if and only if the intersection $U \cap K$ is closed in K for every $K \in \mathcal{K}$ (equivalently, X is a k_ω-space, provided X is the direct limit of a sequence of its compact subsets).

We define a topological space X to be an *$\mathcal{M}_\omega$-space* if X contains a countable collection $\mathcal{M}$ of closed metrizable subsets of X such that a subset U of X is open in X if and only if the intersection $U \cap M$ is closed in M for every $M \in \mathcal{M}$ (equivalently, X is an $\mathcal{M}_\omega$-space, if X is the direct limit of a sequence of its closed metrizable subsets).

It turns out that an existence of a compatible group structure imposes very strict restrictions on the topology of $\mathcal{M}_\omega$-spaces.

Theorem 4. *Suppose a topological group X is an $\mathcal{M}_\omega$-space. If X is not metrizable, then*

(1) *X contains a closed copy of the Fréchet-Urysohn fan;*
(2) *each closed metrizable subset of X is locally compact;*
(3) *X contains an open subgroup H that is a k_ω-space;*
(4) *X is homeomorphic to a product of a k_ω-space and a discrete space;*
(5) *X is homeomorphic to an open subset of a k_ω-space.*

Proof. Suppose X is not metrizable and let e denote the unity of the group X. Let $X = \varinjlim X_n$ be the direct limit of a sequence $\{e\} = X_0 \subset X_1 \subset X_2 \subset \dots$ consisting of closed metrizable subsets of X. To prove 1) we will show that for every n there is m such that e is a limit point of the set $X_m \setminus X_n$ in X_m. Fix n and a decreasing neighborhood base $\{U_i\}_{i=1}^{\infty}$ of e in X_n. Since X is not metrizable, each U_i is not open in X, and thus U_i is not open in some $X_{m(i)}$. Consequently, there is a sequence $\{y_{ij}\}_{j=1}^{\infty} \subset X_{m(i)} \setminus X_n$ convergent to a point $x_i \in U_i$. Let $k(i) = \min\{k \in \mathbb{N} \mid \forall j_0 \in \mathbb{N}\ \exists j \geq j_0 \text{ such that } y_{ij} \in X_k\}$. Passing to a subsequence, if necessary, we may assume that $\{y_{ij}\}_{j=1}^{\infty} \subset X_{k(i)} \setminus X_{k(i)-1}$.

If $m = \sup\{k(i) \mid i \in \mathbb{N}\} < \infty$ then all the points y_{ij}, $i, j \in \mathbb{N}$ lie in the set $X_m \setminus X_n$. Since X_m is metrizable and the sequence $\{x_i\}$ tends to e, we may choose a subsequence $\{z_j\}_{j=1}^{\infty} \subset \{y_{ij} \mid i, j \in \mathbb{N}\}$ convergent to e. Thus e is a limit point of the set $X_m \setminus X_n$ and we are done.

Now suppose $\sup\{k(i) \mid i \in \mathbb{N}\} = \infty$. Using the continuity of the multiplication $*$, find $p \in \mathbb{N}$ such that $U_p * U_p \subset X_k$ for some k. Let i be such that $k(i) > k$ and $i \geq p$. Obviously, the sequence $\{x_i^{-1} * y_{ij}\}_{j=1}^{\infty}$ converges to e. We claim that there exists $j_0 \in \mathbb{N}$ such that $x_i^{-1} * y_{ij} \notin X_n$ for all $j \geq j_0$. Assuming the converse we would find j such that $x_i^{-1} * y_{ij} \in U_p \subset X_n$. Then $y_{ij} \in x_i * U_p \subset U_p * U_p \subset X_k$, a contradiction with $k(i) > k$ and $y_{ij} \in X_{k(i)} \setminus X_{k(i)-1}$. Thus we have proven that $\{x_i^{-1} * y_{ij}\}_{j \geq j_0} \subset X \setminus X_n$ for some j_0. Since this sequence converges to e, it is contained in some X_m.

Now we are ready to construct a closed copy of V in X. Applying the statement proved above, we may construct inductively an increasing number sequence $\{m(i)\}_{i=1}^{\infty}$ and sequences $\{y_{ij}\}_{j=1}^{\infty}$, $i \in \mathbb{N}$, such that

$$\lim_{j \to \infty} y_{ij} = e, \; y_{ij} \in X_{m(i)} \setminus X_{m(i-1)}, \; j \in \mathbb{N}.$$

Evidently, the set $\{e\} \cup \{y_{ij} \mid i, j \in \mathbb{N}\}$ is a closed copy of V in X. Hence a) is proven.

To prove 2), notice that X, being a direct limit of a sequence of metrizable spaces, is a perfectly normal sequential space. Since X contains a copy of the test space V, by Theorems 1 and 2, X contains no closed copy of the test space K. Because every metrizable non-locally compact space contains a closed copy of K, every closed metrizable subset in X must be locally compact.

To prove 3), let us firstly construct an open separable subset $U \subset X$. By 2), each X_n is locally compact. Thus, we may choose inductively open neighborhoods U_n of e in X_n so that the closure $\bar{U}_n$ is compact and $\bar{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$. Since each $\bar{U}_n$ is a (separable) metric compactum, the union $U = \bigcup_{n \in \mathbb{N}} U_n$ is an open separable subset of X. Then its span $H = \operatorname{span}(U)$ is an open separable subgroup of X. Using the separability of H, one can easily show that for every n the locally compact space $H_n = H \cap X_n$ is separable, and thus H_n is a k_ω-space. Then $H = \varinjlim H_n$ is a k_ω-space too. Since the subgroup H is open in X, the decomposition of X onto right residue classes of H just provides us with a homeomorphism of X onto the product $H \times D$ for some discrete space D. Let αD be the one-point compactification. Evidently, $H \times D$ is an open subset of the k_ω-space $H \times \alpha D$, thus (5) follows. □

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Received 15.06.1997