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ON EXTENSION OF CONTRAVARIANT FUNCTORS ONTO THE KLEISLI CATEGORY

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We consider the problem of extending contravariant functors onto the category Kleisli of a monad. The results are applied to the monad in the category of Tychonov spaces generated by the second iteration of the functor C_p (the space of functions in the topology of pointwise convergence).

1°. A monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$, where $T: \mathcal{C} \to \mathcal{C}$ is a covariant functor and $\eta: 1_{\mathcal{C}} \to T$, $\mu: T^2 \to T$ are natural transformations satisfying the conditions: $\mu \circ \eta T = \mu \circ T\eta = 1_T$ and $\mu \circ \mu T = \mu \circ T\mu$. The Kleisli category of $\mathbb{T}$ is the category $\mathcal{C}_{\mathbb{T}}$ defined as follows: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X, Y) = \mathcal{C}(X, TY)$, and the composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y, Z)$ is given by $g * f = \mu Z \circ Tg \circ f$.

Define the functor $I: \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ by $IX = X$, $X \in |\mathcal{C}|$ and $If = \eta Y \circ f$ for $f \in \mathcal{C}(X, Y)$.

The above definitions can be found, e.g., in [BW].

In [V] J. Vinárek considered the following problem. Suppose $F: \mathcal{C} \to \mathcal{C}$ is a covariant functor, is there a covariant functor $\bar{F}: \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ such that $IF = \bar{F}I$ (the problem of extension of functor onto the Kleisli category)?

In this paper we consider the corresponding problem for contravariant functor F.

2°. In what follows we fix a monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$.

The following result is a counterpart of a result of Vinárek [V].

Proposition 1. *There exists a bijective correspondence between the extensions of a contravariant functor F onto the category $\mathcal{C}_{\mathbb{T}}$ and the natural transformations $\xi: F \to TFT$ satisfying the conditions: (i) $TF\eta \circ \xi = \eta F$; (ii) $TF\mu \circ \xi = \mu FT^2 \circ T\xi T \circ \xi$.*

Proof. Suppose there is a natural transformations $\xi: F \to TFT$ such that conditions (i) and (ii) are satisfied. For every $X \in |\mathcal{C}|$ put $\bar{F}X = FX$ and for every $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$ put $\bar{F}f = TFf \circ \xi Y$.

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If $f = Ig$ for $g \in \mathcal{C}(X,Y)$, we obtain

$$\bar{F}f = \bar{F}(\eta Y \circ g) = TF(\eta Y \circ g) \circ \xi Y = TFg \circ TF\eta Y \circ \xi Y = TFg \circ \eta FY = \eta FX \circ Fg = IFg.$$

For $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$ and $g \in \mathcal{C}_{\mathbb{T}}(Y,Z)$ we obtain

$$\begin{aligned}\bar{F}(g * f) &= TF(g * f) \circ \xi Z = TF(\mu Z \circ Tg \circ f) \circ \xi Z = TFf \circ TFTg \circ TF\mu Z \circ \xi Z = \\ &= TFf \circ TFTg \circ \mu FT^2 Z \circ T\xi TZ \circ \xi Z = TFf \circ \mu FTY \circ T^2 FTg \circ T\xi TZ \circ \xi Z = \\ &= TFf \circ \mu FTY \circ T\xi Y \circ TFg \circ \xi Z = \mu FX \circ T^2 Ff \circ T\xi Y \circ TFg \circ \xi Z = \\ &= \mu FX \circ T(TFf \circ \xi Y) \circ TFg \circ \xi Z = \bar{F}f * \bar{F}g.\end{aligned}$$

Summing up we see that $\bar{F}$ is an extending F contravariant endofunctor in $\mathcal{C}_{\mathbb{T}}$.

On the other hand, suppose $\bar{F}\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ is an extension of F onto $\mathcal{C}_{\mathbb{T}}$. Since $1_{TX} \in \mathcal{C}_{\mathbb{T}}(TX,X)$, we see that $\bar{F}1_{TX} \in \mathcal{C}_{\mathbb{T}}(FX,FTX) = \mathcal{C}(FX,TFTX)$. Put $\xi X = \bar{F}1_{TX}$.

Show that $\xi = (\xi X)$ is a natural transformation from F to TFT. Given $f \in \mathcal{C}(X,Y)$ we obtain

$$\begin{aligned}\bar{F}(If * 1_{TX}) &= \bar{F}1_{TX} * \bar{F}If = \bar{F}1_{TX} * IFf = \mu FTX \circ T\xi X \circ \eta FX \circ Ff = \\ &= \mu FTX \circ \eta TFTX \circ \xi X \circ Ff = \xi X \circ Ff\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\bar{F}(If * 1_{TX}) &= \bar{F}\big(\mu Y \circ T(\eta Y \circ f) \circ 1_{TX}\big) = \bar{F}(\mu Y \circ T1_{TY} \circ \eta TY \circ Tf) = \\ &= \bar{F}(1_{TY} * ITf) = \bar{F}ITf * \bar{F}1_{TY} = IFTf * \xi Y = (\eta FTX \circ FTf) * \xi Y = \\ &= \mu FTX \circ T\eta FTX \circ TFTf \circ \xi Y = TFTf \circ \xi Y.\end{aligned}$$

Thus, $TFTf \circ \xi Y = \xi X \circ Ff$.

Show that (i) holds. We have

$$\begin{aligned}\eta FX &= IF1_X = \bar{F}I1_X = \bar{F}(I1_X * I1_X) = \bar{F}(\eta X * \eta X) = \bar{F}(\mu X \circ T\eta X \circ \eta X) = \\ &= \bar{F}\big(\mu X \circ T(1_{TX}) \circ \eta TX \circ \eta X\big) = \bar{F}(1_{TX} * I\eta X) = \bar{F}I\eta X * \bar{F}1_{TX} = \\ &= IF\eta X * \xi X = \mu FX \circ T\eta FX \circ TF\eta X \circ \xi X = TF\eta X \circ \xi X.\end{aligned}$$

Finally, we have to check (ii). We have

$$\begin{aligned}\mu FT^2 X \circ T\xi TX \circ \xi X &= \xi TX * \xi X = \bar{F}(1_{T^2 X}) * \bar{F}(1_{TX}) = \bar{F}(1_{TX} * 1_{T^2 X}) = \\ &= \bar{F}(\mu X \circ T1_{TX} \circ 1_{T^2 X}) = \bar{F}\mu X = \bar{F}(1_{TX} * I\mu X) = \bar{F}I\mu X * \bar{F}1_{TX} = \\ &= IF\mu X * \xi X = \mu FT^2 X \circ T\eta FT^2 X \circ TF\mu X \circ \xi X = TF\mu X \circ \xi X.\end{aligned}$$

Show that the above correspondence is a bijection. Given a natural transformation $\xi = (\xi X)$ satisfying (i) and (ii) consider the extension $\bar{F}$ defined by $\bar{F}f = TFf \circ \xi Y$, where $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$. Then $\bar{F}$ determines the natural transformation $\hat{\xi} = (\hat{\xi} X)$, $\hat{\xi} X = \bar{F}1_{TX}$ and we have $\hat{\xi} X = \bar{F}1_{TX} = TF1_{TX} \circ \xi X = \xi X$.

Conversely, given an extension $\bar{F}$ of F onto the category $\mathcal{C}_{\mathbb{T}}$, consider the natural transformation $\xi = (\xi X)$ defined by $\xi X = \bar{F} 1_{TX}$, $X \in |\mathcal{C}|$. The natural transformation ξ determines the extension $\widehat{F}$ of F onto $\mathcal{C}_{\mathbb{T}}$ by the formula $\widehat{F} f = TFf \circ \xi Y$, $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$. We have

$$\begin{aligned}\widehat{F} f &= TFf \circ \xi Y = TFf \circ \bar{F} 1_{TY} = \mu FX \circ T\eta FX \circ TFf \circ \bar{F} 1_{TY} = \\ &= \mu FX \circ T(\eta FX \circ Ff) \circ \bar{F} 1_{TY} = IFf * \bar{F} 1_{TY} = \bar{F} If * \bar{F} 1_{TY} = \bar{F}(1_{TY} * If) = \\ &= \bar{F}\big(1_{TY} * (\eta TY \circ f)\big) = \bar{F}(\mu Y \circ T1_{TY} \circ \eta TY \circ f) = \bar{F} f.\end{aligned}$$

Remark. From the proof of Proposition 1 we see that a bijective correspondence between extensions $\bar{F}$ of F onto $\mathcal{C}_{\mathbb{T}}$ and natural transformations ξ satisfying (i) and (ii) can be given by: given ξ we set $\bar{F} f = TFf \circ \xi Y$, for $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$; given $\bar{F}$ we set $\xi X = \bar{F} 1_{TX}$.

In this situation, the natural transformation ξ is called associated to the extension $\bar{F}$.

Definition. Suppose F, $F' \colon \mathcal{C} \to \mathcal{C}$ are contravariant functors admitting extensions onto $\mathcal{C}_{\mathbb{T}}$ and $\xi \colon F \to TFT$, $\xi' \colon F \to TF'T$ are the associated natural transformations. A natural transformation $t \colon F \to F'$ is called $\mathbb{T}$-coordinated if $\xi' \circ t = TtT \circ \xi$.

Proposition 2. *A natural transformation $t \colon F \to F'$ determines a natural transformation $\bar{t} = It \colon \bar{F} \to \bar{F}'$ of the extended functors if and only if t is $\mathbb{T}$-coordinated.*

Proof. Suppose t is $\mathbb{T}$-coordinated and $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$. Then

$$\begin{aligned}\bar{F}' f * \bar{t} Y &= \mu F'X \circ T^2 F' f \circ T\xi' Y \circ \eta F' Y \circ tY = \mu F'X \circ T^2 F' f \circ \eta TF'TY \circ \xi' Y \circ tY = \\ &= \mu F'X \circ T^2 F' f \circ \eta TF'TY \circ TtTY \circ \xi Y = \mu F'X \circ \eta TF'X \circ TF' f \circ TtTY \circ \xi Y = \\ &= \mu F'X \circ \eta TF'X \circ T(F' f \circ tTY) \circ \xi Y = \mu F'X \circ \eta TF'X \circ TtX \circ TFf \circ \xi Y = \\ &= \mu F'X \circ T\eta F'X \circ TtX \circ TFf \circ \xi Y = \bar{t} X * \bar{F} f.\end{aligned}$$

Now suppose $\bar{t} \colon \bar{F} \to \bar{F}'$ is a natural transformation. Put $f = 1_{TY} \in \mathcal{C}_{\mathbb{T}}(TY, Y)$; then

$$\begin{aligned}\bar{F}' f * \bar{t} Y &= \mu F'TY \circ T^2 F' f \circ T\xi' Y \circ \eta F' Y \circ tY = \\ &= \mu F'TY \circ T^2 F' 1_{TY} \circ \eta TF'TY \circ \xi' Y \circ tY = \xi' Y \circ tY.\end{aligned}$$

On the other hand, $\bar{t} TY * \bar{F} f = \mu F'TY \circ T\eta F'TY \circ TtTY \circ TF1_{TY} \circ \xi Y = TtTY \circ \xi Y$, and t is $\mathbb{T}$-coordinated.

Recall that $\mathbb{T}$ is said to be projective [V] provided there exists a natural transformation $\pi \colon T \to 1$ (projection) such that $\pi \circ \eta = 1$ and $\pi \circ \mu = \pi \circ \pi T$. The following is a counterpart of a result of Vinárek.

Proposition 3. *For any contravariant functor F and any projective monad $\mathbb{T}$ there exists an extension of F onto the category $\mathcal{C}_{\mathbb{T}}$.*

Proof. Put $\xi = \eta FT \circ F\pi$ (here π denotes the projection), then

$$TF\eta \circ \xi = TF\eta \circ \eta FT \circ F\pi = \eta F \circ F\eta \circ F\pi = \eta F \circ F(\pi \circ \eta) = \eta F.$$

Besides,

$$TF\mu \circ \xi = TF\mu \circ \eta FT \circ F\pi = \eta FT^2 \circ F\mu \circ F\pi = \eta FT^2 \circ F(\pi \circ \mu) = \\ = \eta FT^2 \circ F(\pi \circ \pi T) = \eta FT^2 \circ F\pi T \circ F\pi.$$

and, on the other hand,

$$\mu FT^2 \circ T\xi T \circ \xi = \mu FT^2 \circ T\eta FT^2 \circ TF\pi T \circ \eta FT \circ F\pi = \eta FT^2 \circ F\pi T \circ F\pi.$$

3°. Suppose $C: \mathcal{C} \to \mathcal{C}$ is a contravariant functor such that there exists a natural transformation $\eta: 1 \to C^2$ satisfying the property: $C\eta \circ \eta C = 1_C$. Put $T = C^2$ and define the natural transformation $\mu: T^2 = C^4 \to C^2 = T$ by the formula: $\mu = C\eta C$.

Proposition 4. *$\mathbb{T} = (T, \eta, \mu)$ is a monad on the category $\mathcal{C}$.*

Proof. We have

$$\mu \circ T\eta = C\eta C \circ C^2\eta = C(C\eta \circ \eta C) = 1_{C^2} = 1_T,$$
$$\mu \circ \eta T = C\eta C \circ \eta C^2 = 1_{C^2} = 1_T.$$

Besides,

$$\mu \circ T\mu = C\eta C \circ C^3\eta C = C(C^2\eta C \circ \eta C) = C(\eta C^3 \circ \eta C) = C\eta C \circ C\eta C^3 = \mu \circ \mu T.$$

In the sequel, we denote the monad descrited in Proposition 4 by $\mathbb{C}^2 = (T = C^2, \eta, \mu)$.

Proposition 5. *The natural transformation $\xi = \eta C^3 \circ \eta C: C \to C^5 = TCT$ satisfies conditions (i) and (ii) from Proposition 1.*

Proof. We have $C^3\eta \circ \eta C^3 \circ \eta C = \eta C \circ C\eta \circ \eta C = \eta C$, and this implies (i).

To prove (ii), we see that

$$\mu C^5 \circ C^2\xi C^2 \circ \xi = \mu C^5 \circ C^2\eta C^5 \circ C^2\eta C^3 \circ \eta C^3 \circ \eta C = C^2\eta C^3 \circ \eta C^3 \circ \eta C = \eta C^5 \circ \eta C^3 \circ \eta C,$$

and, on the other hand, $C^3\mu \circ \xi = C^4\eta C \circ \eta C^3 \circ \eta C^3 = \eta C^5 \circ C^2\eta C \circ \eta C = \eta C^5 \circ \eta C^3 \circ \eta C$, thus $\mu C^5 \circ C^2\xi C^2 \circ \xi = C^3\mu \circ \xi$.

Let $\mathcal{T}ych$ denote the category of Tychonov spaces and their continuous maps. For a Tychonov space X we denote by C_pX the space of real-valued functions on X endowed by the topology of pointwise convergence. This construction determines a contravariant functor in $\mathcal{T}ych$: for a map $f: X \to Y$ we have $C_pf(\varphi) = \varphi \circ f$, $\varphi \in C_pY$.

It is well-known that there exists a natural transformation $\eta: 1_{\mathcal{T}ych} \to C_pC_p = C_p^2$. It is defined by the condition: $\eta X(x)(\varphi) = \varphi(x)$, where $x \in X$, $\varphi \in C_pX$.

Lemma. $C_p\eta \circ \eta C_p = 1_{C_p}$.

Proof. For every $x \in X$ and $\varphi \in C_pX$ we have

$$C_p\eta X \circ \eta C_pX(\varphi)(x) = C_p\eta X\big(\eta C_pX(\varphi)\big)(x) = \eta C_pX(\varphi)\big(\eta X(x)\big) = \eta X(x)(\varphi) = \varphi(x).$$

Using Lemma and Proposition 1 we see that the functor $T_p = C_p^2$ determines a monad on the category $\mathcal{T}ych$ (see [PZ]).

Corollary. *The contravariant functor C_p has an extension onto the category $\mathcal{T}ych_{\mathbb{T}}$.*

4°. In this section we consider the problem of extending compositions of functors onto the category $\mathcal{C}_{\mathbb{T}}$.

Proposition 6. *Suppose a covariant functor $G: \mathcal{C} \to \mathcal{C}$ and a contravariant functor $F: \mathcal{C} \to \mathcal{C}$ have an extension onto the Kleisli category $\mathcal{C}_{\mathbb{T}}$ of monad $\mathbb{T} = (T, \eta, \mu)$. Then the composition GF has an extension onto the category $\mathcal{C}_{\mathbb{T}}$.*

Proof. By the cited above Vinárek result [V], there exists a natural transformation $\theta: GT \to TG$ such that $\theta \circ G\eta = \eta G$ and $\mu G \circ T\theta \circ \theta T = \theta \circ G\mu$. By Proposition 1, there exists a natural transformation $\xi: F \to TFT$ such that $TF\eta \circ \xi = \eta F$ and $TF\mu \circ \xi = \mu FT^2 \circ T\xi T \circ \xi$. Put $\widetilde{F} = GF$ and define the natural transformation $\tilde{\xi}: \widetilde{F} \to T\widetilde{F}T$ by the formula: $\tilde{\xi} = \theta FT \circ G\xi$. Then

$$T\widetilde{F}\eta \circ \tilde{\xi} = TGF\eta \circ \theta FT \circ G\xi = \theta F \circ GTF\eta \circ G\xi = \theta F \circ G\eta F = \eta GF = \eta \widetilde{F}$$

and

$$\begin{aligned} \mu\widetilde{F}T^2 \circ T\tilde{\xi}T \circ \tilde{\xi} &= \mu GFT^2 \circ T\theta FT^2 \circ TG\xi T \circ \theta FT \circ G\xi = \\ &= \mu GFT^2 \circ T\theta FT^2 \circ \theta TFT^2 \circ GT\xi T \circ G\xi = \\ &= \theta FT^2 \circ G\mu FT^2 \circ GT\xi T \circ G\xi = \theta FT^2 \circ GTF\mu \circ G\xi = T\widetilde{F}\mu \circ \tilde{\xi}. \end{aligned}$$

It is known (see [Z]) that every finite (and every transfinite, if the category $\mathcal{C}$ is complete) iteration T^α of the functor T has an extension onto the Kleisli category $\mathcal{C}_{\mathbb{T}}$. For the monad $\mathbb{T}$ defined in Section 3° this means that every even iteration $C^{2\alpha}$ of C has an extension onto $\mathcal{C}_{\mathbb{T}}$. Combining Propositions 5 and 6 we see that every odd iteration $C^{2\alpha+1} = C^{2\alpha}C$ can also be extended onto the category $\mathcal{C}_{\mathbb{T}}$. In particular, we have

Proposition 7. *Every iteration C_p^α of the contravariant functor C_p has an extension onto the category $\mathcal{T}ych_{\mathbb{T}}$.*

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